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Navier-Stokes Equations with Random Forcing

Nobuo Yoshida

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0 Introduction

We would like to analyze the turbulence of a viscous fluid in $\mathbb{R}^d$ (physically, $d=3$). Let

\[ u = (u_i(t, x))_{i=1}^{d} \in \mathbb{R}^d \]  
\[ \Pi = \Pi(t, x) \in \mathbb{R} \]  

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be the velocity and the pressure of the fluid at time $t \geq 0$ at the position $x \in \mathbb{R}^d$. For fluids like air and water, it is accepted in hydrodynamics that they satisfy the Navier-Stokes equation:

$$\text{div} u = 0, \quad \partial_t u + (u \cdot \nabla) u = -\nabla \Pi + \nu \Delta u + F, \quad (0.3)$$

where $u \cdot \nabla = \sum_{j=1}^{d} u_j \partial_j$, $\nu > 0$ is a constant, called kinematic viscosity, and $F = F_t(x)$, $(t, x) \in [0, \infty) \times \mathbb{T}^d$ is a given external force. Physical interpretation of (0.3) is the mass conservation, while (0.4) is the motion equation.

On the other hand, since the turbulence is a random phenomenon, we need to bring a certain random factor into the model. To do so, we consider a colored noise, which is “time derivative” of a certain function space valued Brownian motion $W = W_t(x)$ and take $F_t(x) = \partial_t W_t(x)$ in (0.4). This may look too much of an idealization of the real turbulence. However, this way of modeling is common in literatures [F108] and references therein.

Based mainly on [F108], we explain the construction of the weak solution to (0.3)-(0.4) globally in time in the case $F_t(x) = \partial_t W_t(x)$.

1 Physical derivation of the Navier-Stokes equation

We review the heuristic argument to “derive” (0.3)-(0.4) from the physical assumptions. Let $e_1, \ldots, e_d$ be the canonical basis of $\mathbb{R}^d$:

$$e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \ldots, \quad e_d = (0, \ldots, 0, 1). \quad (1.1)$$

Also, it is convenient to introduce the following small box and plaquettes:

$$\square = \left[ -\frac{\delta}{2}, \frac{\delta}{2} \right]^d, \quad \square_i = \{ x \in \square ; x_i = 0 \}, \quad i = 1, \ldots, d, \quad (1.2)$$

where the side-length $\delta > 0$ of the box $\square$ and the plaquette $\square_i$ is supposed to be very small, eventually tending to zero. Let

$$u = (u_i(t, x))_{i=1}^{d}, \quad \rho = \rho(t, x) \geq 0 \quad (1.3)$$

be the velocity and the density of the fluid at time-space $(t, x)$.

1.1 The mass conservation

We first derive (0.3) for a constant density fluid $\rho \equiv \text{const}$. To do so, however, we do not assume that $\rho \equiv \text{const}$. for a moment and consider the mass $m(x+\square)$ of the fluid on the cube $x+\square$ centered at $x$ (cf. (1.2)):

$$m(x+\square) = \int_{x+\square} \rho \cong \rho(x) \delta^d \quad (1.4)$$

Here and often in what follows, we omit the time $t$ in the notation. The time derivative of the mass is given as follows:

$$\partial_t m(x+\square) = \sum_{j=1}^{d} m_j(x), \quad (1.5)$$

2
where
\[
m_j(x) = \frac{(\rho u_j)(x - \frac{\delta}{2}e_j) - \partial_j(\rho u_j)(x)\frac{\delta}{2} + O(\delta^2)}{\delta^{d-1}} \delta^{d-1}
\]
inward flux of the mass through the face \((x - \frac{\delta}{2}e_j) + \square_j\)
\[
- \frac{(\rho u_j)(x + \frac{\delta}{2}e_j) + \partial_j(\rho u_j)(x)\frac{\delta}{2} + O(\delta^2)}{\delta^{d-1}} \delta^{d-1}
\]
outerward flux of the mass through the face \((x + \frac{\delta}{2}e_j) + \square_j\).

By Taylor expanding \((\rho u_j)(x \mp \frac{\delta}{2}e_j)\) above, we see that
\[
m_j(x) = ((\rho u_j)(x) - \partial_j(\rho u_j)(x)\frac{\delta}{2} + O(\delta^2))\delta^{d-1}
\]
- \((\rho u_j)(x) + \partial_j(\rho u_j)(x)\frac{\delta}{2} + O(\delta^2))\delta^{d-1}
\]
= \(-\partial_j(\rho u_j)(x)\delta^d + O(\delta^{d+1})\).

By this and (1.5), we get:
\[
\frac{1}{\delta^d} \partial_t m(x + \square) = -\sum_{j=1}^{d} \partial_j(\rho u_j)(x) + O(\delta)
\]
(1.6)

Note that
\[
\rho(x) = \lim_{\delta \to 0} \frac{1}{\delta^d} m(x + \square).
\]

If we believe that the above limit commutes with \(\partial_t\), we see from (1.6) that
\[
\partial_t \rho + \sum_{j=1}^{d} \partial_j(\rho u_j)(x) = 0.
\]
(1.7)

In particular, for a constant density flow: \(\rho \equiv \text{const}\), (1.7) is reduced to (0.3). Note also that the interchange of the order of \(\lim_{\delta \to 0}\) and \(\partial_t\) assumed above is perfectly correct in this case.

1.2 Force exerted on fluids: the stress tensor

The notion of stress can be thought of as actions, like pushing, pulling and rubbing a door. Then, the action has obviously different effects depending on the side of the door which the action is made on. Therefore, we distinguish the side of the plaquette \(\square_i\): let
\[
\square_i^+ = \text{"the } x_i > 0\text{-side" of } \square_i = \{x \in \square; x_i = 0\}
\]
\[
\square_i^- = \text{the "opposite side" of } \square_i.
\]

Imagine that the plaquette \(\square_i\) is put in a stream with the velocity \(u\). Then forces are exerted on plane \(\square_i\), e.g., pulling, pushing, or rubbing. With this in mind, we introduce:
\[
\tau_i^\square(x) = (\tau_i^\square(x))_{j=1}^{d} = \text{the force exerted on } x + \square_i^+ \text{ by the stream}
\]
\[
- \tau_i^- = \text{the force exerted on } x + \square_i^- \text{ by the stream},
\]
(1.8) (1.9)
where the second equality is, of course, the principle of action-reaction. We then define the stress tensor $\tau(x) = (\tau_{ij}(x))_{i,j=1}^{d}$ by:

$$
\tau_{ij}(x) = \lim_{\delta \searrow 0} \frac{1}{\delta^{d-1}} \tau^\square_{ij}(x).
$$

(1.10)

$\tau_{ij}(x)$ is the $j$-th component of the force exerted on $x$ by the stream from the side $x_i+$. We will assume that

- $\tau$ is of the form:

$$
\tau(x) = -\Pi(x)I + \tau^F(x),
$$

(1.11)

where $\Pi(x) = \Pi(t, x)$ is the pressure (a real function), $I$ is the identity matrix, and $\tau^F(x)$ is the friction term of $\tau(x)$.

- $\tau$ is symmetric, i.e., $\tau_{ij} = \tau_{ji}$, or equivalently, $\tau^F_{ij} = \tau^F_{ji}$.

The symmetry assumption above is based on the conservation of the angular momentum. A typical example of the friction term is provided by the following Stokes law:

$$
\tau^F_{ij} = \mu (\partial_i u_j + \partial_j u_i),
$$

(1.12)

where the constant $\mu > 0$ is the coefficient of friction, and the tensor $\left(\frac{\partial u_i + \partial u_i}{2}\right)$ is called the symmetrized velocity gradient tensor.

Let

$$
f^\square(x) = (f^\square_j(x))_{j=1}^{d}
$$

the force exerted on the outer boundary of $x + \square$ by the stream.

Here, the outer boundary is the union of

$$(x + \frac{\delta}{2}e_i) + \square^+_i, (x - \frac{\delta}{2}e_i) + \square^-_i \quad i = 1, \ldots, d.$$  

Then, it turn out to be reasonable to define the force exerted to a point $x$ by the stream by:

$$
f(x) = (f_j(x))_{j=1}^{d}, \quad \text{where } f_j(x) = \lim_{\delta \searrow 0} \frac{1}{\delta^{d}} f^\square_j(x).
$$

(1.13)

It may appear at first sight that $2d\delta^{d-1}$ is more appropriate in place of $\delta^d$ above. However, we will see later on that $\delta^d$ is indeed the right normalization. We will prove that

$$
f_j = \sum_{i=1}^{d} \partial_i \tau_{ij}.
$$

(1.14)

Before we prove (1.14), we make some remarks. By (1.11), (1.14) reads:

$$
f = -\nabla \Pi + \left(\sum_{i=1}^{d} \partial_i \tau^F_{ij}\right)_{j=1}^{d}.
$$

(1.15)
Moreover, if we suppose that the fluid is of constant density and the Stokes law (1.12) holds, then, since $\text{div} u = 0$,

$$
\sum_{i=1}^{d} \partial_{i}\tau_{ij}^{F} = \mu \sum_{i=1}^{d} (\partial_{i}\partial_{i}u_{j} + \partial_{i}\partial_{j}u_{i}) = \mu \Delta u_{j}.
$$

Thus, (1.15) becomes:

$$
f(x) = -\nabla \Pi + \mu \Delta u. \quad (1.16)
$$

We turn to the proof of (1.14). We have, by (1.8)–(1.10) that

$$
f_{j}^{\square}(x) = \sum_{i=1}^{d} \tau_{ij}^{\square} \left( x + \frac{\delta}{2} e_{i} \right) + \sum_{i=1}^{d} \tau_{ij}^{\square} \left( x - \frac{\delta}{2} e_{i} \right)
$$

On the other hand, by Taylor expanding $\tau_{ij}(x \pm \frac{\delta}{2} e_{i})$ above, we have that

$$
\tau_{ij}(x + \frac{\delta}{2} e_{i}) - \tau_{ij}(x - \frac{\delta}{2} e_{i}) = \left( \tau_{ij}(x) + \partial_{i}\tau_{ij}(x) \frac{\delta}{2} + O(\delta^{2}) \right) - \left( \tau_{ij}(x) - \partial_{i}\tau_{ij}(x) \frac{\delta}{2} + O(\delta^{2}) \right) = \partial_{i}\tau_{ij}(x) \delta + O(\delta^{2}).
$$

Plugging this into (1.17), we have

$$
f_{j}^{\square}(x) \cong \partial_{i}\tau_{ij}(x) \delta^{d} + O(\delta^{d+1})
$$

Thus, if we believe that the approximation $\cong$ is good enough, we have (1.14).

1.3 The motion equation

To derive the motion equation (0.4), we introduce the stream line $x(t) \in \mathbb{R}^{d}, t \geq 0$ define by:

$$
x(t) = x(0) + \int_{0}^{t} u(s, x(s)) ds.
$$

The curve $x(\cdot)$ is the integral curve of the velocity $u$, hence, roughly speaking, it is a position of a particle moving on the stream. The classical Newton’s motion equation is:

$$
\text{mass} \times \text{acceleration} = \text{force},
$$

which, in our case, takes the following form:

$$
\rho(x(t)) \frac{d}{dt} u(t, x(t)) = f(x(t)), \quad (1.18)
$$
where the force $f$ is given by (1.15). We have by the chain rule that

$$\frac{d}{dt} u(t, x(t)) = \partial_t u(t, x(t)) + \sum_{j=1}^d \partial_j u(t, x(t)) \frac{dx_j(t)}{dt} u_j(t, x(t))$$

By the above identity, together with (1.15) and (1.18), we get

$$\rho(\partial_t u + (u \cdot \nabla) u) = -\nabla \Pi + \left( \sum_{i=1}^d \partial_i \tau_{ij}^F \right)_{j=1}^d$$

(1.19)

If we suppose that the fluid is of constant density and the Stokes law (1.12) holds, then, by (1.16), we have that

$$\partial_t u + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla \Pi + \frac{\mu}{\rho} \Delta u$$

(1.20)

where the constant $\nu \overset{\text{def}}{=} \frac{\mu}{\rho}$ is the kinematic viscosity.

2 The mathematical framework in the case of non-random forcing term

From here on, we assume that the container of the fluid is the $d$-dimensional torus:

$$\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0, 1]^d.$$  

This is a part of idealization. The unknown functions of the Navier-Stokes equation (NS) are

- velocity of fluid $u = u_t(x) \in \mathbb{R}^d$, $(t, x) \in [0, \infty) \times \mathbb{T}^d$ with suitable regularity, say $C^2$ in $(t, x)$.

- pressure $\Pi = \Pi_t(x) \in \mathbb{R}$, $(t, x) \in [0, \infty) \times \mathbb{T}^d$ with suitable regularity, say $C^1$ in $(t, x)$.

Given an initial velocity $u_0 : \mathbb{T}^d \rightarrow \mathbb{R}^d$,

$$\text{div} u = 0,$$

$$\partial_t u + (u \cdot \nabla) u = -\nabla \Pi + \nu \Delta u + F,$$

(2.1)

(2.2)

where $\nu > 0$ is a constant, called *kinematic viscosity* and $F = F_t(x)$, $(t, x) \in [0, \infty) \times \mathbb{T}^d$ is a given external force. Physical interpretation of (2.1) and (2.2) were explained in section 1.

2.1 A weak formulation

Let $\mathcal{V}$ be the set of $\mathbb{R}^d$-valued divergence free, mean-zero trigonometric polynomials, i.e., the set of $v : \mathbb{T}^d \rightarrow \mathbb{R}^d$ of the following form:

$$v(x) = \sum_{x \in \mathbb{Z}^d} \bar{v}_x \psi_x(x), \quad x \in \mathbb{T}^d,$$

(2.3)
where $\psi_z(x) = \exp(2\pi iz \cdot x)$ and the coefficients $\hat{v}_z \in \mathbb{R}^d$ satisfy

\begin{align}
\hat{v}_z &= 0 \quad \text{for } z = 0 \text{ and except for finitely many } z \neq 0, \\
\overline{\hat{v}_z} &= \hat{v}_{-z} \quad \text{for all } z, \\
z \cdot \hat{v}_z &= 0 \quad \text{for all } z.
\end{align}

(2.4)

Note that (2.6) implies that: \( \text{div} v = 0 \) for all \( v \in \mathcal{V} \).

We equip the torus $\mathbb{T}^d$ with the Lebesgue measure and denote by $\|f\|_p$ the usual $L_p$-norm of $f \in L_p(\mathbb{T}^d)$. For $\alpha \in \mathbb{R}$ and $v \in \mathcal{V}$ we define

\[ (1 - \Delta)^{\alpha/2}v = \sum_{z \in \mathbb{Z}^d} (1 + 4\pi^2|z|^2)^{\alpha/2} \hat{v}_z \psi_z. \]

(2.3)

We then introduce:

\[ V_{2,\alpha} = \text{the completion of } \mathcal{V} \text{ with respect to the norm } \| \cdot \|_{2,\alpha}, \quad \alpha \in \mathbb{R}, \]

(2.7)

where

\[ \|v\|_{2,\alpha}^2 = \int_{\mathbb{T}^d} |(1 - \Delta)^{\alpha/2}v|^2 = \sum_{z \in \mathbb{Z}^d} (1 + 4\pi^2|z|^2)^{\alpha} |\hat{v}_z|^2. \]

(2.8)

Here are some basic properties of the space $V_{2,\alpha}$:

- Any $v \in V_{2,\alpha}$ is identified with a summation of the form (2.3) with (2.4) replaced by the condition that the last summation in (2.8) converges.
- $V_{2,-\alpha}$ is identified with the set of continuous linear functional on $V_{2,\alpha}$.

(2.9)

cf. Definition 2.1.1 and Exercise 2.1.1 below.

**Definition 2.1.1** Let $E_0, E_1$ be normed vector spaces.

- $E_0 \hookrightarrow E_1$ means that $E_0$ is continuously imbeded into $E_1$, i.e., $E_0 \subset E_1$ with the inclusion map being continuous.

- $E_0 \hookrightarrow\hookrightarrow E_1$ means that $E_0$ is compactly imbeded into $E_1$, i.e., $E_0 \subset E_1$ with the inclusion map being a compact operator.

**Exercise 2.1.1** Recall that any $v \in V_{2,\alpha}$ is identified with a summation of the form (2.3) with (2.4) replaced by the condition that the last summation in (2.8) converges. Let $\alpha \in \mathbb{R}$, $\beta > 0$ and $v \in V_{2,\alpha+\beta}$. Prove that

\[ \|v - I_n v\|_{2,\alpha} \leq (1 + 4\pi^2n^2)^{-\beta/2}\|v\|_{2,\alpha+\beta}, \quad \text{where } I_n v = \sum_{|z| \leq n} \hat{v}_z \psi_z. \]

Then, conclude (2.9) from this.

**Exercise 2.1.2** Prove the following interpolation inequality:

\[ \|v\|_{2,\alpha+(1-\theta)\beta} \leq \|v\|_{2,\alpha}^{\theta}\|v\|_{2,\beta}^{1-\theta} \quad \text{for } \alpha, \beta \in \mathbb{R} \text{ and } \theta \in [0,1]. \]

(2.10)
For $v, w : \mathbb{T}^d \to \mathbb{R}^d$, with $w$ supposed to be differentiable (for a moment), we define a vector field:

$$(v \cdot \nabla)w = \sum_{i=1}^{d} v_i \partial_i w,$$  \hspace{1cm} (2.11)

which is bilinear in $(v, w)$. Later on, we will generalize the definition of the above vector field (cf. (2.18)).

**Lemma 2.1.2** For $v \in \mathcal{V}, w, \varphi \in C^1(\mathbb{T}^d \to \mathbb{R}^d)$,

$$\langle \varphi, (v \cdot \nabla)w \rangle = -\langle w, (v \cdot \nabla)\varphi \rangle,$$  \hspace{1cm} (2.12)

In particular, $\langle w, (v \cdot \nabla)w \rangle = 0$.

Proof: Since $\text{div} v = 0$, we have that

$$\sum_j \partial_j (\varphi_i v_j) = \sum_j (\partial_j \varphi_i) v_j + \varphi_i \sum_j \partial_j v_j.$$  \hspace{1cm} (2.13)

Therefore,

$$\text{LHS } (2.12) = \sum_{i,j} \langle \varphi_i, v_j \partial_j w_i \rangle = - \sum_{i,j} \langle \partial_j (\varphi_i v_j), w_i \rangle$$

$$\overset{1)}{=} - \sum_{i,j} \langle (\partial_j \varphi_i) v_j, w_i \rangle = \text{RHS } (2.12).$$

Suppose that $u, \Pi, F$ in (NS) ((2.1)-(2.2)) have suitable regularity. Then, for a test function $\varphi \in \mathcal{V}$,

$$\partial_t \langle \varphi, u \rangle = \langle \varphi, (u \cdot \nabla)u \rangle + \nu \langle \varphi, \Delta u \rangle - \langle \varphi, \nabla \Pi \rangle + \langle \varphi, F \rangle.$$  \hspace{1cm} (2.14)

Thus, $\ast$ becomes

$$\partial_t \langle \varphi, u \rangle = \langle u, (u \cdot \nabla)\varphi \rangle + \nu \langle \Delta \varphi, u \rangle + \langle \varphi, F \rangle.$$

By integration, we arrive at:

$$\langle \varphi, u_t \rangle = \langle \varphi, u_0 \rangle + \int_0^t \left( \langle u_s, (u_s \cdot \nabla)\varphi \rangle + \nu \langle \Delta \varphi, u_s \rangle + \langle \varphi, F_s \rangle \right) ds.$$  \hspace{1cm} (2.15)

This is a standard weak formulation of (NS) ((2.1)-(2.2)).
2.2 Bounds on the non-linear term

**Lemma 2.2.1** Suppose $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with at least two of them being non-zero, and that $\alpha_1 + \alpha_2 + \alpha_3 \geq \frac{d}{2}$. Then, there exists $C \in (0, \infty)$ such that:

$$|\langle w, (v \cdot \nabla) \varphi \rangle| \leq C \|v\|_{2, \alpha_1} \|w\|_{2, \alpha_2} \|\varphi\|_{2, 1+\alpha_3},$$

(2.14)

for $v, w, \varphi \in C^\infty(T^d \to \mathbb{R}^d)$.

**Proof:** Since the norm $\|\cdot\|_{2, \alpha}$ is increasing in $\alpha$, it is enough to prove (2.16) with $\alpha_i$ replaced by $\tilde{\alpha}_i = \frac{(d/2)\alpha_i}{\alpha_1 + \alpha_2 + \alpha_3}$. Therefore, we may assume without loss of generality that $(\alpha_1, \alpha_2, \alpha_3) \in [0, \frac{d}{2})^3$ and $\alpha_1 + \alpha_2 + \alpha_3 = \frac{d}{2}$.

Let $q_i \in [2, \infty), i = 1, 2, 3$ be defined by $\frac{1}{q_i} = \frac{1}{2} - \frac{\alpha}{d} > 0$. Since

$$\sum_{i,j} |w_i v_j \partial_j \varphi_i| \leq |w| |v| |\nabla \varphi|,$$

we have

$$|\langle w, (v \cdot \nabla) \varphi \rangle| \leq \|v\|_{q_1} \|w\|_{q_2} \|\nabla \varphi\|_{q_3}.$$

We then use the following Sobolev imbedding theorem (e.g.,[Ta96, p. 4, (2.11)]):

$$V_{2, \alpha} \hookrightarrow L_q(T^d \to \mathbb{R}^d), \text{ if } \frac{1}{q} = \frac{1}{2} - \frac{\alpha}{d} \text{ def} > 0.$$

(2.15)

$\square$

We have the following variant of Lemma 2.2.1, which is applicable even when $\alpha_2 = \alpha_3 = 0$:

**Lemma 2.2.2** Let $\alpha_1, \alpha_2, \alpha_3 \geq 0$ be such that $\alpha_1 + \alpha_2 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 \geq \frac{d}{2}$. Then, there exists $C \in (0, \infty)$ such that:

$$|\langle w, (v \cdot \nabla) \varphi \rangle| \leq C \|\varphi\|_{2, 1+\alpha_3} \sqrt{\|v\|_{2, \alpha_1} \|v\|_{2, \alpha_2} \|w\|_{2, \alpha_1} \|w\|_{2, \alpha_2}},$$

(2.16)

for $v, w, \varphi \in C^\infty(T^d \to \mathbb{R}^d)$.

**Proof:** Note that

1) $\|u\|_{2, \alpha_1 + \alpha_2}^{(2.16)} \leq \sqrt{\|u\|_{2, \alpha_1} \|u\|_{2, \alpha_2}}$ for $u \in V_{2, \alpha_1} \cap V_{2, \alpha_2}$.

On the other hand, by (2.14) with $(\alpha_1 + \alpha_2, \frac{\alpha_1 + \alpha_2}{2}, \alpha_3)$ in place of $(\alpha_1, \alpha_2, \alpha_3)$, we have

$$|\langle w, (v \cdot \nabla) \varphi \rangle|^{(2.14)} \leq C \|v\|_{2, \alpha_1 + \alpha_2} \|w\|_{2, \alpha_1 + \alpha_2} \|\varphi\|_{2, 1+\alpha_3} \leq \text{RHS (2.16)}.$$

$\square$

**Remark:** (2.16) gives a generalization of [Te79, p. 292, Lemma 3.4]

Let $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 > 0$, and $\alpha_3 \overset{\text{def}}{=} \left(\frac{d}{2} - \alpha_1 - \alpha_2\right)^+$. (2.17)
Then, $\alpha_i$'s $(i = 1, 2, 3)$ satisfy conditions for Lemma 2.2.2. Let also $v, w \in V_{2,\alpha_1 \vee \alpha_2}$. In view of (2.12), we think of $(v \cdot \nabla)w$ as the following linear functional on $\mathcal{V}$:

$$\varphi \mapsto \langle \varphi, (v \cdot \nabla)w \rangle^{def} = -\langle w, (v \cdot \nabla)\varphi \rangle,$$

which, by (2.16), extends continuously on $V_{2,1+\alpha_3}$. This way, we regard

$$(v \cdot \nabla)w \in V_{2,-1-\alpha_3},$$

with $\| (v \cdot \nabla)w \|_{2,-1-\alpha_3} \leq C \sqrt{\|v\|_{2,\alpha_1} \|v\|_{2,\alpha_2} \|w\|_{2,\alpha_1} \|w\|_{2,\alpha_2}}$. 

(2.18)

Let us consider the case $v = w$ and $\alpha_1 \geq \alpha_2$ (Although $v$ and $w$ are identical, it is convenient to take $\alpha_1 > \alpha_2$, as we will see later on). Note that:

$$\Delta v \in V_{2,\alpha_1-2} \text{ with } \| \Delta v \|_{2,\alpha_1-2} \leq \|v\|_{2,\alpha_1},$$

By this and (2.18), we have that:

$$b(v)^{def} \nu \Delta v - (v \cdot \nabla)v \in V_{2,-\beta(\alpha_1,\alpha_2)}$$

with $\| b(v) \|_{2,-\beta(\alpha_1,\alpha_2)} \leq \nu \|v\|_{2,\alpha_1} + C \|v\|_{2,\alpha_1} \|v\|_{2,\alpha_2}$, 

(2.19)

where

$$\beta(\alpha_1, \alpha_2) = (1 + (\frac{d}{2} - \alpha_1 - \alpha_2)^+) \vee (2 - \alpha_1).$$

(2.20)

With this notation, (2.13) takes the form:

$$\langle \varphi, u_t \rangle = \langle \varphi, u_0 \rangle + \int_0^t \langle \varphi, b(u_s) \rangle ds + \int_0^t \langle \varphi, F_s \rangle ds.$$ 

i.e., 

$$u_t = u_0 + \int_0^t b(u_s) ds + \int_0^t F_s ds$$

(2.21)

as linear functionals on $\mathcal{V}$.

**Lemma 2.2.3** Let $\alpha_1 > 0$ and $\alpha_1 \geq \alpha_2 \geq 0$ for which $\beta(\alpha_1, \alpha_2)$ is defined by (2.20). Then, there exists $C \in (0, \infty)$ such that:

$$\int_0^T \| b(v_t) \|^q_{2,-\beta(\alpha_1,\alpha_2)} dt \leq \int_0^T (\nu + C \|v_t\|_{2,\alpha_2})^q \|v_t\|^q_{2,\alpha_1} dt$$

(2.22)

for any measurable $v : [0, T] \rightarrow V_{2,\alpha}$ and $q \in [1, \infty)$. Moreover, for $\alpha > 0$, the following map is continuous:

$$v \mapsto \int_0^t b(v_s) ds; \quad L_2([0, T] \rightarrow V_{2,\alpha}) \rightarrow C([0, T] \rightarrow V_{2,-\beta(\alpha,\alpha)})$$

Proof: (2.22) is a direct consequence of (2.19). For the rest of this proof, we write $\beta = \beta(\alpha, \alpha)$ for simplicity. Let $v, w \in L_2([0, T] \rightarrow V_{2,\alpha})$. Then,

1) $\sup_{0 \leq t \leq T} \left\| \int_0^t (b(v_s) - b(w_s)) ds \right\|^q_{2,-\beta} \leq \int_0^T \|b(v_s) - b(w_s)\|^q_{2,-\beta} ds.$
On the other hand, for $\varphi \in V_{2,-\beta}$,

$$
\langle \varphi, b(v_s) - b(w_s) \rangle \overset{(2.19)}{=} \nu \langle \Delta \varphi, v_s - w_s \rangle - \langle v_s, (v_s \cdot \nabla)\varphi \rangle + \langle w_s, (w_s \cdot \nabla)\varphi \rangle,
$$

which implies that:

$$
\|b(v_s) - b(w_s)\|_{2,-\beta} \leq \nu + C\|v_s\|_{2,\alpha} + C\|w_s\|_{2,\alpha}.
$$

Plugging this into 1), we arrive at:

$$
\sup_{0 \leq t \leq T} \left| \int_0^t (b(v_s) - b(w_s)) ds \right|_{2,-\beta}
\leq \sqrt{3} \left( \int_0^T (\nu^2 + C^2\|v_s\|_{2,\alpha}^2 + C^2\|w_s\|_{2,\alpha}^2) ds \right)^{1/2} \left( \int_0^T \|v_s - w_s\|_{2,\alpha}^2 ds \right)^{1/2},
$$

which implies the desired continuity.

**Remark:** By (2.22) for $q = 1$ and $(\alpha_1, \alpha_2) = (1,1)$, we see that

$$
v \in L_2([0, T] \to V_{2,1}) \implies b(v) \in L_1([0, T] \to V_{2,-\beta(1,1)})
$$

On the other hand, by (2.22) for $q = 2$ and $(\alpha_1, \alpha_2) = (1,0)$, we see that

$$
v \in L_2([0, T] \to V_{2,1}) \cap L_\infty([0, T] \to V_{2,0}) \implies b(v) \in L_2([0, T] \to V_{2,-\beta(1,0)}).
$$

Note also that:

$$
\beta(1,1) = \begin{cases} 1 & \text{if } d \leq 4, \\ \frac{d}{2} - 1 & \text{if } d \geq 5 \end{cases}, \quad \beta(1,0) = \begin{cases} 1 & \text{if } d = 2, \\ \frac{d}{2} & \text{if } d \geq 3 \end{cases}.
$$

### 3 The stochastic Navier-Stokes equation

The construction of a weak solution to the Navier-Stokes equation (2.1)–(2.2) goes back to classical results by J. Leray [Le33, Le34a, Le34b] and E. Hopf [Ho50]. Here, following [Fl08], we consider the case in which the external force is given by a colored noise.

#### 3.1 Introduction of the noise

Throughout this subsection, let $H$ be a separable Hilbert space, and $\Gamma : H \to H$ be a bounded self-adjoint, non-negative definite operator. We suppose in addition that $\Gamma$ is of trace class, that is, the following summation converges for any CONS $\{\varphi_n\}_{n \geq 1}$ of $H$:

$$
\text{tr}(\Gamma) \overset{\text{def}}{=} \sum_{n \geq 1} \langle \varphi_n, \Gamma \varphi_n \rangle.
$$

The number defined above is called the trace of $\Gamma$ and is independent of the choice of the CONS [RS72, p.206, Theorem VI.18].
Definition 3.1.1 Let \((\Omega, \mathcal{F}, P)\) be a probability space.

a) A r.v. \(B = (B_t)_{t\geq 0}\) with values in \(C([0, \infty) \to \mathbb{R}^d)\) is called a standard \(d\)-dimensional Brownian motion (abbreviated by BM\(^d\) below) if, for each \(\theta \in \mathbb{R}^d\) and \(0 \leq s < t\),

\[
E \left[ \exp \left( i \theta \cdot (B_t - B_s) \right) | \mathcal{G}_{s}^{B} \right] = \exp \left( -\frac{t-s}{2} |\theta|^2 \right), \text{ a.s.} \tag{3.2}
\]

where \(\mathcal{G}_{s}^{B}\) denotes the \(\sigma\)-field generated by \((B_u)_{u\leq s}\). (cf. the complement at the end of this subsection for a definition of the conditional expectation.)

b) A r.v. \(W = (W_t)_{t\geq 0}\) with values in \(C([0, \infty) \to H)\) is called a \(H\)-valued Brownian motion with the covariance operator \(\Gamma\) (abbreviated by BM\((H, \Gamma)\) below) if, for each \(\varphi \in H\) and \(0 \leq s < t\),

\[
E \left[ \exp \left( i \varphi \cdot (W_t - W_s) \right) | \mathcal{G}_{s}^{W} \right] = \exp \left( -\frac{t-s}{2} \langle \varphi, \Gamma \varphi \rangle \right), \text{ a.s.} \tag{3.3}
\]

where \(\mathcal{G}_{s}^{W}\) denotes the \(\sigma\)-field generated by \((W_u)_{u\leq s}\).

**Remark:** The distributional time derivative \(\partial_t W_t\) of a BM\((H, \Gamma)\) \(W_t\) is sometimes called the colored noise.

**Exercise 3.1.1** Let \(W_t\) be as in Definition 3.1.1 b) and \(H_0 \subset H\) be a \(d\)-dimensional subspace of \(H\) such that \(\Gamma H_0 \subset H_0\) with the orthogonal projection \(\pi_0\). Then, conclude from (3.3) that

\((\pi_0 W_t)_{t\geq 0}\) and \((\sigma B_t)_{t\geq 0}\) have the same law,

where \((B_t)_{t\geq 0}\) is BM\(^d\) on \(H_0\) (identified with \(\mathbb{R}^d\)) and \(\sigma : H_0 \to H_0\) is a square root of \(\Gamma|_{H_0}\). In particular, for each \(\varphi \in H\), the process \(\langle \varphi, W_t \rangle\), \(t \geq 0\) is of the following form:

\[
\langle \varphi, W_t \rangle = \sqrt{\langle \varphi, \Gamma \varphi \rangle} B_t, \quad t \geq 0,
\]

where \(B\) is a BM\(^1\).

**Complement:** Let \(X \in L_1(P)\) and \(\mathcal{G}\) be a sub \(\sigma\)-field of \(\mathcal{F}\). We define the conditional expectation \(E[X|\mathcal{G}]\) of \(X\), given \(\mathcal{G}\). An implicit definition is given by declaring that \(Y = E[X|\mathcal{G}]\) is the unique \(\mathcal{G}\)-measurable r.v. in \(L^1(P)\) such that:

1) \[E[Y 1_G] = E[X 1_G] \quad \text{for any } G \in \mathcal{G}.\]

Another definition is given by explicitly writing down \(E[X|\mathcal{G}]\) as a certain Radon Nikodym derivative, which proves that the r.v. \(Y\) as referred to above does exist. To do so, we introduce the following signed measure:

\[E^X(F) \overset{\text{def}}{=} E[X1_F], \quad F \in \mathcal{F}.\]

Since \(E^X|_{\mathcal{G}}\) is absolutely continuous with respect to \(P|_{\mathcal{G}}\), we can define:

\[E[X|\mathcal{G}] = \frac{dE^X|_{\mathcal{G}}}{dP|_{\mathcal{G}}},\]

where the RHS stands for the Radon Nikodym derivative. Then, it is clear that \(Y = E[X|\mathcal{G}]\) satisfies 1).
Let us relate the above abstract definition with the elementary conditional expectation of $X \in L_1(P)$, given an event $A \in \mathcal{F}$ with $0 < P(A) < 1$:

$$E[X|A] = \frac{E[X1_A]}{P(A)}.$$

For the $\sigma$-field $\mathcal{G} = \{A, A^c, \emptyset, \Omega\}$, it is clear that

$$E[X|\mathcal{G}] = E[X|A]1_A + E[X|A^c]1_{A^c}.$$

### 3.2 The existence theorem for the stochastic Navier-Stokes equation

We recall (2.19)-(2.21).

**Theorem 3.2.1** Let

1. $\Gamma : V_{2,0} \rightarrow V_{2,0}$ be a self-adjoint, non-negative definite operator of trace class, $\Delta \Gamma = \Gamma \Delta$ and;
2. $\mu_0$ be a Borel probability measure on $V_{2,0}$ such that $m_0 \equiv \int \|v\|^2_2 d\mu_0(v) < \infty$.

Then, there exist a process $(X, Y) = (X_t, Y_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, P)$, where

- $X = (X_t)_{t \geq 0}$ takes values in
  $$L_{2,1oc}(0, \infty) \cap L_{\infty,1oc}(0, \infty) \cap C([0, \infty) \rightarrow V_{2,-\beta(1,1)}),$$  
  (3.4)
  with $\beta(1,1) = 1$ for $d \leq 4$ and $\beta(1,1) = \frac{d}{2} - 1$ for $d \geq 5$. cf. (2.25);
- $Y = (Y_t)_{t \geq 0}$ is a $BM(V_{2,0}, \Gamma)$ (cf. Definition 3.1.1).

The couple $(X, Y)$ is a weak solution to the Navier-Stokes equation with the initial law $\mu_0$ in the sense that:

1. $P(X_0 \in \cdot) = \mu_0$;
2. $Y_{t+} - Y_t$ and $\{\langle \varphi, X_s \rangle ; s \leq t, \varphi \in \mathcal{V}\}$ are independent for any $t \geq 0$;
3. $\langle \varphi, X_t \rangle = \langle \varphi, X_0 \rangle + \int_0^t \langle \varphi, b(X_s) \rangle ds + \langle \varphi, Y_t \rangle$, for all $\varphi \in \mathcal{V}$ and $t \geq 0$.

Moreover, the following a priori bounds hold true: for any $T > 0$,

$$E\left[\|X_T\|^2_2 + 2\nu \int_0^T \|X_t\|^2_{2,1} dt\right] \leq m_0 + \text{tr}(\Gamma)T,$$

(3.8)

$$E\left[\sup_{t \leq T} \|X_t\|^2_2\right] \leq (1 + T)C < \infty,$$

(3.9)

with $C \in (0, \infty)$ depending only on $\text{tr}(\Gamma)$, and $m_0$.

**Remark:**

1. The integral $\int_0^t \langle \varphi, b(X_s) \rangle ds$ in (3.7) is well defined because of (2.23) (or (2.24)) and (3.4).

2. The bound (3.8) is sometimes referred to as the energy balance inequality. The interpretation is that

$$\frac{1}{2}\|X_T\|^2_2 = \text{the kinetic energy},$$

$$\nu \int_0^T \|X_t\|^2_{2,1} dt = \text{the energy dissipated by the friction},$$

$$\frac{1}{2}\text{tr}(\Gamma)T = \text{the energy injected from outside (by the colored noise).}$$
Although the validity of the equality is not known in general, the equality does hold at the level of finite dimensional approximation (see (5.10) below).

**Theorem 3.2.2** For $d = 2$, the weak solution in Theorem 3.2.1 is pathwise unique in the sense: if $(X, Y)$ and $(\tilde{X}, Y)$ are two solutions on a common probability space $(\Omega, \mathcal{F}, P)$ with a common $BM(V_{2,0}, \Gamma)$ $Y$ such that $X_0 = \tilde{X}_0$ a.s., then,

$$P(X_t = \tilde{X}_t \text{ for all } t \geq 0) = 1.$$ 

4 The Itô theory for beginners

In this section, we will explain elements in Itô’s stochastic calculus without going much into proofs. In what follows, $(\Omega, \mathcal{F}, P)$ is a probability space and $B = (B_t)_{t \geq 0}$ denotes a $BM^r$.

4.1 Stochastic integrals with respect to the Brownian motion

We fix some notation and terminology:

- A family $X = (X_t)_{t \geq 0}$ of r.v.’s indexed by $t \geq 0$ (most commonly interpreted as “time”) is called a process. A process $X$ is said to be continuous if $t \mapsto X_t$ is continuous a.s.

- Let $(\mathcal{F}_t)_{t \geq 0}$ be a family of sub $\sigma$-fields which are increasing in $t \geq 0$, as such a filtration. We assume that it is right-continuous in the sense that:

$$\bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t, \quad t \geq 0. \quad (4.1)$$

- In general, a process $X = (X_t)_{t \geq 0}$ is said to be $(\mathcal{F}_t)$-adapted, if $X_t$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

- We assume that $B = (B_t)_{t \geq 0}$ is a $BM^r$ with respect to $(\mathcal{F}_t)$, that is, $B$ is $(\mathcal{F}_t)$-adapted and

$$E[\exp(i\theta \cdot (B_t - B_s)) | \mathcal{F}_s] = \exp \left( -\frac{t-s}{2} |\theta|^2 \right), \quad \text{a.s.} \quad (4.2)$$

for each $\theta \in \mathbb{R}^r$ and $0 \leq s < t$. We also assume that

$$\mathcal{N}^B \subset \mathcal{F}_t, \quad t \geq 0, \quad (4.3)$$

where $\mathcal{N}^B$ is the null-set with respect to $B$ define as follows:

$$\mathcal{G}_t^B = \sigma(B_s, s \leq t), \quad 0 \leq t < \infty, \quad \mathcal{G}_\infty^B = \sigma(\cup_{t \geq 0} \mathcal{G}_t^B),$$

$$\mathcal{N}^B = \{ N \subset \Omega, \exists \tilde{N} \in \mathcal{G}_\infty^B, N \subset \tilde{N}, P(\tilde{N}) = 0 \}.$$ 

An example of such $(\mathcal{F}_t)_{t \geq 0}$ is given by the argumented filtration defined by:

$$\mathcal{F}_t = \sigma(\mathcal{G}_t^B \cup \mathcal{N}^B), \quad 0 \leq t < \infty. \quad (4.4)$$

See [KS91,pp.90–91] for the proof the properties (4.1)–(4.2) of the argumented filtration. On the other hand, $\mathcal{G}_t^B$ is not right-continuous [KS91,p.89, Problem 7.1].
Definition 4.1.1 (Stopping times) A r.v. $\tau : \Omega \to [0, \infty]$ is called a stopping time if
\[ \{ \tau \leq t \} \in \mathcal{F}_t \text{ for all } t \geq 0. \tag{4.5} \]

Example 4.1.2 Let $\Gamma \subset \mathbb{R}^r$ and define
\[ \tau(\Gamma) = \inf \{ t > 0 : B_t \in \Gamma \}. \]
It is known that $\tau(\Gamma)$ is a stopping time if $\Gamma \subset \mathbb{R}^r$ is a Borel set. This is not difficult to prove if $\Gamma$ is either open or closed. Here, in the proof, one sees how the right continuity of $\mathcal{F}_t$ is used.

Consider the following condition\(^2\) for a r.v. $\tau : \Omega \to [0, \infty]$;
\[ \{ \tau < t \} \in \mathcal{F}_t \text{ for all } t \geq 0. \tag{4.6} \]
Then, this is equivalent to (4.5). In fact, we have
\begin{enumerate}
\item \( \{ \tau < t \} = \bigcup_{n \geq 1} \{ \tau \leq t - \frac{1}{n} \}, \)
\item \( \{ \tau > t \} = \bigcap_{m \geq 1} \bigcup_{n \geq m} \{ \tau \geq t - \frac{1}{n} \}. \)
\end{enumerate}
We see from 1) that (4.5) implies (4.6), while the converse can be seen from 2) and the right continuity of $\mathcal{F}_t$.

The observation above can be used to prove that $\tau(\Gamma)$ defined in Example 4.1.2 is a stopping time for an open set $\Gamma$. We prove that $\tau(\Gamma)$ satisfies (4.6) as follows:
\[ \{ \tau(\Gamma) < t \} = \bigcup_{s \in (0,t)} \{ B_s \in \Gamma \} = \bigcup_{s \in \mathbb{Q} \cap (0,t)} \{ B_s \in \Gamma \} \in \mathcal{F}_t, \]
where, to get the second equality, we have used that $\Gamma$ is open and that $s \mapsto B_s$ is continuous.

Exercise 4.1.1 Prove that $\tau(\Gamma)$ defined in Example 4.1.2 is a stopping time if $\Gamma$ is closed. Hint: There is a sequence of open sets $G_1 \supset G_2 \supset \ldots$ such that $\Gamma = \bigcap_{m \geq 1} G_m$.

We now define some classes of integrands for the stochastic integral.

Definition 4.1.3 (Integrands for stochastic integral) We define a function space $\Phi$ as the totality of $\varphi : [0, \infty) \times \Omega \to \mathbb{R} ((s, \omega) \mapsto \varphi_s(\omega))$ such that\(^3\):
\[ \varphi|_{[0,t] \times \Omega} \text{ is } \mathcal{B}([0, t]) \otimes \mathcal{F}_t \text{ measurable for all } t \geq 0. \]

We also define
\begin{align*}
\Phi_2 &= \{ \varphi \in \Phi ; E \int_0^t |\varphi_s|^2 ds < \infty \text{ for all } t > 0 \}, \tag{4.7} \\
\Phi_2^{loc} &= \{ \varphi \in \Phi ; \int_0^t |\varphi_s|^2 ds < \infty, P\text{-a.s. for all } t > 0 \}. \tag{4.8}
\end{align*}
Clearly, $\Phi_2 \subset \Phi_2^{loc} \subset \Phi$.

\(^2\)A r.v. $\tau$ with this condition is called an optional time. We see from the argument of this remark that a stopping time is always an optional time, and that the converse is true when the filtration is right continuous.

\(^3\)This property is called progressive measurability.
Example 4.1.4 Let $g : \mathbb{R}^r \to \mathbb{R}$ be Borel measurable and

$$
\varphi_s(\omega) = g(B_s(\omega)).
$$

Then,

- If $g$ is bounded, then $\varphi \in \Phi_2$.
- If $\sup_K |g| < \infty$ for any bounded set $K \subset \mathbb{R}^r$ (in particular, if $g \in C(\mathbb{R}^r)$), then $\varphi \in \Phi_2^{loc}$.

**Theorem 4.1.5** For $\varphi \in \Phi_2^{loc}$, there are continuous processes (called the stochastic integral with respect to the Brownian motion)

$$
\left( \int_0^t \varphi_s dB_s^i \right)_{t \geq 0} \quad i = 1, \ldots, r
$$

(4.9)

with the following properties;

a) If

$$
\varphi_s(\omega) = \xi_a(\omega)1_{(a,b]}(s)
$$

(4.10)

where $0 \leq a < b$ and $\xi_a$ is a bounded, $\mathcal{F}_a$-measurable r.v., then

$$
\int_0^t \varphi_s dB_s^i = \xi_a(\omega)(B_{t \wedge b}^i - B_{t \wedge a}^i).
$$

(4.11)

b) For $t \geq 0$, $\alpha, \beta \in \mathbb{R}$ and $\varphi, \psi \in \Phi_2^{loc}$,

$$
\int_0^t (\alpha \varphi_s + \beta \psi_s) dB_s^i = \alpha \int_0^t \varphi_s dB_s^i + \beta \int_0^t \psi_s dB_s^i.
$$

(4.12)

c) If $\varphi, \psi \in \Phi_2$ and $t \geq 0$, then,

$$
E \left[ \left( \int_0^t \varphi_s dB_s^i \right) \left( \int_0^t \psi_s dB_s^j \right) \right] = \delta_{ij} E \int_0^t \varphi_s \psi_s ds < \infty,
$$

(4.13)

$$
E \left[ \int_0^t \varphi_u dB_u^i \bigg| \mathcal{F}_s \right] = \int_0^s \varphi_u dB_u^i \text{ whenever } 0 \leq s \leq t.
$$

(4.14)

We now indicate how the construction of the integrals (4.9) goes (See [KS91, Section 3.2] for details).

**Step 1:** Let $\Phi_0$ be the set of linear combinations of r.v.'s of the form (4.10). We proceed as follows:

1) For $\varphi \in \Phi_0$, define the integral (4.9) by (4.11) and (4.12).

2) Properties (4.13)–(4.14) hold for $\varphi, \psi \in \Phi_0$ (not difficult to see).

**Step 2:** We define the integral (4.9) for $\varphi \in \Phi_2$. To do so, we note that $\Phi_2$ is a Fréchet space generated by the semi-norms:

$$
\left( E \int_0^T |\varphi_s|^2 ds \right)^{1/2}, \quad T = 1, 2, ...
$$

We also introduce:
Definition 4.1.6 A process $M = (M_t)_{t \geq 0}$ is said to be a martingale, if:

- $(\mathcal{F}_t)$-adapted, $M_t \in L_1(P)$ for all $t \geq 0$;
- $E[M_t|\mathcal{F}_s] = M_s$ whenever $0 \leq s < t$.  \hfill (4.15)

A martingale $M$ is said to be square integrable, if $E[M_T^2] < \infty$ for all $T > 0$. Let

$$\mathcal{M}_2 = \text{the set of continuous, square-integrable martingales.}$$

Then, $\mathcal{M}_2$ is a a Fréchet space generated by the semi-norms:

$$E \left[ \sup_{s \leq T} M_s^2 \right]^{1/2}, \ T = 1, 2, ...$$

(cf. (4.16) below). We define:

$$I(\varphi)_t = \int_0^t \varphi_s dB^i_s, \ \varphi \in \Phi_0, \ t \geq 0. \hfill (4.9)$$

We make the following observations:

1) From what we saw in Step 1.2,

$$E[I(\varphi)^2_T] = E \int_0^T |\varphi_s|^2 ds, \ I(\varphi) \in \mathcal{M}_2, \ \text{for } \varphi \in \Phi_0$$

2) $\Phi_0$ is dense in $\Phi_2$ (cf. [IW89, p.46, Lemma 1.1]). Thus, by 1) above, $I$ extends uniquely to a uniformly continuous mapping $I : \Phi_2 \rightarrow \mathcal{M}_2$. This justifies the definition of the integral (4.9) for $\varphi \in \Phi_2$:

$$\int_0^t \varphi_s dB^i_s \overset{\text{def}}{=} I(\varphi)_t, \ t \geq 0. \hfill (4.10)$$

Properties (4.12)-(4.14) for $\varphi \in \Phi_2$ is then automatic from the construction.

Step 3: We define the integral (4.9) for $\varphi \in \Phi_2^{\text{loc}}$. For $\varphi \in \Phi_2^{\text{loc}}$, we consider

$$\tau^{(n)} = n \land \inf \left\{ t > 0; \int_0^t |\varphi_s|^2 ds \geq n \right\} \hfill (\tau^{(n)} \nearrow \infty)$$

Then, $\varphi^{(n)} \in \Phi_2$. We then define the integrals (4.9) by

$$\int_0^t \varphi_s dB^i_s = \int_0^t \varphi_s^{(n)} dB^i_s, \ \text{for } t \leq \tau^{(n)}. \hfill (4.11)$$

This finishes the construction.

Finally, we mention the following useful inequality:
Theorem 4.1.7 (Doob’s $L^2$-maximal inequality) For a square-integrable martingale $M$,

$$E \left[ \sup_{0 \leq s \leq t} M_s^2 \right] \leq 4E[M_t^2].$$ \hfill (4.16)

In particular, if $\varphi \in \Phi_2$, then

$$E \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \varphi_u dB_u^i \right|^2 \right] \leq 4 \int_0^t |\varphi_s|^2 ds.$$ \hfill (4.17)

For a proof, see e.g. [IW89, p.33, Theorem 6.10], [KS91, p.13, 3.8 Theorem].

4.2 Itô’s formula for semi-martingales

Definition 4.2.1 Let $(\mathcal{F}_t)$ be a right-continuous filtration and $B = (B_t)_{t \geq 0}$ be a BM with respect to $(\mathcal{F}_t)$ (cf. (4.1)-(4.3)).

► An $\mathbb{R}^d$-valued process $X = (X_t)_{t \geq 0}$ is said to be a semi-martingale if it is of the following form:

$$X_t = X_0 + \int_0^t \sigma_s dB_s + \int_0^t b_s ds,$$ \hfill (4.18)

or more precisely,

$$X_t^i = X_0^i + \sum_{j=1}^r \int_0^t \sigma_s^{ij} dB_s^j + \int_0^t b_s^i ds, \quad i = 1, \ldots, d.$$ \hfill (4.19)

where

- $X_0$ is a $\mathcal{F}_0$-measurable r.v.;
- $\sigma = (\sigma^{ij})$ is a matrix with $\sigma^{ij} \in \Phi_2^{loc}$ (cf. (4.8));
- $b = (b_t)_{t \geq 0}$ is an $(\mathcal{F}_t)$-adapted process such that $t \mapsto b_t$ is continuous.

► For the semi-martingale (4.18) and a process $(\varphi_t)_{t \geq 0}$, we define:

$$\int_0^t \varphi_s dX_s^i = \sum_{j=1}^r \int_0^t \varphi_s \sigma_s^{ij} dB_s^j + \int_0^t \varphi_s b_s^i ds, \quad i = 1, \ldots, d,$$ \hfill (4.20)

if each integral on the RHS is well defined, i.e.,

$$\varphi \sigma^{ij} \in \Phi_2^{loc} \quad \text{and} \quad \int_0^t |\varphi_s b_s^i| ds < \infty \ \text{a.s.} \ i, j = 1, \ldots, d.$$

The integral (4.19) is called the stochastic integral with respect to the semi-martingale (4.18).

► For a semi-martingale (4.18), we define the bracket processes by:

$$\langle X^i, X^j \rangle_t = \sum_{k=1}^r \int_0^t \sigma_s^{ik} \sigma_s^{jk} ds, \quad i, j = 1, \ldots, d.$$ \hfill (4.21)
Theorem 4.2.2 (Itô's formula for semi-martingales) Suppose that $X$ is a semi-martingale given by (4.18) and $f \in C^{2}(\mathbb{R}^{d})$. Then, $P$-a.s.,

$$f(X_{t}) - f(X_{0}) = \sum_{i=1}^{d} \int_{0}^{t} \partial_{i} f(X_{s}) dX_{s}^{i} + \frac{1}{2} \sum_{i,j=1}^{d} \int_{0}^{t} \partial_{i} \partial_{j} f(X_{s}) d\langle X^{i}, Y^{j} \rangle_{s}, \quad \text{for all } t \geq 0.$$  

(4.21)

The proof goes along the following line (e.g.[IW89, pp.67-71], [KS91, pp.150-153]). Let $d = r = 1$ for simplicity, and $0 = t_{0} < t_{1} < ... < t_{n} = t$ be the division for which $\delta_{n} \overset{\text{def}}{=} \max_{1 \leq k \leq n} (t_{k} - t_{k-1}) \to 0 \quad (n \to \infty)$. For the indices to be read easily, we write $\tilde{X}_{k} = X_{t_{k}}$. Then, by Taylor expanding $f$ around $\tilde{X}_{k-1}$, we have:

$$f(\tilde{X}_{k}) - f(\tilde{X}_{k-1}) = f'(\tilde{X}_{k-1}) \triangle_{k} + \frac{1}{2} f''(\tilde{X}_{k-1} + \theta_{k} \triangle_{k}) \triangle_{k}^{2}$$

where $\triangle_{k} = \tilde{X}_{k} - \tilde{X}_{k-1}$ and $\theta_{k} \in (0, 1)$. This implies that:

$$f(X_{t}) - f(X_{0}) = \sum_{k=1}^{n} f'(\tilde{X}_{k-1}) \Delta_{k} + \frac{1}{2} \sum_{k=1}^{n} f''(\tilde{X}_{k-1} + \theta_{k} \Delta_{k}) \Delta_{k}^{2}.$$

By verifying

$$\lim_{n \to \infty} I_{n} = \int_{0}^{t} f'(X_{s}) dX_{s} \quad \text{and} \quad \lim_{n \to \infty} J_{n} = \int_{0}^{t} f''(X_{s}) d\langle X, X \rangle_{s},$$

in an appropriate sense, one obtains (4.21) for $d = r = 1$. The extension to general $d, r$ is straightforward.

Example 4.2.3 For the semi-martingale (4.18), we have:

$$|X_{t}|^{2} - |X_{0}|^{2} = 2M_{t} + \int_{0}^{t} (2X_{s} \cdot b_{s} + |\sigma_{s}|^{2}) ds, \quad \text{with } M_{t} = \sum_{1 \leq i \leq d} \int_{0}^{t} X_{s}^{i} \sigma_{s}^{ij} dB_{s}^{j}. \quad (4.22)$$

Here, and in what follows, $|\sigma|^{2} = \sum_{1 \leq i \leq d} (\sigma^{ij})^{2}$. Suppose in particular that

$$E[|X_{0}|^{2}] \leq m_{0} < \infty, \quad X_{t} \cdot b_{t} \leq C, \quad |\sigma_{t}|^{2} \leq C, \quad (4.23)$$

where $m_{0}$ and $C$ is a non-random constant. Then, for any $t > 0$,

$$E[|X_{t}|^{2}] = E[|X_{0}|^{2}] + E \int_{0}^{t} (2X_{s} \cdot b_{s} + |\sigma_{s}|^{2}) ds, \quad (4.24)$$

$$E \left[ \sup_{s \leq t} |X_{s}|^{2} \right] \leq E[|X_{0}|^{2}] + C't, \quad (4.25)$$

where the constant $C'$ depends only on $m_{0}$ and $C$. 

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Proof: Note that
\[ \partial_{i}|x|^{2} = 2x^{i}, \quad \partial_{i}\partial_{j}|x|^{2} = 2\delta_{i,j}. \]
Thus, we see from Itô's formula that:
\[
|X_{t}|^{2} - |X_{0}|^{2} = \sum_{j=1}^{d} \int_{0}^{t} 2X_{s}^{j} \cdot dX_{s}^{j} + \frac{1}{2} \sum_{i,j=1}^{d} \int_{0}^{t} 2\delta_{i,j} d\langle X^{i}, X^{j} \rangle_{s},
\]
with
\[
I = 2M_{t} + 2 \int_{0}^{t} X_{s} \cdot b(X_{s})ds,
\]
\[
J = \sum_{1 \leq i \leq d} \langle X^{i}, X^{i} \rangle_{t}^{(4.20)} \int_{0}^{t} \sum_{i,k=1}^{d} (\sigma_{s}^{ik})^{2} ds.
\]
This proves (4.22). We next assume (4.23) to show (4.24)-(4.25). This will be straightforward, once we know that \( M \) is a square-integrable martingale. However, we have to settle this technical point first. We start by showing that:

1) \( E[|X_{t}|^{2}] \leq m_{0} + 3Ct, \)

Since \( X \) is continuous and \( |X_{0}| < \infty \) a.s., we have that:
\[
e_{n} \overset{\text{def}}{=} \inf\{t ; |X_{t}| \geq n\} \nearrow \infty, \quad \text{as} \quad n \nearrow \infty.
\]
Note also that:
\[
M_{t\wedge e_{n}} = \sum_{1 \leq i \leq d} \int_{0}^{t \wedge e_{n}} X_{s}^{i} \sigma_{s}^{ij} dB_{s}^{j} = \sum_{1 \leq i \leq d} \int_{0}^{t} 1_{\{s \leq e_{n}\}} X_{s}^{i} \sigma_{s}^{ij} dB_{s}^{j}
\]
and that \( 1_{\{s \leq e_{n}\}} X_{s}^{i} \sigma_{s}^{ij} \in \Phi_{2}. \) These and (4.14) imply that \( E[M_{t\wedge e_{n}}] = 0. \) Combining this with:

2) \( |X_{t}|^{2} \overset{(4.22),(4.23)}{\leq} |X_{0}|^{2} + 2M_{t} + 3Ct, \)

we have that:
\[
E[|X_{t\wedge e_{n}}|^{2}] \leq m_{0} + 3Ct.
\]
Thus, 1) follows from Fatou's lemma. 1) and (4.23) imply that:
\[
X_{t}^{i} \sigma_{s}^{ij} \in \Phi_{2}.
\]
Then, \( E[M_{t}] = 0 \) by (4.14). Thus, (4.24) follows from (4.22) taking expectation. We next show that:

3) \( E \left[ \sup_{s \leq t} |M_{s}|^{2} \right] \leq C_{1}(t + t^{2}). \)

To do so, we start by noting that:
4) \[ \sum_{j} (\sum_{i} X_{i}^{j} \sigma_{s}^{ij})^2 = |\sigma_{s}^{*} X_{s}|^2 \leq |\sigma_{s}|^2 |X_{s}|^2. \]

Then,

\[
E \left[ \sup_{s \leq t} |M_{s}|^2 \right] \leq 4E \left[ |M_{t}|^2 \right] \leq 4 \sum_{j} E \int_{0}^{t} \left( \sum_{i} X_{i}^{j} \sigma_{s}^{ij} \right)^2 ds \leq 4 \int_{0}^{t} |\sigma_{s}|^2 |X_{s}|^2 ds \leq 4 \left( m_{0} t + \frac{3C}{2} t^2 \right). \]

we then get (4.22) as follows:

\[
E \left[ \sup_{s \leq t} |X_{s}|^2 \right] \leq m_0 + 2E \left[ \sup_{s \leq t} |M_{s}|^2 \right]^{1/2} + 3C t \leq m_0 + C_{2} t.
\]

\[\square\]

Example 4.2.4 (Itô's formula for the Brownian motion) Suppose that \( f \in C^{2}(\mathbb{R}^{d}) \). Then, \( P \)-a.s.,

\[ f(B_{t}) - f(0) = \sum_{1 \leq i \leq r} \int_{0}^{t} \partial_{i} f(B_{s}) dB_{s}^{i} + \frac{1}{2} \int_{0}^{t} \Delta f(B_{s}) ds, \quad \text{for all } t \geq 0. \]  

(4.26)

Proof: A special case of (4.21) with \( d = r, \sigma^{ij} = \delta^{ij}, \) and \( b \equiv 0. \) \[\square\]

4.3 Stochastic differential equations: an existence and uniqueness theorem

Let \( \sigma \in C(\mathbb{R}^{d} \rightarrow \mathbb{R}^{d} \otimes \mathbb{R}^{r}), b \in C(\mathbb{R}^{d} \rightarrow \mathbb{R}^{d}) \) and \( \xi \) be an \( \mathbb{R}^{d} \)-valued r.v. We consider a stochastic differential equation (SDE):

\[
X_{t} = \xi + \int_{0}^{t} \sigma(X_{s}) dB_{s} + \int_{0}^{t} b(X_{s}) ds. \]  

(4.27)

or more precisely,

\[ X_{t}^{i} = \xi^{i} + \sum_{j=1}^{r} \int_{0}^{t} \sigma^{ij}(X_{s}) dB_{s}^{j} + \int_{0}^{t} b^{i}(X_{s}), \quad i = 1, \ldots, d. \]

We define:

\[
E_{t}^{\xi,B} = \sigma(\xi, B_{s}, s \leq t), \quad 0 \leq t < \infty, \quad G_{\infty}^{\xi,B} = \sigma \left( \cup_{t \geq 0} E_{t}^{\xi,B} \right), \]

\[\mathcal{N}^{\xi,B} = \{ N \subset \Omega, \exists \tilde{N} \in G_{\infty}^{\xi,B}, N \subset \tilde{N}, P(\tilde{N}) = 0 \}, \]

and

\[ F_{t}^{\xi,B} = \sigma \left( E_{t}^{\xi,B} \cup \mathcal{N}^{\xi,B} \right), \quad 0 \leq t < \infty. \]  

(4.28)

We now state the following existence and uniqueness theorem:
Theorem 4.3.1 Referring to (4.27), suppose that

$$m_{0} \overset{\text{def}}{=} E[|\xi|^{2}] < \infty$$

and that there exist $K, L_{n} \in (0, \infty), n = 1, 2, ...$ such that:

$$|\sigma(x) - \sigma(y)|^{2} + |b(x) - b(y)|^{2} \leq L_{n}|x - y|^{2} \quad \text{if } |x|, |y| \leq n, \quad (4.29)$$

$$|\sigma(x)|^{2} + 2x \cdot b(x) \leq K(1 + |x|^{2}), \quad x \in \mathbb{R}^{d}. \quad (4.30)$$

Then, there exists a unique process $X$ such that:

a) $X_{t}$ is $\mathcal{F}_{t}^{\xi, B}$-measurable for all $t \geq 0$ (cf. (4.28));

b) the SDE (4.27) is satisfied.

Proof: By [IW89, p.178, Theorem 3.1], the condition (4.29) ensures existence of the unique solution admitting the possibility of explosion at finite time:

$$\lim_{t \nearrow \tau} |X_{t}| = \infty, \quad \text{for some } \tau < \infty.$$ 

However, such possibility is excluded by the condition (4.30) [IW89, p.177, Theorem 2.4]. \hfill \square

5 The Galerkin approximation

5.1 The approximating SDE

For each $z \in \mathbb{Z}^{d}\setminus\{0\}$, let $\{e_{z,j}\}_{j=1}^{d-1} \subset \mathbb{R}^{d}$ be an orthonormal basis of the hyperplane:

$$\{x \in \mathbb{R}^{d}; z \cdot x = 0\}$$

and let:

$$\psi_{z,j}(x) = \begin{cases} \sqrt{2}e_{z,j} \cos(2\pi z \cdot x), & j = 1, ..., d - 1, \\ \sqrt{2}e_{z,|j|} \sin(2\pi z \cdot x), & j = -1, ..., -(d - 1) \end{cases}, \quad x \in \mathbb{T}^{d}. \quad (5.1)$$

Then,

$$\{\psi_{z,j}; z \in \mathbb{Z}^{d}\setminus\{0\}, j = \pm 1, ..., \pm(d - 1)\}$$

is an orthonormal basis of $V_{2,0}$. We also introduce:

$$\begin{align*}
\mathcal{V}_{n} &= \text{the linear span of } \{\psi_{z,j}; (z, j) \text{ with } z \in [-n, n]^{d}\}, \\
\mathcal{P}_{n} &= \text{the orthogonal projection } : L^{2}(\mathbb{T}^{d} \rightarrow \mathbb{R}^{d}) \rightarrow \mathcal{V}_{n}. \quad (5.2)
\end{align*}$$

Using the orthonormal basis (5.1), we identify $\mathcal{V}_{n}$ with $\mathbb{R}^{N}, N = \dim \mathcal{V}_{n}$. Let $\mu_{0}$ and $\Gamma : V_{2,0} \rightarrow V_{2,0}$ be as in Theorem 3.2.1. Let also $\xi$ be a r.v. such that $P(\xi \in \cdot) = \mu_{0}$. Finally, let $W_{t}$ be a BM($V_{0}, \Gamma$) defined on a probability space $(\Omega, \mathcal{F}, P)$. Then, $\mathcal{P}_{n}W_{t}$ is identified with an $N$-dimensional Brownian motion with covariance matrix $\Gamma \mathcal{P}_{n}$. Then, we consider the following approximation of (3.7):

$$X_{t}^{n} = X_{0}^{n} + \int_{0}^{t} \mathcal{P}_{n}b(X_{s}^{n})ds + \mathcal{P}_{n}W_{t} \quad t \geq 0, \quad (5.3)$$
where $X_0^n = \mathcal{P}_n \xi$. Let:

$$X_t^{n,z,j} = \langle \psi_{z,j}, X_t^n \rangle$$

and

$$W_t^{z,j} = \langle \psi_{z,j}, W_t \rangle$$

be the $(z,j)$-coordinates of $X_t^n$ and $W_t$. Then, (5.3) reads:

$$X_t^{n,z,j} = X_0^{n,z,j} + \int_0^t b^{z,j}(X_s^n) ds + W_t^{z,j},$$

where

$$b^{z,j}(v) = \langle v, (v \cdot \nabla) \psi_{z,j} \rangle + \nu \langle v, \nabla \psi_{z,j} \rangle, \quad v \in \mathcal{V}_n.$$  

Let $\gamma_{z,j} \geq 0$ be such that $\Gamma \psi_{z,j} = \gamma_{z,j} \psi_{z,j}$ and $I_n = \{(z,j) ; |z| \leq n, \gamma_{z,j} > 0\}$. Then,

$$B_t^{z,j} = \frac{W_t^{z,j}}{\sqrt{\gamma_{z,j}}}, \quad (z,j) \in I_n$$

are independent BM's and

$$\mathcal{P}_n W_t = \sum_{(z,j) \in I_n} W_t^{z,j} \psi_{z,j} = \sum_{(z,j) \in I_n} \sqrt{\gamma_{z,j}} B_t^{z,j} \psi_{z,j}.$$  

Thus, the SDE (5.3) can be thought of as a special case of (4.27), where

$$\sigma(\cdot)$$ is a constant diagonal matrix with $|\sigma(\cdot)|^2 = \text{tr}(\Gamma \mathcal{P}_n)$.  

Also by (5.6),

the drift $\mathcal{P}_n b(v)$ is a polynomial in $v \in \mathcal{V}_n$ of degree two.

Moreover, for $v \in \mathcal{V}_n$,

$$\langle v, \mathcal{P}_n b(v) \rangle = \langle v, \nu \Delta v + (v \cdot \nabla)v \rangle \overset{\text{Lemma 2.1.2}}{=} \nu \langle v, \Delta v \rangle = -\nu \|\nabla v\|^2 \leq 0.$$  

We see from (5.7)–(5.9) above that the SDE (5.3) satisfies the assumptions (4.29)–(4.30) of Theorem 4.3.1, and hence admits a unique solution. The solution is then a semi-martingale of the form (4.18) for which the assumption (4.23) of Example 4.2.3 is valid. Therefore, for any $T > 0$,

$$E \left[ \|X_T^n\|^2 + 2\nu \int_0^T \|X_t^n\|^2 dt \right] = E[\|X_0^n\|^2] + \text{tr}(\Gamma \mathcal{P}_n)T,$$

$$E \left[ \sup_{t \leq T} \|X_t^n\|^2 \right] \leq (1 + T^2)C < \infty,$$

where $C = C(\Gamma, m_0) \in (0, \infty)$.

We will summarize the above considerations as Theorem 5.1.1 below. To do so, we define:

$$G_t^{\xi,W} = \sigma(\xi, W_s, s \leq t), \quad 0 \leq t < \infty, \quad G_\infty^{\xi,W} = \sigma \left( \bigcup_{t \geq 0} G_t^{\xi,W} \right),$$

$$\mathcal{N}_t^{\xi,W} = \{N \in \Omega, \exists \tilde{N} \in G_\infty^{\xi,W}, N \subset \tilde{N}, P(\tilde{N}) = 0\},$$

and

$$\mathcal{F}_t^{\xi,W} = \sigma \left( G_t^{\xi,W} \cup \mathcal{N}_t^{\xi,W} \right), \quad 0 \leq t < \infty.$$  

**Theorem 5.1.1** Let $W$, $\xi$, and $\mathcal{F}_t^{\xi,W}$ as above. Then, for each $n$, there exists a unique process $X^n_t$ such that:

a) $X^n_t$ is $\mathcal{F}_t^{\xi,W}$-measurable for all $t \geq 0$;

b) (5.3), (5.10) and (5.11) are satisfied;
5.2 Compact imbedding lemmas

We will need some compact imbedding lemmas from [FG95]. We first introduce:

Definition 5.2.1 Let $p \in [1, \infty)$, $T \in (0, \infty)$, and $E$ be a Banach space.

a) We let $L_{p,1}(0, T) \rightarrow E$ denote the Sobolev space of all $u \in L_{p}(0, T) \rightarrow E$ such that:

$$u(t) = u(0) + \int_{0}^{t} u'(s)ds, \text{ for almost all } t \in [0, T]$$

with some $u(0) \in E$ and $u'(\cdot) \in L_{p}(0, T) \rightarrow E$. We endow the space $L_{p,1}(0, T) \rightarrow E$ with the norm $\|u\|_{L_{p,1}(0, T) \rightarrow E}$ defined by

$$\|u\|_{L_{p,1}(0, T) \rightarrow E}^{p} = \int_{0}^{T} (|u(t)|_{E}^{p} + |u'(t)|_{E}^{p})dt.$$

b) For $\alpha \in (0, 1)$, we let $L_{p,\alpha}(0, T) \rightarrow E$ denote the Sobolev space of all $u \in L_{p}(0, T) \rightarrow E$ such that:

$$\int_{0<s<t<T} \frac{|u(t) - u(s)|_{E}^{p}}{|t-s|^{1+\alpha p}}dsdt < \infty.$$

We endow the space $L_{p,\alpha}(0, T) \rightarrow E$ with the norm $\|u\|_{L_{p,\alpha}(0, T) \rightarrow E}$ defined by

$$\|u\|_{L_{p,\alpha}(0, T) \rightarrow E}^{p} = \int_{0}^{T} |u(t)|_{E}^{p}dt + \int_{0<s<t<T} \frac{|u(t) - u(s)|_{E}^{p}}{|t-s|^{1+\alpha p}}dsdt.$$

Remark: Note that:

$$\int_{0<s<t<T} \frac{dsdt}{|t-s|^{1+\lambda}} = \begin{cases} \infty & \text{if } \lambda \geq 0, \\ \frac{T^{1+\lambda}}{(1+\lambda)|\lambda|} & \text{if } \lambda < 0 \end{cases}$$

Therefore, roughly speaking, a function in $L_{p,\alpha}(0, T) \rightarrow E$ is, “Hölder continuous with the exponent bigger than $\alpha$”.

Exercise 5.2.1 Prove that $L_{p,\beta}(0, T) \rightarrow E \hookrightarrow L_{p,\alpha}(0, T) \rightarrow E$ if $0 < \alpha < \beta \leq 1$.

Lemma 5.2.2 [FG95, p.370, Theorem 2.1] Let:

- $E_{1}, \ldots, E_{n}$ and $E$ be Banach spaces such that each $E_{i} \hookrightarrow \hookrightarrow E$, $i = 1, \ldots, n$.
- $p_{1}, \ldots, p_{n} \in (1, \infty)$, $\alpha_{1}, \ldots, \alpha_{n} \in (0, 1)$ are such that $p_{i}\alpha_{i} > 1$, $i = 1, \ldots, n$.

Then, for any $T > 0$,

$$L_{p_{1},\alpha_{1}}(0, T) \rightarrow E_{1} + \ldots + L_{p_{n},\alpha_{n}}(0, T) \rightarrow E_{n} \hookrightarrow \hookrightarrow C([0, T] \rightarrow E).$$

Lemma 5.2.3 [FG95, p.372, Theorem 2.2] Let:

$$E_{0} \hookrightarrow \hookrightarrow E \hookrightarrow E_{1}$$

be Banach spaces such that the first imbedding is compact, and $E_{0}, E_{1}$ are reflexive. Then, for any $p \in (1, \infty)$, $\alpha \in (0, 1)$ and $T > 0$,

$$L_{p}(0, T) \rightarrow E_{0} \cap L_{p,\alpha}(0, T) \rightarrow E_{1} \hookrightarrow \hookrightarrow L_{p}(0, T) \rightarrow E.$$
5.3 Regularity of the noise

Let $H$ be a separable Hilbert space, and $\Gamma : H \to H$ be a non-negative self-adjoint operator of trace class, as in section 3.1. By the Hilbert-Schmidt theorem [RS72, p.203, Theorem VI.16], there exist a CONS $(\varphi_n)_{n \geq 1}$ of $H$ and numbers $\gamma_n \geq 0$ such that:

$$\Gamma \varphi_n = \gamma_n \varphi_n, \quad n \geq 1. \quad (5.14)$$

Let $W$ be a BM$(H, \Gamma)$. Then, the processes:

$$B^k \overset{\text{def}}{=} \langle W, \varphi_k \rangle / \sqrt{\gamma_k}, \quad k \in I \overset{\text{def}}{=} \{ k \in \mathbb{N} ; \gamma_k > 0 \}$$

are independent BM$^1$'s. Let $(B^k)_{k \in \mathbb{N} \setminus I}$ be independent BM$^1$'s which are independent of $(B^k)_{k \in I}$. Then, $(W, \varphi_k) = \sqrt{\gamma_k} B^k$ for all $k \in \mathbb{N}$, and thus,

$$W_t = \sum_{k=0}^{\infty} \langle W_t, \varphi_k \rangle \varphi_k = \sum_{k=0}^{\infty} \sqrt{\gamma_k} B^k_t \varphi_k, \quad t \geq 0.$$

Let us consider the finite summation:

$$W^n_t = \sum_{k=0}^{n} \langle W_t, \varphi_k \rangle \varphi_k = \sum_{k=0}^{n} \sqrt{\gamma_k} B^k_t \varphi_k, \quad t \geq 0, \quad (5.15)$$

Lemma 5.3.1 Referring to (5.15), for any $p \in [1, \infty)$, $\alpha \in [0, 1/2)$ and $T > 0$, there exists $C = C_{\alpha,p,T} \in (0, \infty)$ such that:

$$\sup_{n \geq 0} E[\|W^n_t\|^{p}_{L_{p_\alpha}[0,T] \rightarrow H}] \leq C \text{tr}(\Gamma)^{p/2}. \quad (5.16)$$

Proof: We first prepare an exponential moment bound. Let $\epsilon \in (0,1)$, $\lambda, t \geq 0$ be such that $0 \leq \lambda t \gamma_k \leq 1 - \epsilon$ for all $k \in \mathbb{N}$. Then,

$$1) \quad E \left[ \exp \left( \frac{t}{2} \|W^n_t\|^2 \right) \right] = \prod_{k=0}^{n} \frac{1}{\sqrt{1 - \lambda t \gamma_k}} \leq \exp \left( \frac{\lambda t}{2 \epsilon} \text{tr}(\Gamma) \right).$$

Since $\|W^n_t\|^2 = \sum_{k=0}^{n} \gamma_k |B_t^k|^2$,

$$E \left[ \exp \left( \frac{t}{2} \|W^n_t\|^2 \right) \right] = \prod_{k=0}^{n} E \left[ \exp \left( \frac{\lambda \gamma_k}{2} |B_t^k|^2 \right) \right]$$

We next observe for any $\delta \in [0,1-\epsilon]$ that

$$\frac{1}{1-\delta} = 1 + \frac{\delta}{1-\delta} \leq 1 + \frac{\delta}{\epsilon} \leq e^t.$$

Hence, considering $\delta = \lambda t \gamma_k$ and taking the square root, and then the product over $k = 0, \ldots, n$, we have

$$\prod_{k=0}^{n} \frac{1}{\sqrt{1 - \lambda t \gamma_k}} \leq \exp \left( \frac{\lambda t}{2 \epsilon} \text{tr}(\Gamma) \right).$$

Thus, we get 1). Then, it is not difficult (Exercise 5.3.1 below) to see from 1) that
2) \( E[\|W_t^n - W_s^n\|^p] \leq C_p (\text{tr}(\Gamma)t)^{p/2} \) for any \( p \in (0, \infty) \),

with \( C_p \in (0, \infty) \) depending only on \( p \). Noting that

\[
E[\|W_t^n - W_s^n\|^p] = E[\|W_{t-s}^n\|^p] \leq C_p (\text{tr}(\Gamma)(t-s))^{p/2}, \quad s < t,
\]

we get

\[
E \int_{0<s<t<T} \frac{\|W_t^n - W_s^n\|^p}{(t-s)^{1+\alpha p}} dsdt \leq C_p \text{tr}(\Gamma)^{p/2} \int_{0<s<t<T} \frac{dsdt}{(t-s)^{1+(\alpha-\frac{1}{2})p}} \leq C_{p,\alpha} \text{tr}(\Gamma)^{p/2} T^{1+(\frac{1}{2}-\alpha)p}.
\]

This and 2) imply (5.16).

**Exercise 5.3.1** Conclude 2) from 1) in the proof of Lemma 5.3.1. Hint: Take \( \lambda = \frac{1}{2\text{tr}(\Gamma)t} \) in 1).

### 5.4 A digression on tightness

Let \( X^n = (X^n_t)_{t \geq 0} \in \mathcal{V} \) be the unique solution of (5.3) for the Galerkin approximation. In section 5.5, we will find a "convergent subsequence", the limit of which eventually solves (3.7). This can be done by showing that the laws of \( X^n, n \in \mathbb{N} \) are tight (see Definition 5.4.1). This subsection serves as a collection of notions and facts regarding the tightness, which we will use in section 5.5.

Throughout this subsection, let \( S = (S, \rho) \) be a separable metric space and \( (\Omega, \mathcal{F}, P) \) be a probability space.

**Definition 5.4.1** A sequence \( \{X_n : \Omega \rightarrow S\}_{n \in \mathbb{N}} \) of r.v.'s (or more precisely, the laws of these r.v.'s) are said to be tight, if, for any \( \epsilon \in (0, 1) \), there exists a relatively compact set \( K \subset S \) such that:

\[
\inf_{n \in \mathbb{N}} P(X_n \in K) \geq 1 - \epsilon.
\]

Here is a common way to check the tightness:

**Lemma 5.4.2** Let \( \{X_n : \Omega \rightarrow S\}_{n \in \mathbb{N}} \) be r.v.'s. Suppose that there exists a function \( F : S \rightarrow [0, \infty) \) such that:

- the set \( K_R \overset{\text{def}}{=} \{x \in S ; F(x) \leq R\} \) is relatively compact for all \( R > 0 \);
- \( \sup_{n \in \mathbb{N}} E[F(X_n)] \leq C < \infty. \)

Then, \( \{X_n\}_{n \in \mathbb{N}} \) are tight.

**Proof:** We then have that:

\[
\sup_{n \in \mathbb{N}} P(X_n \not\in K_R) = \sup_{n \in \mathbb{N}} P(F(X_n) > R) \leq \sup_{n \in \mathbb{N}} \frac{E[F(X_n)]}{R} \leq \frac{C}{R} \rightarrow 0.
\]

This proves the tightness. ⊓⊔

Once we are able to check that a sequence of r.v.'s is tight, we have the following consequence:
Lemma 5.4.3 Suppose that $S$ is complete and that a sequence $\{X_n : \Omega \to S\}_{n \in \mathbb{N}}$ of r.v.’s are tight. Then, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, a sequence $n(k) \nearrow \infty$ of integers, and a sequence

$$\{\tilde{X}_k : \tilde{\Omega} \to S\}_{k \in \mathbb{N} \cup \{\infty\}}$$

of r.v.’s such that:

$$\tilde{P}(\tilde{X}_k \in \cdot) = P(X_{n(k)} \in \cdot) \text{ for all } k \in \mathbb{N};$$

$$\lim_{k \to \infty} \tilde{X}_k = \tilde{X}_\infty, \tilde{P}\text{-a.s.}$$

Proof: This is a consequence of Prohorov’s theorem [IW89, p.7, Theorem 2.6] and Skorohod’s representation theorem [IW89, p.9, Theorem 2.7].

Lemma 5.4.4 Suppose that $(S_j, \rho_j)$ ($j = 1, \ldots, m$) are complete separable metric spaces such that all of $S_j$ ($j = 1, \ldots, m$) are subsets of a common set. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables with values in $S \triangleq \bigcap_{j=1}^m S_j$ which is tight in each of $(S_j, \rho_j)$, $j = 1, \ldots, m$ separately. Then, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, a sequence $n(k) \nearrow \infty$ of integers, and a sequence

$$\{\tilde{X}_k : \tilde{\Omega} \to S\}_{k \in \mathbb{N} \cup \{\infty\}}$$

of r.v.’s such that:

$$\tilde{P}(\tilde{X}_k \in \cdot) = P(X_{n(k)} \in \cdot) \text{ for all } k \in \mathbb{N};$$

$$\lim_{k \to \infty} \sum_{j=1}^m \rho_j(X, \tilde{X}_k) = 0 \text{ a.s.}$$

Proof: By induction, it is enough to consider the case of $m = 2$. Let $\epsilon > 0$ be arbitrary. Then, for $j = 1, 2$, there exists a compact subset $K_j$ of $S_j$ such that:

$$P(X_n \in K_j) \geq 1 - \epsilon, \text{ for all } j = 1, 2 \text{ and } n = 1, 2, \ldots$$

Now, a very simple, but crucial observation is that $K_1 \cap K_2$ is compact in $S_1 \cap S_2$ with respect to the metric $\rho_1 + \rho_2$. Also,

$$P(X_n \in K_1 \cap K_2) \geq 1 - 2\epsilon, \text{ for all } j = 1, 2 \text{ and } n = 1, 2, \ldots$$

These imply that $(X_n)$ is tight in $S_1 \cap S_2$ with respect to the metric $\rho_1 + \rho_2$. Thus, the lemma follows from Lemma 5.4.3.

5.5 Convergence of the approximation along a subsequence

Let $X^n = (X^n_t)_{t \geq 0} \in \mathcal{V}$ be the unique solution of (5.3) for the Galerkin approximation. Recall the notation from (2.25):

$$\beta(1,0) = \begin{cases} 1 & \text{if } d = 2, \\ \frac{d}{2} & \text{if } d \geq 3 \end{cases}$$

Proposition 5.5.1 For $\alpha \in [0,1)$ and $\beta > \beta(1,0)$ (cf. (2.25)), Then, there exist a process $X$ and a sequence $(\tilde{X}^k)_{k \geq 1}$ of processes defined on a probability space $(\Omega, \mathcal{F}, P)$ such that the following properties are satisfied:
a) The process $X$ takes values in

$$C([0, \infty) \to V_{2,-\beta}) \cap L_{2,\text{loc}}([0, \infty) \to V_{2,\alpha}).$$  \hspace{1cm} (5.17)

b) For some sequence $n(k) \nearrow \infty$, $\tilde{X}^k$ has the same law as $X^{n(k)}$ and

$$\lim_{k \to \infty} \tilde{X}^k = X \text{ in the metric space (5.17), P-a.s.} \hspace{1cm} (5.18)$$

We divide the proof of Proposition 5.5.1 into the series of lemmas: To prepare the proof of these lemmas, we write (5.3) as:

$$X^n_t = X^n_0 + J^n_t + W^n_t \text{ with } J^n_t = \int_0^t \mathcal{P}_n b(X^n_s)ds. \hspace{1cm} (5.19)$$

**Lemma 5.5.2** Let $\beta(1,0)$ and $J^n_t$ be as in (2.25) and (5.19). Then, there exists $C_T \in (0, \infty)$ such that:

$$\sup_{n \geq 1} E[\Vert \sqrt{J^n_t} \Vert_{L_{2,1}([0,T] \to V_{2,-\beta(1,0)})}] \leq C_T < \infty. \hspace{1cm} (5.20)$$

Proof: It is not difficult to see that:

1) \hspace{1cm} $\int_0^T \Vert b(X^n_s) \Vert_{2,-\beta(1,0)}^2 \leq \int_0^T \nu + C \sup_{s \leq T} \Vert X^n_s \Vert_{2,1}^2 ds.$

2) \hspace{1cm} $\int_0^T \nu + C \sup_{s \leq T} \Vert X^n_s \Vert_{2,1}^2 ds \leq (\nu + C \sup_{s \leq T} \Vert X^n_s \Vert_{2,1})^2 \int_0^T \Vert X^n_s \Vert_{2,1}^2 ds.$

Since $\mathcal{P}_n$ is contraction on $V_{2,\alpha}$ for any $\alpha \in \mathbb{R}$, we can combine the above bounds and (5.10)–(5.11) to obtain (5.20) as follows:

$$E \left[ \Vert J^n_t \Vert_{L_{2,1}([0,T] \to V_{2,-\beta(1,0)})}^{1/2} \right] \leq C_T E \left[ (\nu + C \sup_{s \leq T} \Vert X^n_s \Vert_{2,1})^{1/2} \left( \int_0^T \Vert X^n_s \Vert_{2,1}^2 ds \right)^{1/2} \right] \
\leq C_T E \left[ (\nu + C \sup_{s \leq T} \Vert X^n_s \Vert_{2,1}) \left( \int_0^T \Vert X^n_s \Vert_{2,1}^2 ds \right)^{1/2} \right] \
\leq C_T < \infty.$$

**Exercise 5.5.1** Let everything be as in Definition 5.2.1 a) and suppose that $u(0) = 0$. Prove then that

$$\Vert u \Vert_{L_p([0,T] \to \mathbb{R})}^p \leq C_T \int_0^T \Vert u'(s) \Vert_{E}^p ds.$$

**Lemma 5.5.3** Let $\beta > \beta(1,0)$. Then, $\{X^n\}_{n=1}^{\infty}$ are tight on $C([0, \infty) \to V_{2,-\beta}).$
Proof. It is enough to prove the following for each fixed $T > 0$:

1) $(X^n_t)_{t \leq T}, \ n = 1, 2, ...$ are tight on $C([0, T] \to V_{2,-\beta})$.

To see this, we set:

$$\mathcal{S} = L_{2,1}([0, T] \to V_{2,-\beta(1,0)}) + L_{p,\alpha}([0, T] \to V_{2,0}), \ \text{with } \alpha \in (0, 1/2), p > 1/\alpha.$$

The idea is to take $\| \cdot \|_\mathcal{S}$ as the function $F$ in Lemma 5.4.2. We have that

2) $$\sup_n E[\|X^n_0 + J^n\|_{L_{2,1}([0,T] \to V_{2,-\beta(1,0)})}] \overset{(5.20)}{=} C_T < \infty$$

On the other hand,

3) $$\sup_n E[\|W^n\|_{L_{p,\alpha}([0,T] \to V_{2,0})}] \overset{(5.16)}{=} C_T < \infty.$$

We conclude from 2)–3) and the decomposition (5.19) that

$$\sup_n E[\|X^n\|_\mathcal{S}] \leq C_T < \infty$$

On the other hand, we see from Lemma 5.2.2 that

$$\mathcal{S} \hookrightarrow C([0, T] \to V_{2,-\beta})$$

hence that the set:

$$\{X; \|X^n\|_\mathcal{S} \leq R\}$$

is relatively compact in $C([0, T] \to V_{2,-\beta})$. Thus, we have the tightness 1) by Lemma 5.4.2. 

Lemma 5.5.4 Suppose that $\alpha \in [0, 1)$. Then, $(X^n)_{n=1}^\infty$ are tight on $L_{2,loc}([0, \infty) \to V_{2,\alpha})$.

Proof. It is enough to prove the following for each fixed $T > 0$:

1) $(X^n_t)_{t \leq T}, \ n = 1, 2, ...$ are tight on $L_2([0, T] \to V_{2,\alpha})$.

To see this, we set:

$$\mathcal{I} = L_2([0, T] \to V_{2,1}) \cap L_{2,\gamma}([0, T] \to V_{2,-\beta(1,0)}), \ \text{with } \gamma \in (0, 1/2).$$

The idea is to take $\| \cdot \|_\mathcal{I}$ as the function $F$ in Lemma 5.4.2. We have that

2) $$\sup_n E[\|X^n\|_{L_2([0,T] \to V_{2,1})}^2] = \sup_n E[\int_0^T \|X^n_t\|_{2,1}^2 dt] \overset{(5.10)}{=} C_T < \infty$$

On the other hand,

$$\sup_n E[\|X^n\|_{L_{2,\gamma}([0,T] \to V_{2,-\beta(1,0)})}]$$

$$\leq \sup_n E[\|X^n_0 + J^n\|_{L_{2,\gamma}([0,T] \to V_{2,-\beta(1,0)})}] + \sup_n E[\|W^n\|_{L_{2,\gamma}([0,T] \to V_{2,0})}]$$

$$\overset{(5.16),(5.20)}{=} C_T < \infty.$$
We conclude from this and 2) that
\[ \sup_n E[\|X^n\|_I] \leq C_T < \infty. \]

On the other hand, we will see from Lemma 5.2.3 that
\[ I \hookrightarrow L_2([0, T] \to V_{2,\alpha}), \]
hence that the set:
\[ \{ X ; \|X^n\|_I \leq R \} \]
is relatively compact in \( L_2([0, T] \to V_{2,\alpha}) \). Thus, we have the tightness 1) by Lemma 5.4.2. □

Finally, Proposition 5.5.1 follows from Lemma 5.5.3–Lemma 5.5.4 and Lemma 5.4.4.

6 Proof of Theorem 3.2.1 and Theorem 3.2.2

6.1 Proof of Theorem 3.2.1

Let \( X \) and \( \tilde{X}^k \) be as in Proposition 5.5.1. We will verify that \( X \) takes values in the metric space (3.4) as well as properties (3.5)–(3.9) for \( X \). (3.5) can easily be seen. In fact,
\[ \tilde{X}_0^k \rightarrow X_0 \quad \text{a.s. in } V_{2,-\beta}, \]
\[ \tilde{X}_0^k \text{law}= X_0^{n(k)} = \mathcal{P}_{n(k)} \xi \rightarrow \xi \quad \text{a.s. in } V_{2,0}. \]
Thus the laws of \( X_0 \) and \( \xi \) are identical. To see (3.8)–(3.9), note that:
\[ \|v_T\|_2^2, \sup_{t \leq T} \|v_t\|_2^2, \int_0^T \|v_t\|_{2,1}^2 dt \]
are lower semi-continuous functions of \( v \) on the metric space (5.17). Thus, (3.8)–(3.9) follow from (5.10)–(5.11) and Proposition 5.5.1 via Fatou’s lemma.

To show (3.6)–(3.7), we prepare the following:

Lemma 6.1.1 Let \( \varphi \in \mathcal{V} \) and \( T > 0 \). Then,
\[ \lim_{k \to \infty} \int_0^T \langle \varphi, (\tilde{X}_t^k \cdot \nabla)\tilde{X}_t^k \rangle dt = \int_0^T \langle \varphi, (X_t \cdot \nabla)X_t \rangle dt \quad \text{in probability,} \quad (6.1) \]
\[ \lim_{k \to \infty} \int_0^T \langle \Delta \varphi, \tilde{X}_t^k \rangle dt = \int_0^T \langle \Delta \varphi, X_t \rangle dt \quad \text{a.s.,} \quad (6.2) \]
\[ \lim_{k \to \infty} \int_0^T \langle \varphi, \mathcal{P}_{n(k)}b(\tilde{X}_t^k) \rangle dt = \int_0^T \langle \varphi, b(X_t) \rangle dt \quad \text{in probability.} \quad (6.3) \]

Proof: (6.1): Since,
\[ \tilde{X}_t^k \cdot \nabla \tilde{X}_t^k - X_t \cdot \nabla X_t = (\tilde{X}_t^k - X_t) \cdot \nabla \tilde{X}_t^k + X_t \cdot \nabla (\tilde{X}_t^k - X_t), \]
we have
\[ \int_0^T |\langle \varphi, \tilde{X}_t^k \cdot \nabla \tilde{X}_t^k - X_t \cdot \nabla X_t \rangle| dt \leq I_1 + I_2, \]

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where

\[
I_1 = \int_0^T |\langle \varphi, (\tilde{X}_t^k - X_t) \cdot \nabla \tilde{X}_t^k \rangle| \, dt, \quad \text{and} \quad I_2 = \int_0^T |\langle \varphi, X_t \cdot \nabla (\tilde{X}_t^k - X_t) \rangle| \, dt.
\]

To bound \(I_1\), we take

\[
\alpha_1 = \alpha \in (0, 1 \wedge \frac{d}{2}), \quad \alpha_2 = 0, \quad \alpha_3 = \frac{d}{2} - \alpha \in (0, \frac{d}{2}).
\]

in Lemma 2.2.1. Then, by (2.14), we have that

\[
|\langle \varphi, (\tilde{X}_t^k - X_t) \cdot \nabla \tilde{X}_t^k \rangle| \leq C \|\tilde{X}_t^k - X_t\|_{2,\alpha} \|\tilde{X}_t^k\|_2 \|\varphi\|_{2,1+\alpha_3}
\]

and hence that,

\[
I_1 \leq C \|\varphi\|_{2,1+\alpha_3} \sup_{t \leq T} \|\tilde{X}_t^k\|_2 \int_0^T \|\tilde{X}_t^k - X_t\|_{2,\alpha} \, dt.
\]

By (5.11) and Proposition 5.5.1,

\[
\sup_{k \geq 1} \mathbb{E}[\sup_{t \leq T} \|\tilde{X}_t^k\|_2^2] < \infty \quad \text{and} \quad \lim_{k \to \infty} \int_0^T \|\tilde{X}_t^k - X_t\|_{2,\alpha}^2 \, dt = 0 \quad \text{P-a.s.}
\]

Then, it is easy to conclude from these that \(\lim_{k \to \infty} I_1 = 0\) in probability (Exercise 6.1.1 below). To bound \(I_2\), we take

\[
\alpha_1 = 0, \quad \alpha_2 = \alpha \in (0, 1 \wedge \frac{d}{2}), \quad \alpha_3 = \frac{d}{2} - \alpha \in (0, \frac{d}{2})
\]

in Lemma 2.2.1. On the other hand, we have by (2.14) that

\[
|\langle \varphi, X_t \cdot \nabla (\tilde{X}_t^k - X_t) \rangle| \leq C \|X_t\|_2 \|\tilde{X}_t^k - X_t\|_{2,\alpha} \|\varphi\|_{2,1+\alpha_3}
\]

and hence that,

\[
I_2 \leq C \|\varphi\|_{2,1+\alpha_3} \sup_{t \leq T} \|X_t\|_2 \int_0^T \|\tilde{X}_t^k - X_t\|_{2,\alpha} \, dt.
\]

By (3.9) and Proposition 5.5.1,

\[
\mathbb{E}[\sup_{t \leq T} \|X_t\|_2^2] < \infty \quad \text{and} \quad \lim_{k \to \infty} \int_0^T \|\tilde{X}_t^k - X_t\|_{2,\alpha} \, dt = 0 \quad \text{P-a.s.}
\]

Then, it is easy to conclude from these that \(\lim_{k \to \infty} I_2 = 0\) in probability (Exercise 6.1.1 below).

(6.2): This is an easy consequence of Proposition 5.5.1.

(6.3) follows from (6.1) and (6.2). Since \(\varphi \in \mathcal{V}\) is fixed and \(k\) is tending to \(\infty\), we do not have to care about \(\mathcal{P}_{n(k)}\) here. \(\square\)

**Exercise 6.1.1** Let \(X_n, Y_n\) be r.v.'s such that \(\{X_n\}_{n \geq 1}\) are tight and \(Y_n \to 0\) in probability. Prove then that \(X_n Y_n \to 0\) in probability.

We see (3.6)–(3.7) from the following:
Lemma 6.1.2 Let:
\[ Y_t = Y_t(X) = X_t - X_0 - \int_0^t b(X_s)ds, \quad t \geq 0. \] (6.4)

Then, \( Y \) is a BM\((V_{2,0}, \Gamma)\). Moreover, \( Y_{t+} - Y_t \) and \( \{ (\varphi, X_s) ; s \leq t, \varphi \in \mathcal{V} \} \) are independent for any \( t \geq 0 \).

It is enough to prove that for each \( \varphi \in \mathcal{V} \) and \( 0 \leq s < t \),

1) \[ E \left[ \exp \left( i (\varphi, Y_t - Y_s) \right) \right] | \mathcal{G}_s] = \exp \left( -\frac{t-s}{2} \langle \varphi, \Gamma \varphi \rangle \right), \text{ a.s.} \]

where \( \mathcal{G}_s = \sigma((\varphi, X_u) ; u \leq s, \varphi \in \mathcal{V}) \). We set

\[ F(X) = f((\varphi_1, X_{u_1}), ..., (\varphi_n, X_{u_n})), \]

where \( f \in C_b(\mathbb{R}^n), 0 \leq u_1 < ... < u_n \leq s \) and \( \varphi_1, ..., \varphi_n \in \mathcal{V} \) are chosen arbitrary in advance. Then, 1) can be verified by showing that

2) \[ E \left[ \exp \left( i (\varphi, Y_t - Y_s) \right) F(X) \right] = \exp \left( -\frac{t-s}{2} \langle \varphi, \Gamma \varphi \rangle \right) E[F(X)]. \]

Let:
\[ Y_t^k = \tilde{X}_t^k - \tilde{X}_0^k - \int_0^t \mathcal{P}_{n(k)} b(\tilde{X}_s^k)ds, \quad t \geq 0. \]

We then see from Theorem 5.1.1 that

3) \[ E \left[ \exp \left( i (\varphi, Y_t^k - Y_s^k) \right) F(\tilde{X}^k) \right] = \exp \left( -\frac{t-s}{2} \langle \varphi, \Gamma \mathcal{P}_{n(k)} \varphi \rangle \right) E[F(\tilde{X}^k)], \]

Moreover, we have for any \( \varphi \in \mathcal{V} \),

\[ \lim_{k \to \infty} \langle \varphi, Y_t^k - Y_s^k \rangle \stackrel{(5.18),(6.3)}{=} \langle \varphi, Y_t - Y_s \rangle \text{ in probability,} \]

and hence

\[ \lim_{k \to \infty} \text{LHS of 3)} = \text{LHS of 2).} \]

On the other hand,

\[ \lim_{k \to \infty} \text{RHS of 3)} \stackrel{(5.18)}{=} \text{RHS of 2).} \]

These prove 2). \[ \square \]

Finally, we prove that \( X \) takes values in the metric space \((3.4)\). It follows from \((3.9)\) that

\[ X \in L_{2,\text{loc}}([0, \infty) \to V_{2,1}) \cap L_{\infty,\text{loc}}([0, \infty) \to V_{2,0}). \]

Thus, it remains to show that \( X \in C([0, \infty) \to V_{2,-\beta(1,1)}) \). We see from Lemma 2.2.3 that:

\[ \int_0^t b(X_s)ds \in C([0, \infty) \to V_{2,-\beta(1,1)}) \text{ if } X \in L_2([0, \infty) \to V_{2,1}). \]

On the other hand, \( Y \in C([0, \infty) \to V_{2,0}). \) These show that \( X \in C([0, \infty) \to V_{2,-\beta(1,1)}) \). \[ \square \]
6.2 Proof of Theorem 3.2.2

Here, we can follow the argument of [Te79, p. 294, Theorem 3.2] almost verbatim. We will present it for the convenience of the readers.

We need technical lemmas:

**Lemma 6.2.1** [Te79, pp. 60–61, Lemma 1.2] Let $H$ and $V$ be a Hilbert spaces such that:

$$V \hookrightarrow H \hookrightarrow V^*.$$  

Suppose that $f \in L_2([0, T] \rightarrow V)$ has derivative $f'$ in $L_2([0, T] \rightarrow V^*)$. Then,

$$\frac{d}{dt} |f|^2_H = 2
\langle f, f' \rangle_{V^*},$$  

(6.5)

in the distributional sense on $(0, T)$.

**Lemma 6.2.2** For any $T > 0$, there exists $C_T \in (0, \infty)$ such that:

$$E \left[ \int_0^T \|b(X_t)\|_{2,-\beta(1,0)} \right] \leq C_T < \infty.$$  

(6.6)

Proof: Using (3.9), the lemma can be shown in the same way as Lemma 5.5.2. \qed

Let $X$ and $\tilde{X}$ be as in the assumptions of Theorem 3.2.2 and

$$Z_t = X_t - \tilde{X}_t = \int_0^t (b(X_s) - b(\tilde{X}_s)) ds.$$  

Then,

1) $Z \in L_{2,\text{loc}}([0, \infty) \rightarrow V_{2,1})$

and by Lemma 6.2.2,

2) $\partial_t Z = b(X) - b(\tilde{X}) \in L_{2,\text{loc}}([0, \infty) \rightarrow V_{2,-\beta(1,0)})$

Since $\beta(1, 0) = 1$, we see from 2) and Lemma 6.2.1 (applied to $f = Z$ and $V = V_{2,1}$) that

3) $\frac{1}{2} \frac{d}{dt} \|Z\|^2 \leq \langle Z_t, b(X_t) - b(\tilde{X}_t) \rangle = -I_t - J_t$

in the distributional sense, where

$$I_t = \langle Z_t, (X_t \cdot \nabla)X_t - (\tilde{X}_t \cdot \nabla)\tilde{X}_t \rangle,$$

$$J_t = \nu \langle \nabla Z_t, \nabla X_t - \nabla \tilde{X}_t \rangle = \nu \|\nabla Z_t\|^2_2.$$  

On the other hand, since $\tilde{X}_t = X_t - Z_t$, we see that

$$\langle Z_t, (\tilde{X}_t \cdot \nabla)\tilde{X}_t \rangle \overset{\text{Lemma 2.1.2}}{=} \langle Z_t, (\tilde{X}_t \cdot \nabla)X_t \rangle = \langle Z_t, ((X_t - Z_t) \cdot \nabla)X_t \rangle,$$

and hence that

$$I_t = \langle Z_t, (Z_t \cdot \nabla)X_t \rangle.$$  

We now apply Lemma 2.2.2 with $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, 0)$. Note that these $\alpha_i$ satisfy the assumption of Lemma 2.2.2 only when $d = 2$.  

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4) \[ |I_t| \leq C_3 \|Z_t\|_{2,1} \|Z_t\|_2 X_t \|Z_t\|_2 \leq \nu \|Z_t\|_{2,1}^2 + C_4 \|X_t\|_{2,1} \|Z_t\|_2^2. \]

We see from 3)-4) that
\[ \frac{1}{2} \frac{d}{dt} \|Z_t\|_2^2 \leq C_4 \|X_t\|_{2,1} \|Z_t\|_2^2. \]

This implies, via Gronwall’s lemma (We need an appropriate generalization, since the derivative above is in the distributional sense.) that
\[ \|Z_t\|_2^2 \leq \|Z_0\|_2^2 \exp(\int_0^t \|X_s\|_{2,1}^2 ds). \]

This proves that \( \|Z_t\|_2 \equiv 0. \)

\[ \square \]

7 Appendix

**Lemma 7.0.3** Suppose that a CONS \( \{\varphi_n\}_{n \geq 1} \) of \( H \) and numbers \( \gamma_n \geq 0 \) satisfy (5.14).

a) Let \( \{B^k\}_{k \in \mathbb{N}} \) be independent standard BM\(^1\)’s. Then, the process
\[ W^m_t = \sum_{k=0}^{m} \sqrt{\gamma_k} B^k_t \varphi_k, \quad t \geq 0, \]
converges to a BM\((H, \Gamma) W \) in the sense that:
\[ \lim_{n \to \infty} E \left[ \sup_{t \leq T} \|W^n_t - W_t\|^2 \right] = 0 \quad \text{for any } T > 0. \]

b) For any BM\((H, \Gamma) W \), there are independent standard BM\(^1\)’s such that (7.2) holds with the process defined by (5.15).

Proof: a): Let us show that

1) \((W^m)_n \in \mathbb{N} \) is a Cauchy sequence with respect to seminorms:
\[ \|W\|_t = E \left[ \sup_{s \leq t} \|W_s\|^2 \right]^{1/2}, \quad t \in (0, \infty). \]

In fact, for \( m < n, \)
\[ \|W^n_t - W^m_t\|^2 = \sum_{m < k \leq n} \gamma_k |B^k_t|^2. \]

By this and Doob’s \( L^2 \)-maximal inequality,
\[ E \left[ \sup_{s \leq t} \|W^n_s - W^m_s\|^2 \right] \leq \sum_{m < k \leq n} \gamma_k E \left[ \sup_{s \leq t} |B^k_s|^2 \right] \leq 4t \sum_{m < k \leq n} \gamma_k \to 0. \]

By 1), there exists a random variable \( W \) with values in \( C([0, \infty) \to H) \) such that (7.2) holds. It is easy to see from this that for \( 0 \leq s < t, \)
\[ \lim_{n \to \infty} \exp(i \langle \varphi, W^n_t - W^n_s \rangle) = \exp(i \langle \varphi, W_t - W_s \rangle) \quad \text{in } L^1(P), \]
and hence

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\[ \lim_{n \to \infty} E \left[ \exp \left( i \langle \varphi, W_t^n - W_s^n \rangle \right) \mid \mathcal{G}_s^W \right] = E \left[ \exp \left( i \langle \varphi, W_t - W_s \rangle \right) \mid \mathcal{G}_s^W \right] \text{ in } L^1(P). \]

On the other hand,

\[
E \left[ \exp \left( i \langle \varphi, W_t^n - W_s^n \rangle \right) \mid \mathcal{G}_s^W \right] = \prod_{k=0}^{n} E \left[ \exp \left( i \sqrt{k} \langle \varphi, \varphi_k \rangle (B_t^k - B_s^k) \right) \right] = \prod_{k=0}^{n} \exp \left( -\frac{t-s}{2} \gamma_k \langle \varphi, \varphi_k \rangle^2 \right)^{n \to \infty} \exp \left( -\frac{t-s}{2} \langle \varphi, \Gamma \varphi \rangle \right).
\]

By this and 2), we have (3.3).

b): Processes:

\[ B^k \overset{\text{def}}{=} \langle W, \varphi_k \rangle / \sqrt{k}, \quad k \in I \overset{\text{def}}{=} \{ k \in \mathbb{N} ; \gamma_k > 0 \} \]

are independent BM's. Let \( \{ B^k \}_{k \in I} \) be independent BM's which are independent of \( \{ B^k \}_{k \in I} \). Then, \( \langle W, \varphi_k \rangle = \sqrt{k} B^k \) for all \( k \in \mathbb{N} \), and hence (5.15) holds. \( \square \)

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References


