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REGULARITY PROPERTIES OF DISCRETE MAXIMAL OPERATORS IN METRIC SPACES

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ABSTRACT. We discuss the action of so-called discrete maximal operators on Sobolev, Hölder and Campanato spaces on metric measure spaces equipped with a doubling measure and a Poincaré inequality. The discrete maximal operators have better regularity properties than the standard maximal operators and hence they are more flexible tools in the metric context.

1. INTRODUCTION

By the maximal function theorem of Hardy, Littlewood and Wiener, the Hardy-Littlewood maximal operator is bounded on L^p-spaces when $1 < p \leq \infty$. Fot $p = 1$, there is a corresponding weak type estimate. The action of the maximal operator on some other function spaces is rather well understood as well. This note discusses boundedness properties of maximal operators in Sobolev, Hölder and Campanato spaces defined on metric measure spaces. The emphasis is on oscillation estimates for the maximal functions. In the Euclidean case, many of these estimates follow from the fact that the maximal operator commutes with translations or that the underlying space is linear, see [2], [6], [10], [12] and [14]. Clearly this property is not available in the metric context. There is also an unexpected obstruction in the metric case, as the examples in [4] show. Indeed, it may happen that even the standard Hardy-Littlewood maximal function of a Lipschitz continuous function may fail to be continuous. For this reason, we consider so-called discrete maximal functions, which are constructed in terms of coverings and partitions of unities. The discrete fractional maximal functions are comparable to the standard ones provided the measure is doubling. Hence for all practical purposes, it does not matter which one we choose. The main advantage is that the discrete maximal functions seem to behave better as far as regularity is concerned. This note is based on the original research articles [1], [2], [7], [11] and [15]. Most of the proofs can be found in these references, but we discuss some new aspects and represent some of the arguments here.

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2. Preliminaries

2.1. **Doubling measures.** Let $X = (X, d, \mu)$ be a complete metric space endowed with a metric d and a Borel regular measure μ such that $0 < \mu(B(x,r)) < \infty$ for all open balls

$$B(x,r) = \{y \in X : d(y,x) < r\}$$

with $r > 0$.

The measure μ is said to be doubling, if there exists a constant $c_\mu \geq 1$, called the doubling constant of μ, such that

$$\mu(B(x,2r)) \leq c_\mu \mu(B(x,r)),$$

for all $x \in X$ and $r > 0$. Note that an iteration of the doubling property implies, that if $B(x,R)$ is a ball in X, $y \in B(x,R)$ and $0 < r \leq R < \infty$, then

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq c\Big(\frac{r}{R}\Big)^Q \tag{2.1}$$

for some $c = c(c_\mu)$ and $Q = \log c_\mu / \log 2$. The exponent Q serves as a counterpart of dimension related to the measure.

The measure is Ahlfors Q-regular, if

$$c^{-1} r^Q \leq \mu(B(x,r)) \leq c r^Q$$

for every $x \in X$ and $0 < r \leq \operatorname{diam}(X)$. In case only the lower bound holds in the Alhfors regularity condition, then we say that the measure satisfies the measure lower bound condition.

2.2. **Upper gradients.** A nonnegative Borel function g on X is said to be an upper gradient of a function $u : X \to [-\infty, \infty]$, if for all rectifiable paths $\gamma : [0,1] \to X$ we have

$$|u(\gamma(0)) - u(\gamma(1))| \leq \int_\gamma g\, ds, \tag{2.2}$$

whenever both $u(\gamma(0))$ and $u(\gamma(1))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. The assumption that g is a Borel function is needed in the definition of the path integral. If g is merely a μ-measurable function and (2.2) holds for p-almost every path with $p \geq 1$, then g is said to be a p-weak upper gradient of u. By saying that (2.2) holds for p-almost every path we mean that it fails only for a path family with zero p-modulus. A family Γ of curves is of zero p-modulus if there is a non-negative Borel measurable function $\rho \in L^p(X)$ such that for all curves $\gamma \in \Gamma$, the path integral $\int_\gamma \rho\, ds$ is infinite.

By redefining a p-weak upper gradient on a set of measure zero we obtain an upper gradient of the same function. If g is a p-weak upper gradient of u, then there is a sequence g_i, $i = 1, 2, \ldots$, of upper gradients of u such that g_i converges to g in $L^p(X)$ as $i \to \infty$. Hence every p-weak upper gradient can be approximated by upper gradients in the $L^p(X)$-norm. If u has an upper gradient that belongs to $L^p(X)$ with $p > 1$, then it has a minimal p-weak upper gradient g_u in the sense that for every p-weak upper gradient g of u, $g_u \le g$ almost everywhere.

2.3. **Newtonian spaces.** We define the first order Sobolev spaces on the metric space X using the p-weak upper gradients. These spaces are called Newtonian spaces. For $u \in L^p(X)$ with $p \ge 1$, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. The Newtonian space on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if

$$\|u - v\|_{N^{1,p}(X)} = 0.$$

The same definition applies to subsets of X as well. The notion of a p-weak upper gradient is used to prove that $N^{1,p}(X)$ is a Banach space. For the properties of Newtonian spaces we refer to and [3], [17] and [18].

2.4. **Capacity.** The p-capacity of a set $E \subset X$ is the number

$$\operatorname{cap}_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E. We say that a property regarding points in X holds p-quasieverywhere, and denote p-q.e., if the set of points for which the property does not hold has capacity zero. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ p-q.e. Moreover, if $u, v \in N^{1,p}(X)$ and $u = v$ μ-almost everywhere, then $u \sim v$. Hence, the capacity is the correct gauge for distinguishing between two Newtonian functions.

Let E be a μ-measurable subset of X. The Sobolev space with zero boundary values is the space

$$N^{1,p}_0(E) = \{u|_E : u \in N^{1,p}(X) \text{ and } u = 0 \ p\text{-q.e. in } X \setminus E\}.$$

The space $N^{1,p}_0(E)$ equipped with the norm inherited from $N^{1,p}(X)$ is a Banach space, see [9].

2.5. **Poincaré inequality.** We say that X supports a weak $(1,p)$-Poincaré inequality if there exist constants $c > 0$ and $\tau \geq 1$ such that for all balls $B(x,r) \subset X$, for all locally integrable functions u on X and for all p-weak upper gradients g of u,

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq cr \Big(\fint_{B(x,\tau r)} g^p \, d\mu \Big)^{1/p},$$

where we denote

$$u_{B(x,r)} = \fint_{B(x,r)} u \, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

Note that since p-weak upper gradients can be approximated by upper gradients in the $L^p(X)$-norm, it would be enough to require the Poincaré inequality for upper gradients only.

By the Hölder inequality it is easy to see that if X supports a weak $(1,p)$-Poincaré inequality, then it supports a weak $(1,q)$-Poincaré inequality for every $q > p$. If X is complete and μ doubling, then it is shown in [8] that a weak $(1,p)$-Poincaré inequality implies a weak $(1,q)$-Poincaré inequality for some $q < p$. Thus $(1,p)$-Poincaré inequality has a deep self improving property.

2.6. **General assumptions.** Throughout the work, we assume that X is complete, μ is doubling and X supports a weak $(1,p)$-Poincaré inequality. This implies, for example, that Lipschitz functions are dense in $N^{1,p}(X)$ and that the Sobolev embedding theorem holds, see [3]. In some of the results, we make additional assumptions that will be specified at each occurrance.

3. The discrete maximal function

This section is devoted to the definition and basic properties of the discrete Hardy-Littlewood type maximal function.

3.1. **Covering of the space.** Let $r > 0$. Since the measure is doubling there are balls $B(x_i, r)$, $i = 1, 2, \dots$, such that

$$X = \bigcup_{i=1}^{\infty} B(x_i, r)$$

and

$$\sum_{i=1}^{\infty} \chi_{B(x_i, 6r)} \leq N < \infty.$$

This means that the dilated balls $B(x_i, 6r)$, $i = 1, 2, \dots$, are of bounded overlap. The constant N depends only on the doubling constant and, in particular, it is independent of r.

3.2. **Partition of unity.** We construct a partition of unity subordinate to the covering $B(x_i, r)$, $i = 1, 2, \dots$, of X. Indeed, there is a family of functions ψ_i, $i = 1, 2, \dots$, such that $0 \le \psi_i \le 1$, $\psi_i = 0$ in $X \setminus B(x_i, 6r)$, $\psi_i \ge \nu$ in $B(x_i, 3r)$, ψ_i is Lipschitz with constant L/r_i with $\nu > 0$ and L depending only on the covering, and

$$\sum_{i=1}^{\infty} \psi_i(x) = 1$$

for every $x \in X$. The partition of unity can be constructed by first choosing auxiliary cutoff functions φ_i so that $0 \le \varphi_i \le 1$, $\varphi_i = 0$ on $X \setminus B(x_i, 6r)$, $\varphi_i = 1$ in $B(x_i, 3r)$ and each φ_i is Lipschitz continuous with constant $1/r$. For example, we can take

$$\varphi_i(x) = \begin{cases} 1, & x \in B(x_i, 3r), \\ 2 - \dfrac{d(x, x_i)}{3r}, & x \in B(x_i, 6r) \setminus B(x_i, 3r), \\ 0, & x \in X \setminus B(x_i, 6r). \end{cases}$$

Then we define the functions ψ_i, $i = 1, 2, \dots$, in the partition of unity by

$$\psi_i(x) = \frac{\varphi_i(x)}{\sum_{j=1}^{\infty} \varphi_j(x)}.$$

It is not difficult to verify that these functions satisfy the required properties.

3.3. **Discrete convolution.** Let $f \in L^1_{\mathrm{loc}}(X)$. We define an approximation of f at the scale of $3r$ by setting

$$f_r(x) = \sum_{i=1}^{\infty} \psi_i(x) f_{B(x_i, 3r)}$$

for every $x \in X$. The function f_r is called the discrete convolution of f. The partition of unity and the discrete convolution are standard tools in harmonic analysis on homogeneous spaces, see for example [5] and [16].

Next we recall the basic properties of the discrete convolution. The easy proofs are left for the interested reader.

Remark 3.1. (1) The function f_r is Lipschitz continuous for every $r > 0$.

(2) $f_r \to f$ μ-almost everywhere in X as $r \to 0$.

(3) If $f \in L^p(X)$ for some $1 \le p \le \infty$, then there is a constant $c = c(c_\mu, p)$ such that

$$\|f_r\|_{L^p(X)} \le c \|f\|_{L^p(X)}.$$

Moreover, the discrete convolution approximates f in the $L^p(X)$-norm as $r \to 0$ if $1 \le p < \infty$.

3.4. **The maximal function.** Let r_j, $j = 1, 2, \dots$, be an enumeration of the positive rationals. For every radius r_j we choose covering balls $B(x_i, r_j)$, $i = 1, 2, \dots$, of X as above. Observe that for each radius there are many possible choices for the covering but we simply take one of those. We define the discrete maximal function of $f \in L^1_{\mathrm{loc}}(X)$ by

$$M^* f(x) = \sup_j |f|_{r_j}(x)$$

for every $x \in X$. Observe that the defined maximal operator depends on the chosen coverings. However, this is not a serious matter, since our estimates are independent of the chosen coverings.

As a supremum of continuous functions, the discrete maximal function is lower semicontinuous and hence measurable. It is also clear from the definition that the discrete maximal operator is homogeneous in the sense that if $\alpha \in \mathbb{R}$, then

$$M^*(\alpha f)(x) = |\alpha| M^* f(x)$$

for every $x \in X$. Moreover, the discrete maximal operator is sublinear, which means that

$$M^*(f + g)(x) \le M^* f(x) + M^* g(x)$$

for every $x \in X$. By Remark 3.1, we also have

$$|f(x)| = \lim_{t \to 0} |f|_t(x) \le M^* f(x)$$

for μ-almost every $x \in X$.

The discrete maximal function is closely related to the standard Hardy-Littlewood maximal function. Indeed, by Lemma 3.1 in [11] there is a constant $c = c(c_\mu) \ge 1$ such that

$$c^{-1} M f(x) \le M^* f(x) \le c M f(x) \tag{3.2}$$

for every $x \in X$, where

$$M f(x) = \sup_{r>0} \fint_{B(x,r)} |f| \, d\mu.$$

In this definition, we consider balls that are centered at x, but we obtain a noncentered maximal function by taking the supremum over all balls containing x. For doubling measures, these maximal functions are comparable and it does not matter which one we choose.

By the maximal function theorem for doubling measures (see [5]) we see that the Hardy-Littlewood maximal operator is bounded on $L^p(X)$ when $1 < p \le \infty$ and maps $L^1(X)$ into the weak $L^1(X)$. Since the maximal operators are comparable by (3.2) we conclude that the same results hold for the discrete maximal operator. In particular, there is a constant $c = c(p, c_\mu)$ such that

$$\|M^* f\|_{L^p(X)} \le c\|Mf\|_{L^p(X)} \le c\|f\|_{L^p(X)} \tag{3.3}$$

whenever $p > 1$. If $p = 1$ there is a constant $c = c(c_\mu)$ such that the weak type estimate

$$\mu(\{M^* f > \lambda\}) \le \mu(\{cMf > \lambda\}) \le \frac{c}{\lambda}\int_X |f|\, d\mu \tag{3.4}$$

holds for every $\lambda > 0$.

Remark 3.5. It is also possible to define a local maximal function in subdomains of X. The definition of the local maximal function is rather similar to that of the global maximal function. The main difference is that a Whitney type covering lemma is used in the construction of the discrete convolution instead of the covering of the space with balls of the same radii, see [1].

4. The discrete fractional maximal function

Let $0 \le \alpha \le Q$, where Q is as in (2.1). The fractional maximal function of $f \in L^1_{\text{loc}}(X)$ is defined as

$$M_\alpha f(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |f|\, d\mu.$$

For $\alpha = 0$, we have the usual Hardy-Littlewood maximal function.

Let the balls $B(x_i, r_j)$, $i = 1, 2, \dots$, be a covering of X as above. The discrete fractional maximal function of $f \in L^1_{\text{loc}}(X)$ is

$$M^*_\alpha f(x) = \sup_j |f|^\alpha_{r_j}(x)$$

for every $x \in X$. For $\alpha = 0$, we obtain the discrete Hardy-Littlewood type maximal function. See [7] for more on the discrete fractional maximal function.

The following versions of the maximal function theorem hold for the fractional maximal function. We present the simple proofs here although the results are well-known for the experts.

Theorem 4.1. *Assume that the measure is doubling and that the measure lower bound condition holds. Let $p > 1$ and assume that $0 < \alpha < Q/p$. Then there is a constant c, depending only on the the doubling constant, constant in the measure lower bound, p and α, such that*

$$\|M_\alpha f\|_{L^{p^*}(X)} \le c\|f\|_{L^p(X)},$$

for every $f \in L^p(X)$ with $p^* = Qp/(Q - \alpha p)$.

Proof. Let $u \in L^p(X)$, $x \in X$ and $r > 0$. As $1/p^* = 1/p - \alpha/Q$, the measure lower bound and Hölder's inequality imply that

$$
\begin{aligned}
r^\alpha \fint_{B(x,r)} |u|\,d\mu &= \frac{r^\alpha}{\mu(B(x,r))} \int_{B(x,r)} |u|^{p/p^*} |u|^{\alpha p/Q}\,d\mu \\
&\le c\mu(B(x,r))^{(\alpha-Q)/Q} \Big(\int_{B(x,r)} |u|^{(p/p^*)\cdot Q/(Q-\alpha)}\,d\mu \Big)^{(Q-\alpha)/Q} \\
&\qquad \cdot \Big(\int_{B(x,r)} |u|^p\,d\mu \Big)^{\alpha/Q} \\
&\le c \Big(\fint_{B(x,r)} |u|^{(p/p^*)\cdot Q/(Q-\alpha)}\,d\mu \Big)^{(Q-\alpha)/Q} \Big(\int_X |u|^p\,d\mu \Big)^{\alpha/Q} \\
&\le c \big(M|u|^{(p/p^*)\cdot Q/(Q-\alpha)}(x) \big)^{(Q-\alpha)/Q} \Big(\int_X |u|^p\,d\mu \Big)^{\alpha/Q}.
\end{aligned}
$$

By taking the supremum over the radii on the left-hand side, we have

$$
M_\alpha u(x) \le c\big(M|u|^{(p/p^*)\cdot Q/(Q-\alpha)}(x) \big)^{(Q-\alpha)/Q} \Big(\int_X |u|^p\,d\mu \Big)^{\alpha/Q}.
$$

By integrating the estimate above and using the Hardy-Littlewood maximal function theorem with the exponent $p^*(Q-\alpha)/Q > 1$, we arrive at

$$
\begin{aligned}
\Big(\int_X (M_\alpha u)^{p^*}\,d\mu \Big)^{1/p^*} &\le c \Big(\int_X |u|^p\,d\mu \Big)^{1/p^*} \Big(\int_X |u|^p\,d\mu \Big)^{\alpha/Q} \\
&\le c \Big(\int_X |u|^p\,d\mu \Big)^{1/p}.
\end{aligned}
$$

This proves the claim. □

Remark 4.2. Let $\alpha = Q/p$. Under the same assumptions as in the previous theorem, we have

$$
\|M_\alpha f\|_{L^\infty(X)} \le c\|f\|_{L^p(X)},
$$

where the constant c depends only on the constant in the measure lower bound and p. By Hölder's inequality, we have that

$$
\begin{aligned}
r^\alpha \fint_{B(x,r)} |f|\,d\mu &\le \Big(r^{\alpha p} \fint_{B(x,r)} |f|^p\,d\mu \Big)^{1/p} \\
&\le c \Big(\int_{B(x,r)} |f|^p\,d\mu \Big)^{1/p} \le c\|f\|_{L^p(X)}
\end{aligned}
$$

for every $x \in X$ and $r > 0$. By taking the supremum over all radii $r > 0$ on the left-hand side, we obtain

$$
M_\alpha f(x) \le c\|f\|_{L^p(X)}
$$

for every $x \in X$ and the claim follows.

Then we recall a weak type estimate for the fractional maximal operator.

Theorem 4.3. *Assume that the measure is doubling and that the measure lower bound condition holds. Let $0 < \alpha < Q$. Then there is a constant c, depending only on the the doubling constant, the constant in the measure lower bound and α, such that*

$$\mu(\{M_\alpha f > \lambda\}) \le c\Big(\frac{\|f\|_{L^1(X)}}{\lambda}\Big)^{Q/(Q-\alpha)},$$

for every $f \in L^1(X)$.

Proof. Let $\lambda > 0$ and let $E_\lambda = \{M_\alpha u > \lambda\}$. For every $x \in E_\lambda$, there is r_x such that

$$r_x^\alpha \fint_{B(x,r_x)} |u|\, d\mu > \lambda.$$

By the measure lower bound, we have

$$r_x^{Q-\alpha} \le C\frac{\mu(B(x,r_x))}{r_x^\alpha} \le \int_{B(x,r_x)} |u|\, d\mu \le \|u\|_{L^1(X)},$$

and consequently, the radii r_x are uniformly bounded in E_λ. By the standard covering argument, we obtain a countable subcollection such that the balls $B(x_i, r_i)$, $i = 1, 2, \dots$, are pairwise disjoint and

$$E_\lambda \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_i).$$

By the measure lower bound, we also have

$$\lambda < r_i^\alpha \fint_{B(x_i,r_i)} |u|\, d\mu \le c\mu(B(x_i, r_i))^{(\alpha-Q)/Q} \int_{B(x_i,r_i)} |u|\, d\mu,$$

from which we conclude that

$$\mu(B(x_i, r_i))^{(Q-\alpha)/Q} \le \frac{c}{\lambda} \int_{B(x_i,r_i)} |u|\, d\mu$$

for every $i = 1, 2, \dots$ This implies that

$$\begin{aligned}
\mu(E_\lambda) &\le \sum_{i=1}^{\infty} \mu(B(x_i, 5r_i)) \le c\sum_{i=1}^{\infty} \mu(B(x_i, r_i)) \\
&\le c\Big(\sum_{i=1}^{\infty} \mu(B(x_i, r_i))^{(Q-\alpha)/Q}\Big)^{Q/(Q-\alpha)} \\
&\le c\Big(\sum_{i=1}^{\infty} \frac{1}{\lambda} \int_{B(x_i,r_i)} |u|\, d\mu\Big)^{Q/(Q-\alpha)} \le c\Big(\frac{\|u\|_{L^1(X)}}{\lambda}\Big)^{Q/(Q-\alpha)}.
\end{aligned}$$

□

The discrete fractional maximal function is comparable to the standard fractional maximal function, see [7].

Lemma 4.4. *Assume that the measure is doubling. Let $f \in L^1_{\mathrm{loc}}(X)$. Then there is a constant $c = c(c_\mu) \geq 1$ such that*

$$c^{-1} M_\alpha f(x) \leq M^*_\alpha f(x) \leq c M_\alpha f(x)$$

for every $x \in X$.

Again this implies that the previous L^p-bounds for the fractional maximal operator also hold for the discrete fractional maximal operator.

5. Sobolev space estimates

Our goal is to show that the discrete maximal operator preserves the smoothness of the function and that, under relatively mild conditions on the measure, the discrete fractional maximal smoothens the function.

We begin with a result for the discrete convolution. For the proof, we refer to [1], [2] and [11].

Lemma 5.1. *Suppose that $u \in N^{1,p}(X)$ with $p > 1$ and let $r > 0$. Then $u_r \in N^{1,p}(X)$ and there is a constant $c = c(c_\mu, p)$ and $q < p$ such that $c(Mg^q)^{1/q}$ is a p-weak upper gradient of u_r whenever g is a p-weak upper gradient of u.*

Remark 5.2. If $u \in N^{1,p}(X)$ with $p > 1$, then by the previous lemma $u_r \in N^{1,p}(X)$ for every $r > 0$. By Remark 3.1 we see that $u_r \to u$ in $L^p(X)$ and pointwise μ-almost everywhere as $r \to 0$. However, one dimensional examples show that u_r does not, in general, converge to u as $r \to 0$ in the Newtonian space $N^{1,p}(X)$. This can be seen by considering such partitions of unity in the construction of the maximal function that every component at all scales is constant in a set of large measure.

Now we are ready to conclude that the discrete maximal operator preserves Newtonian spaces. We use the following simple fact in the proof: Suppose that u_i are functions and g_i are p-weak upper gradients of u_i, $i = 1, 2, \dots$, respectively. Let $u = \sup_i u_i$ and $g = \sup_i g_i$. If u is finite μ-almost everywhere, then g is a p-weak upper gradient of u. For the proof, we refer to [3].

The next result shows that the discrete maximal operator is bounded in the Newtonian space.

Theorem 5.3. *If $u \in N^{1,p}(X)$ with $p > 1$, then $M^*u \in N^{1,p}(X)$. In addition, there is a constant $c = c(c_\mu, p)$ such that*

$$\|M^*u\|_{N^{1,p}(X)} \leq c\|u\|_{N^{1,p}(X)}.$$

Proof. By (3.3) we see that $M^*u \in L^p(X)$ and, in particular, $M^*u < \infty$ μ-almost everywhere in X. Since

$$M^*u(x) = \sup_j |u|_{r_j}(x)$$

and by the preceding lemma $c(Mg^q)^{1/q}$ is an upper gradient of $|u|_{r_j}$ for every j, we conclude that it is an upper gradient of M^*u. Here we also used the fact that every p-weak upper gradient of u will do as a p-weak upper gradient of $|u|$ as well. The claim follows from the maximal function theorem. □

Remark 5.4. The discrete maximal operator defined in a subdomain also preserves the boundary values in the Sobolev sense. In particular, the discrete maximal operator preserves Newtonian spaces with zero boundary values, see [2].

Next we study the behaviour of the discrete fractional maximal function in Newtonian spaces. The first result shows that the discrete fractional maximal function of a Sobolev function belongs to a Sobolev space with the Sobolev conjugate exponent. These results have been originally studied in [7], but we reproduce some details here.

Theorem 5.5. *Assume that the measure is doubling and that the measure lower bound condition holds. Let $u \in N^{1,p}(X)$ and $0 < \alpha < Q/p$. Then $M_\alpha^*u \in N^{1,p^*}(X)$ with $p^* = Qp/(Q - \alpha p)$. Moreover, there is a constant c, depending only on the doubling constant, the constant in the measure lower bound, p and α, such that*

$$\|M_\alpha^*u\|_{N^{1,p^*}(X)} \leq c\|u\|_{N^{1,p}(X)}.$$

Proof. Let $u \in N^{1,p}(X)$ and let $g \in L^p(X)$ be a weak upper gradient of u. By Theorem 4.1, we have

$$\|M_\alpha^*u\|_{L^{p^*}(X)} \leq c\|u\|_{L^p(X)}.$$

For the weak upper gradient, let $x, y \in B(x_j, r)$, and let

$$I_j = \{i : B(x_i, 6r) \cap B(x_j, r) \neq \emptyset\}.$$

By the bounded overlap of the balls $B(x_i, 6r)$, the set I_j is finite and the cardinality does not depend on j. By the L/r-Lipschitz continuity of functions ψ_i and by the $(1,q)$-Poincaré inequality, which follows from the $(1,p)$-Poincaré inequality for some $1<q<p$, we have

$$\begin{aligned}\big||u|_r^\alpha(x)-|u|_r^\alpha(y)\big| &= r^\alpha\Big|\sum_{i=1}^\infty\big(|u|_{B(x_i,3r)}-|u|_{B(x_j,3r)}\big)\big(\psi_i(x)-\psi_i(y)\big)\Big|\\ &\le cr^{\alpha-1}d(x,y)\sum_{i\in I_j}\big||u|_{B(x_i,3r)}-|u|_{B(x_j,3r)}\big|\\ &\le cr^{\alpha-1}d(x,y)\fint_{B(x_j,10r)}\big||u|-|u|_{B(x_j,10r)}\big|\,d\mu\\ &\le cr^\alpha d(x,y)\Big(\fint_{B(x_j,10\lambda r)}g^{p'}\,d\mu\Big)^{1/p'}.\end{aligned}$$

Since the pointwise Lipschitz constant of a function is a weak upper gradient, we see that

$$g_r(x)=cr^\alpha\sum_{j=1}^\infty\Big(\fint_{B(x_j,10\lambda r)}g^{p'}\,d\mu\Big)^{1/p'}\chi_{B(x_j,6r)}(x)$$

is a weak upper gradient of $|u|_r^\alpha$. Moreover, by the bounded overlap of the balls,

$$\begin{aligned}g_r(x)&\le c\sum_{j=1}^\infty\Big(r^{\alpha p'}\fint_{B(x_j,10\lambda r)}g^{p'}\,d\mu\Big)^{1/p'}\chi_{B(x_j,6r)}(x)\\ &\le c\big(M^*_{\alpha p'}g^{p'}(x)\big)^{1/p'}.\end{aligned}$$

By the same argument as in the proof of Theorem 5.3, we conclude that $\big(M^*_{\alpha p'}g^{p'}\big)^{1/p'}$ is a weak upper gradient of $M^*_\alpha u$. Since $g^{p'}\in L^{p/p'}(X)$ and $p/p'>1$, Theorem 4.1 implies that

$$\|\big(M^*_{\alpha p'}g^{p'}\big)^{1/p'}\|_{L^{p^*}(X)}\le c\|g\|_{L^p(X)}$$

and the claim follows. □

The following theorem is a generalization of the main result of [14] to the metric setting. It shows that the discrete fractional maximal operator is a smoothing operator. More precisely, the discrete fractional maximal function of an L^p-function has a weak upper gradient and both u and the weak upper gradient belong to a higher Lebesgue space than u.

Theorem 5.6. *Assume that the measure is doubling and that the measure lower bound condition holds. Let $u\in L^p(X)$ with $1<p<Q$ and*

$$1\le\alpha<Q/p,\quad p^*=Qp/(Q-\alpha p)\quad\text{and}\quad q=Qp/(Q-(\alpha-1)p).$$

Then $M^*_{\alpha-1}u$ is a weak upper gradient of $M^*_\alpha u$. Moreover, there is a constant c, depending only on the doubling constant, the constant in the measure lower bound, p and α, such that

$$\|M^*_\alpha u\|_{L^{p^*}(X)} \le c\|u\|_{L^p(X)} \quad \textit{and} \quad \|M^*_{\alpha-1}u\|_{L^q(X)} \le c\|u\|_{L^p(X)}.$$

Proof. We begin by considering $|u|^\alpha_r$. By Lemma 4.4, we have

$$|u|^\alpha_r(x) = r^\alpha |u|_r(x) \le M^*_\alpha u(x) \le cM_\alpha u(x)$$

for every $x \in X$. Then we consider the weak upper gradient of $|u|^\alpha_r$. Since

$$|u|^\alpha_r(x) = r^\alpha \sum_{i=1}^\infty \psi_i(x)|u|_{B(x_i,3r)},$$

each ψ_i is L/r-Lipschitz continuous and has a support in $B(x_i, 6r)$, the function

$$g_r(x) = Lr^{\alpha-1}\sum_{i=1}^\infty |u|_{B(x_i,3r)}\chi_{B(x_i,6r)}(x)$$

is a weak upper gradient of $|u|^\alpha_r$. If $x \in B(x_i, r)$, then $B(x_i, 3r) \subset B(x, 9r) \subset B(x_i, 15r)$ and

$$|u|_{B(x_i,3r)} \le c \fint_{B(x,9r)} |u|\, d\mu.$$

The bounded overlap property of the balls $B(x_i, 6r)$, $i = 1, 2, \dots$, implies that

$$g_r(x) \le cr^{\alpha-1}\fint_{B(x,9r)} |u|\, d\mu \le cM_{\alpha-1}u(x) \le cM^*_{\alpha-1}u(x)$$

and consequently $M^*_{\alpha-1}u$ is a weak upper gradient of $|u|^\alpha_r$ as well.

By Lemma 4.4 and Theorem 4.1, $M^*_\alpha u$ belongs to $L^{p^*}(X)$ and hence $M^*_\alpha u$ is finite almost everywhere. As

$$M^*_\alpha u(x) = \sup_j |u|^\alpha_{r_j}(x),$$

and because $M^*_{\alpha-1}u$ is an upper gradient of $|u|^\alpha_{r_j}$ for every $j = 1, 2, \dots$, we conclude that it is an upper gradient of $M^*_\alpha u$ as well. The norm bounds follow from Theorem 4.1. □

6. Oscillation estimates

6.1. **Hölder continuity.** The next result shows that the discrete maximal function M^*f is Hölder continuous with the same exponent as f. In particular, if f is Lipschitz continuous, then M^*f is also Lipschitz continuous. Recall that $f \in C^{0,\beta}(X)$ means that f is a Hölder continuous function with exponent $0 < \beta \le 1$, that is,

$$|f(x) - f(y)| \le cd(x,y)^\beta$$

for all $x, y \in X$.

Theorem 6.1. *Let $f \in C^{0,\alpha}(X)$ with $0 < \alpha \le 1$. Then $M^*f \in C^{0,\beta}(X)$, provided M^*f is not identically infinity in X.*

Proof. Fix a scale $r > 0$ and let $x, y \in X$. We begin by proving that the discrete convolution f_r is Hölder continuous. We consider two cases. First we assume that $d(x,y) > r$. By the definition of the discrete convolution we have

$$|f_r(x) - f_r(y)| \le |f(x) - f(y)| + \sum_{i=1}^{\infty} \psi_i(x)|f_{B(x_i,3r)} - f(x)|$$
$$+ \sum_{i=1}^{\infty} \psi_i(y)|f_{B(x_i,3r)} - f(y)|.$$

The terms in the sums are non-zero only if $x \in B(x_i, 6r)$ or $y \in B(x_i, 6r)$ for some i. If $x \in B(x_i, 6r)$ for some i, then by Hölder continuity of f we have

$$|f_{B(x_i,3r)} - f(x)| \le cr^\beta.$$

Similarly, if $y \in B(x_i, 6r)$ for some i, then

$$|f_{B(x_i,3r)} - f(y)| \le cr^\beta.$$

Since the balls $B(x_i, 6r)$, $i = 1, 2, \dots$, are of bounded overlap and f is Hölder continuous, we arrive at

$$|f_r(x) - f_r(y)| \le cd(x,y)^\beta + cr^\beta$$

Since $d(x,y) > r$, we have

$$|f_r(x) - f_r(y)| \le cd(x,y)^\beta$$

and we are done.

Then we assume that $d(x,y) \le r$. By the definition of the discrete convolution we have

$$|f_r(x) - f_r(y)| \le \sum_{i=1}^{\infty} |\psi_i(x) - \psi_i(y)||f_{B(x_i,3r)} - f(x)|.$$

The term in the sum is non-zero only if $x \in B(x_i, 6r)$ or $y \in B(x_i, 6r)$ for some i. If $x \in B(x_i, 6r)$, then

$$|f_{B(x_i,3r)} - f(x)| \le cr^\beta$$

as above. On the other hand, if $y \in B(x_i, 6r)$, then $x \in B(x_i, 7r)$ because $d(x,y) \le r$ and we again have

$$|f_{B(x_i,3r)} - f(x)| \le cr^{\beta}.$$

Since there are only a bounded number indices for which the term in the sum is non-zero we arrive at

$$\sum_{i=1}^{\infty} |\psi_i(x) - \psi_i(y)||f_{B(x_i,3r)} - f(x)| \le cd(x,y)r^{\beta-1} \le cd(x,y)^{\beta}.$$

Here we also used Lipschitz continuity of ψ_i. This shows that f_r is Hölder continuous.

Let us prove now that the discrete maximal function preserves Hölder continuity. Without loss of generality we may assume that $M^*f(x) \ge M^*f(y)$.

Let $\varepsilon > 0$. Choose $r_\varepsilon > 0$ so that

$$|f|_{r_\varepsilon}(x) > M^*f(x) - \varepsilon.$$

Then

$$M^*f(x) - M^*f(y) \le |f|_{r_\varepsilon}(x) - |f|_{r_\varepsilon}(y) + \varepsilon \le cd(x,y)^{\beta} + \varepsilon.$$

Since the left hand side is independent of ε the theorem follows by letting $\varepsilon \to 0$. □

Remark 6.2. The proof of the previous theorem shows that the discrete maximal operator is bounded in the space of Hölder continuous functions.

Remark 6.3. Similar arguments as above can be used to show that the discrete maximal operator preserves continuity, provided it is not identically infinity.

The next results shows that the fractional maximal function of a Hölder continuous function is Hölder continuous with a better exponent or a Lipschitz function. This also reflects the smoothing property of the discrete fractional maximal operator.

Theorem 6.4. *Let $u \in C^{0,\beta}(X)$ with $0 < \beta \le 1$. If $\alpha + \beta \le 1$, then $M^*_\alpha u \in C^{0,\alpha+\beta}(X)$, provided M^*f is not identically infinity in X.*

Proof. Let $r > 0$. We begin by proving the claim for $|u|^{\alpha}_r$. Let $x, y \in X$. Assume first that $d(x,y) > r$. Then

$$\big||u|^{\alpha}_r(x) - |u|^{\alpha}_r(y)\big| \le r^{\alpha}\Big(|u(x) - u(y)| + \sum_{i=1}^{\infty} \psi_i(x)\big||u|_{B(x_i,3r)} - |u(x)|\big|$$
$$+ \sum_{i=1}^{\infty} \psi_i(y)\big||u|_{B(x_i,3r)} - |u(y)|\big|\Big).$$

In the first sum, $\psi_i(x) \neq 0$ only if $x \in B(x_i, 6r)$. For such i, by the Hölder continuity of u, we have

$$\big||u|_{B(x_i,3r)} - |u(x)|\big| \le cr^\beta.$$

A similar estimate holds for terms of second sum when $y \in B(x_i, 6r)$. The bounded overlap of the balls $B(x_i, 6r)$, $i = 1, 2, \dots$, and the Hölder continuity of u imply that

$$\big||u|_r^\alpha(x) - |u|_r^\alpha(y)\big| \le cr^\alpha\big(d(x,y)^\beta + r^\beta\big) \le cd(x,y)^{\alpha+\beta}.$$

Assume then that $d(x,y) \le r$. Now

$$\big||u|_r^\alpha(x) - |u|_r^\alpha(y)\big| \le r^\alpha\Big(\sum_{i=1}^{\infty} |\psi_i(x) - \psi_i(y)|\big||u|_{B(x_i,3r)} - |u(x)|\big|\Big),$$

where $\psi_i(x) - \psi_i(y) \neq 0$ only if $x \in B(x_i, 6r)$ or $y \in B(x_i, 6r)$. If $y \in B(x_i, 6r)$, then the assumption $d(x,y) \le r$ implies that $x \in B(x_i, 7r)$. Hence for such i, as above,

$$\big||u|_{B(x_i,3r)} - |u(x)|\big| \le cr^\beta.$$

By the L/r-Lipschitz-continuity of the functions ψ_i and the bounded overlap of the balls $B(x_i, 6r)$, we have

$$\big||u|_r^\alpha(x) - |u|_r^\alpha(y)\big| \le cr^\alpha d(x,y) r^{\beta-1},$$

where, if $\alpha + \beta \le 1$,

$$r^\alpha d(x,y) r^{\beta-1} \le d(x,y)^{\alpha+\beta}.$$

The claim for $|u|_r^\alpha$ follows from this.

Then we prove the claim for $M_\alpha^* u$. We may assume that $M_\alpha^* u(x) \ge M_\alpha^* u(y)$. Let $\varepsilon > 0$ and let $r_\varepsilon > 0$ such that

$$|u|_{r_\varepsilon}^\alpha(x) > M_\alpha^* u(x) - \varepsilon.$$

Then, by the first part of the proof,

$$M_\alpha^* u(x) - M_\alpha^* u(y) \le |u|_{r_\varepsilon}^\alpha(x) - |u|_{r_\varepsilon}^\alpha(y) + \varepsilon \le cd(x,y)^{\alpha+\beta} + \varepsilon,$$

if $\alpha + \beta < 1$. By letting $\varepsilon \to 0$, we obtain

$$|M_\alpha^* u(x) - M_\alpha^* u(y)| \le cd(x,y)^{\alpha+\beta}.$$

□

6.2. **Campanato spaces.** In this section, we study the behaviour of the discrete fractional maximal operator in Campanato spaces. Most of the results are originally considered in [7], but we reproduce some of the arguments here. Let $1 \leq p < \infty$ and $\beta \in \mathbb{R}$. A function $u \in L^1_{\mathrm{loc}}(X)$ belongs to the Campanato space $\mathcal{L}^{p,\beta}(X)$, if

$$\|u\|_{\mathcal{L}^{p,\beta}(X)} = \sup r^{-\beta}\Big(\fint_{B(x,r)} |u - u_{B(x,r)}|^p \, d\mu\Big)^{1/p} < \infty,$$

where the supremum is taken over all $x \in X$ and $r > 0$.

Let $1 \leq p < \infty$ and $\beta \in \mathbb{R}$. A function $u \in L^1_{\mathrm{loc}}(X)$ belongs to the Morrey space $\mathcal{M}^{p,\beta}(X)$, if

$$\|u\|_{\mathcal{M}^{p,\beta,\kappa}(X)} = \sup r^{-\beta}\Big(\fint_{B(x,r)} |u|^p \, d\mu\Big)^{1/p} < \infty,$$

where the supremum is taken over all $x \in X$ and $r > 0$. Observe, that $\|\cdot\|_{\mathcal{M}^{p,\beta}(X)}$ is a norm in the Morrey space, but $\|\cdot\|_{\mathcal{L}^{p,\beta}(X)}$ is merely a seminorm in the Campanato space.

Morrey spaces, Campanato spaces, functions of bounded mean oscillation (BMO) and functions in $C^{0,\beta}(X)$ have the following connections:

- $\mathcal{M}^{p,\beta}(X) \subset \mathcal{L}^{p,\beta}(X)$,
- $\mathcal{L}^{p,\beta}(X) = \mathcal{M}^{p,\beta}(X)$ if $-Q/p < \beta < 0$ (here we identify functions that differ only by an additive constant),
- $\mathcal{L}^{1,0}(X) = BMO(X)$, and
- $\mathcal{L}^{p,\beta}(X) = C^{0,\beta}(X)$ if $0 < \beta \leq 1$.

The following technical lemma will be useful for us.

Lemma 6.5. *Assume that $u \in \mathcal{L}^{p,\beta}(X)$. Let $x \in X$, $0 < 2 < R$ and $y \in B(x, 2R)$. If $\beta < 0$, then*

$$|u_{B(y,r)} - u_{B(x,R)}| \leq c r^{\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)}. \tag{6.6}$$

If $\beta = 0$, then

$$|u_{B(y,r)} - u_{B(x,R)}| \leq c \log \frac{6R}{r} \|u\|_{\mathcal{L}^{p,0}(X)}. \tag{6.7}$$

The constant c depends only on the doubling constant.

Proof. Let k be the smallest index such that $2^k r \geq 3R$. Then $B(x,R) \subset B(y, 2^k r)$ and

$$\begin{aligned}
&|u_{B(y,r)} - u_{B(x,R)}| \\
&\leq \sum_{i=1}^{k} |u_{B(y,2^i r)} - u_{B(y,2^{i-1}r)}| + |u_{B(y,2^k r)} - u_{B(x,R)}| \\
&\leq \sum_{i=1}^{k} \fint_{B(y,2^{i-1}r)} |u - u_{B(y,2^i r)}|\, d\mu + \fint_{B(x,R)} |u - u_{B(y,2^k r)}|\, d\mu \\
&\leq c \sum_{i=1}^{k} \fint_{B(y,2^i r)} |u - u_{B(y,2^i r)}|\, d\mu + c \fint_{B(y,2^k r)} |u - u_{B(y,2^k r)}|\, d\mu \\
&\leq c r^{\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)} \Big(\sum_{i=1}^{\infty} 2^{i\beta} + 2^{k\beta} \Big) \leq c r^{\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)},
\end{aligned}$$

where c depends only on the doubling constant and the sum converges since $\beta < 0$. This proves (6.6).

The proof of (6.7) is quite similar. Indeed, by the choice of k, we have $2^k r \leq 6R$ and consequently

$$\begin{aligned}
&|u_{B(y,r)} - u_{B(x,R)}| \\
&\leq c \sum_{i=1}^{k} \fint_{B(y,2^i r)} |u - u_{B(y,2^i r)}|\, d\mu + c \fint_{B(y,2^k r)} |u - u_{B(y,2^k r)}|\, d\mu \\
&\leq ck \|u\|_{\mathcal{L}^{p,0}(X)} \leq c \log \frac{6R}{r} \|u\|_{\mathcal{L}^{p,0}(X)}.
\end{aligned}$$

□

According to the next result, the discrete fractional maximal operator maps functions in Campanato spaces to Hölder continuous functions.

Theorem 6.8. *Let $\alpha > 0$, $0 \leq \alpha + \beta \leq 1$ and let $u \in \mathcal{L}^{p,\beta}(X)$. Then there is a constant c, depending only on the doubling constant p and α and β, such that*

$$\|M_\alpha^* u\|_{C^{0,\alpha+\beta}(X)} \leq c \|u\|_{\mathcal{L}^{p,\beta}(X)}.$$

Proof. Let $r > 0$. We begin by proving the claim for $|u|_r^\alpha$. Let $x, y \in X$. Assume first that $r < d(x,y)$. Let $B = B(x, 4d(x,y))$. Then

$$\begin{aligned}
&\big| |u|_r^\alpha(x) - |u|_r^\alpha(y) \big| \\
&\leq \big| |u|_r^\alpha(x) - r^\alpha |u|_B \big| + \big| r^\alpha |u|_B - |u|_r^\alpha(y) \big| \\
&\leq r^\alpha \Big(\sum_{i=1}^{\infty} \psi_i(x) \big| |u|_{B(x_i,3r)} - |u|_B \big| + \sum_{i=1}^{\infty} \psi_i(y) \big| |u|_{B(x_i,3r)} - |u|_B \big| \Big).
\end{aligned}$$

In the first sum, $\psi_i(x) \neq 0$ only if $x \in B(x_i, 6r)$ and in the second sum, only if $y \in B(x_i, 6r)$. If $\beta < 0$, we use the bounded overlap of the balls $B(x_i, 6r)$, $i = 1, 2, \dots$ and (6.6) and we have

$$\big| |u|_r^\alpha(x) - |u|_r^\alpha(y) \big| \le cr^{\alpha+\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)} \le cd(x,y)^{\alpha+\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)}.$$

Similarly, if $\beta = 0$, estimate (6.7) implies that

$$\begin{aligned} \big| |u|_r^\alpha(x) - |u|_r^\alpha(y) \big| &\le cr^\alpha \log \frac{cd(x,y)}{r} \|u\|_{\mathcal{L}^{p,\beta}(X)} \\ &= cd(x,y)^\alpha \Big(\frac{r}{cd(x,y)} \Big)^\alpha \log \frac{cd(x,y)}{r} \|u\|_{\mathcal{L}^{p,\beta}(X)} \\ &\le cd(x,y)^\alpha \|u\|_{\mathcal{L}^{p,\beta}(X)}. \end{aligned}$$

If $r \ge d(x,y)$, then

$$\begin{aligned} \big| |u|_r^\alpha(x) - |u|_r^\alpha(y) \big| &\le r^\alpha \Big(\sum_{i=1}^\infty |\psi_i(x) - \psi_i(y)| \big| |u|_{B(x_i,3r)} - |u|_{B(x,10r)} \big| \Big) \\ &\le cr^{\alpha+\beta-1} d(x,y) \|u\|_{\mathcal{L}^{p,\beta}(X)} \\ &\le cd(x,y)^{\alpha+\beta} \|u\|_{\mathcal{L}^{p,\beta}(X)}. \end{aligned}$$

The claim for $M_\alpha^* u$ follows as in the proof of Theorem 6.4.

If $\beta > 0$, then $\mathcal{L}^{p,\beta}(X) = C^{0,\beta}(X)$ and the result follows from Theorem 6.4. This completes the proof. □

We conclude with two results for the discrete maximal functions in the space of functions of bounded mean oscillation, denoted by $BMO(X)$. A function $f \in L^1_{\mathrm{loc}}(X)$ belongs to $BMO(X)$ if

$$\|f\|_{BMO(X)} = \sup \fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu < \infty,$$

where the supremum is taken over all $x \in X$ and $r > 0$. The proof of the following theorem can be found in [1].

Theorem 6.9. *If $f \in BMO(X)$, then $M^* f \in BMO(X)$ provided $M^* f$ is not identically infinity.*

The proof of the previous result applies a theorem by Coifman and Rochberg, which states that $(Mu)^\gamma$, the Hardy-Littlewood maximal function of u raised to any power $0 < \gamma < 1$, is a Muckenhoupt A_1-weight whenever Mu is not identically infinity. This means that there exists a constant c such that

$$\fint_{B(x,r)} (Mu)^\gamma \, d\mu \le c \operatorname*{ess\,inf}_{B(x,r)} (Mu)^\gamma$$

for every ball $B(x,r)$ in X. For the fractional maximal function, we obtain the result even without taking the power. For the proof, we refer to [7].

Theorem 6.10. *Let $0 < \alpha < Q$. Assume that $u \in L^1_{\mathrm{loc}}(X)$ is such that $M^*_\alpha u$ is not identically infinity. Then $M^*_\alpha u$ is a Muckenhoupt A_1-weight, that is,*

$$\fint_{B(x,r)} M^*_\alpha u \, d\mu \le c \operatorname*{ess\,inf}_{B(x,r)} M_\alpha u$$

for every ball $B(x,r)$ in X. The constant c does not depend on u.

Remark 6.11. Under the assumptions of the previous theorem, we also have

$$\fint_{B(x,r)} (M^*_\alpha u)^\gamma \, d\mu \le c \operatorname*{ess\,inf}_{B(x,r)} (M^*_\alpha u)^\gamma$$

for $0 < \gamma \le 1$ by Hölder's inequality.

Remark 6.12. By standard arguments, the previous theorem also implies that $\log M^*_\alpha u$ belongs to $BMO(X)$.

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