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Kyoto University
Navier-Stokes Equations with Random Forcing

Nobuo Yoshida

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0 Introduction

We would like to analyze the turbulence of a viscous fluid in $\mathbb{R}^d$ (physically, $d = 3$). Let

$$ u = (u_i(t, x))_{i=1}^d \in \mathbb{R}^d $$

$$ \Pi = \Pi(t, x) \in \mathbb{R} $$

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1Division of Mathematics Graduate School of Science Kyoto University, Kyoto 606-8502, Japan. email: nobuo@math.kyoto-u.ac.jp URL: http://www.math.kyoto-u.ac.jp/nobuo/
be the velocity and the pressure of the fluid at time \( t \geq 0 \) at the position \( x \in \mathbb{R}^d \). For fluids like air and water, it is accepted in hydrodynamics that they satisfy the **Navier-Stokes equation**:

\[
\begin{align*}
\text{div} u &= 0, \\
\partial_t u + (u \cdot \nabla) u &= -\nabla \Pi + \nu \Delta u + F,
\end{align*}
\]

(0.3) (0.4)

where \( u \cdot \nabla = \sum_{j=1}^{d} u_j \partial_j \), \( \nu > 0 \) is a constant, called **kinematic viscosity**, and \( F = F_t(x), (t, x) \in [0, \infty) \times \mathbb{T}^d \) is a given external force. Physical interpretation of (0.3) is the mass conservation, while (0.4) is the motion equation.

On the other hand, since the turbulence is a random phenomenon, we need to bring a certain random factor into the model. To do so, we consider a **colored noise**, which is “time derivative” of a certain function space valued Brownian motion \( W = W_t(x) \) and take \( F_t(x) = \partial_t W_t(x) \) in (0.4). This may look too much of an idealization of the real turbulence. However, this way of modeling is common in literatures \[F108\] and references therein.

Based mainly on \[F108\], we explain the construction of the weak solution to (0.3)–(0.4) globally in time in the case \( F_t(x) = \partial_t W_t(x) \).

### 1 Physical derivation of the Navier-Stokes equation

We review the heuristic argument to “derive” (0.3)–(0.4) from the physical assumptions. Let \( e_1, ..., e_d \) be the canonical basis of \( \mathbb{R}^d \):

\[
\begin{align*}
e_1 &= (1, 0, ..., 0), & e_2 &= (0, 1, 0, ..., 0), & \ldots, & e_d &= (0, \ldots, 0, 1).
\end{align*}
\]

(1.1)

Also, it is convenient to introduce the following small box and plaquettes:

\[
\square = \left[ -\frac{\delta}{2}, \frac{\delta}{2} \right]^d, \quad \square_i = \{x \in \square ; x_i = 0\}, \quad i = 1, \ldots, d,
\]

(1.2)

where the side-length \( \delta > 0 \) of the box \( \square \) and the plaquette \( \square_i \) is supposed to be very small, eventually tending to zero. Let

\[
u = (u_i(t, x))_{i=1}^{d}, \quad \rho = \rho(t, x) \geq 0
\]

(1.3)

be the velocity and the density of the fluid at time-space \( (t, x) \).

#### 1.1 The mass conservation

We first derive (0.3) for a constant density fluid \( \rho \equiv \text{const.} \). To do so, however, we do not assume that \( \rho \equiv \text{const.} \) for a moment and consider the mass \( m(x + \square) \) of the fluid on the cube \( x + \square \) centered at \( x \) (cf. (1.2)):

\[
m(x + \square) = \int_{x+\square} \rho \cong \rho(x) \delta^d
\]

(1.4)

Here and often in what follows, we omit the time \( t \) in the notation. The time derivative of the mass is given as follows:

\[
\partial_t m(x + \square) = \sum_{j=1}^{d} m_j(x),
\]

(1.5)
where
\[ m_j(x) = \underbrace{(\rho u_j)(x - \frac{\delta}{2}e_j)}_{\text{inward flux of the mass}} \delta^{d-1} - \underbrace{(\rho u_j)(x + \frac{\delta}{2}e_j)}_{\text{outerward flux of the mass}} \delta^{d-1} \]
through the face \( (x - \frac{\delta}{2}e_j) \square_j \) and through the face \( (x + \frac{\delta}{2}e_j) \square_j \)

By Taylor expanding \((\rho u_j)(x \mp \frac{\delta}{2}e_j)\) above, we see that
\[
m_j(x) = ((\rho u_j)(x) - \partial_j(\rho u_j)(x)\frac{\delta}{2} + O(\delta^2))\delta^{d-1} - ((\rho u_j)(x) + \partial_j(\rho u_j)(x)\frac{\delta}{2} + O(\delta^2))\delta^{d-1}
= -\partial_j(\rho u_j)(x)\delta^d + O(\delta^{d+1}).
\]

By this and (1.5), we get:
\[
\frac{1}{\delta^d} \partial_t m(x + \square) = -\sum_{j=1}^{d} \partial_j(\rho u_j)(x) + O(\delta)
\]
(1.6)

Note that
\[
\rho(x) = \lim_{\delta \searrow 0} \frac{1}{\delta^d} m(x + \square).
\]

If we believe that the above limit commutes with \(\partial_t\), we see from (1.6) that
\[
\partial_t \rho + \sum_{j=1}^{d} \partial_j(\rho u_j)(x) = 0.
\]
(1.7)

In particular, for a constant density flow: \(\rho \equiv \text{const}\), (1.7) is reduced to (0.3). Note also that the interchange of the order of \(\lim_{\delta \searrow 0}\) and \(\partial_t\) assumed above is perfectly correct in this case.

1.2 Force exerted on fluids: the stress tensor

The notion of stress can be thought of as actions, like pushing, pulling and rubbing a door. Then, the action has obviously different effects depending on the side of the door which the action is made on. Therefore, we distinguish the side of the plaquette \(\square_i\): let
\[
\square_i^+ = \text{"the } x_i > 0\text{-side" of } \square_i = \{x \in \square; x_i = 0\}
\]
\[
\square_i^- = \text{the "opposite side" of } \square_i.
\]

Imagine that the plaquette \(\square_i\) is put in a stream with the velocity \(u\). Then forces are exerted on plane \(\square_i\), e.g., pulling, pushing, or rubbing. With this in mind, we introduce:

\[
\tau_i^+(x) = (\tau_i^+(x))_{j=1}^d = \text{the force exerted on } x + \square_i^+ \text{ by the stream}
\]
\[
= -\text{the force exerted on } x + \square_i^- \text{ by the stream,}
\]
(1.8)
where the second equality is, of course, the principle of action-reaction. We then define the stress tensor \( \tau(x) = (\tau_{ij}(x))_{i,j=1}^{d} \) by:

\[
\tau_{ij}(x) = \lim_{\delta \searrow 0} \frac{1}{\delta^{d-1}} \tau_{ij}^{\square}(x).
\]

(1.10)

\( \tau_{ij}(x) \) is the \( j \)-th component of the force exerted on \( x \) by the stream from the side \( x_{i}+ \). We will assume that

- \( \tau \) is of the form:
  \[
  \tau(x) = -\Pi(x)I + \tau^{F}(x),
  \]
  (1.11)
  where \( \Pi(x) = \Pi(t, x) \) is the pressure (a real function), \( I \) is the identity matrix, and \( \tau^{F}(x) \) is the friction term of \( \tau(x) \).

- \( \tau \) is symmetric, i.e., \( \tau_{ij} = \tau_{ji} \), or equivalently, \( \tau_{ij}^{F} = \tau_{ji}^{F} \).

The symmetry assumption above is based on the conservation of the angular momentum. A typical example of the friction term is provided by the following Stokes law:

\[
\tau_{ij}^{F} = \mu(\partial_{i}u_{j} + \partial_{j}u_{i}),
\]

(1.12)

where the constant \( \mu > 0 \) is the coefficient of friction, and the tensor \( \left( \frac{\partial u_{i} + \partial u_{i}}{2} \right) \) is called the symmetrized velocity gradient tensor.

Let

\[
f^{\square}(x) = (f_{j}^{\square}(x))_{j=1}^{d}
\]

the force exerted on the outer boundary of \( x + \square \) by the stream.

Here, the outer boundary is the union of

\[
(x + \frac{\delta}{2}e_{i}) + \square_{i}^{+}, \ (x - \frac{\delta}{2}e_{i}) + \square_{i}^{-} \quad i = 1, \ldots, d.
\]

Then, it turn out to be reasonable to define the force exerted to a point \( x \) by the stream by:

\[
f(x) = (f_{j}(x))_{j=1}^{d}, \quad \text{where} \quad f_{j}(x) = \lim_{\delta \searrow 0} \frac{1}{\delta^{d}} f_{j}^{\square}(x).
\]

(1.13)

It may appear at first sight that "\( 2d\delta^{d-1} \)" is more appropriate in place of \( \delta^{d} \) above. However, we will see later on that \( \delta^{d} \) is indeed the right normalization. We will prove that

\[
f_{j} = \sum_{i=1}^{d} \partial_{i} \tau_{ij}.
\]

(1.14)

Before we prove (1.14), we make some remarks. By (1.11), (1.14) reads:

\[
f = -\nabla \Pi + \left( \sum_{i=1}^{d} \partial_{i} \tau_{ij}^{F} \right)_{j=1}^{d}.
\]

(1.15)
Moreover, if we suppose that the fluid is of constant density and the Stokes law (1.12) holds, then, since $\text{div} u = 0$,
\[
\sum_{i=1}^{d} \partial_{i} r_{ij}^{F} = \mu \sum_{i=1}^{d} (\partial_{i} \partial_{i} u_{j} + \partial_{i} \partial_{j} u_{i}) = \mu \Delta u_{j}.
\]
Thus, (1.15) becomes:
\[
f(x) = -\nabla \Pi + \mu \Delta u. \tag{1.16}
\]
We turn to the proof of (1.14). We have, by (1.8)–(1.10) that
\[
f_{j}^{\square}(x) = \sum_{i=1}^{d} \tau_{ij}^{\square} \left( x + \frac{\delta}{2} e_{i} \right) + \sum_{i=1}^{d} \tau_{ij}^{\square} \left( x - \frac{\delta}{2} e_{i} \right)
\]
the force exerted on $(x + \frac{\delta}{2} e_{i}) + \square_{i}^{+}$ the force exerted on $(x - \frac{\delta}{2} e_{i}) + \square_{i}^{-}$
\[
\cong \sum_{i=1}^{d} (\tau_{ij}(x + \frac{\delta}{2} e_{i}) - \tau_{ij}(x - \frac{\delta}{2} e_{i})) \delta^{d-1}. \tag{1.17}
\]
On the other hand, by Taylor expanding $\tau_{ij}(x \pm \frac{\delta}{2} e_{i})$ above, we have that
\[
\tau_{ij}(x + \frac{\delta}{2} e_{i}) - \tau_{ij}(x - \frac{\delta}{2} e_{i})
\]
\[
= \left( \tau_{ij}(x) + \partial_{i} \tau_{ij}(x) \frac{\delta}{2} + O(\delta^{2}) \right) - \left( \tau_{ij}(x) - \partial_{i} \tau_{ij}(x) \frac{\delta}{2} + O(\delta^{2}) \right)
\]
\[
= \partial_{i} \tau_{ij}(x) \delta + O(\delta^{2}).
\]
Plugging this into (1.17), we have
\[
f_{j}^{\square}(x) \cong \partial_{i} \tau_{ij}(x) \delta^{d} + O(\delta^{d+1})
\]
Thus, if we believe that the approximation $\cong$ is good enough, we have (1.14).

1.3 The motion equation
To derive the motion equation (0.4), we introduce the stream line $x(t) \in \mathbb{R}^{d}, t \geq 0$ define by:
\[
x(t) = x(0) + \int_{0}^{t} u(s, x(s)) ds.
\]
The curve $x(\cdot)$ is the integral curve of the velocity $u$, hence, roughly speaking, it is a position of a particle moving on the stream. The classical Newton’s motion equation is:
\[
\text{mass} \times \text{acceleration} = \text{force},
\]
which, in our case, takes the following form:
\[
\rho(x(t)) \frac{d}{dt} u(t, x(t)) = f(x(t)), \tag{1.18}
\]
where the force \( f \) is given by (1.15). We have by the chain rule that
\[
\frac{d}{dt}u(t, x(t)) = \partial_t u(t, x(t)) + \sum_{j=1}^{d} \partial_j u(t, x(t)) \frac{dx_j(t)}{dt} = \partial_t u(t, x(t)) + (u \cdot \nabla)u(t, x(t)).
\]

By the above identity, together with (1.15) and (1.18), we get
\[
\rho (\partial_t u + (u \cdot \nabla)u) = -\nabla \Pi + \left( \sum_{i=1}^{d} \partial_i \tau_{ij}^{F} \right). \tag{1.19}
\]

If we suppose that the fluid is of constant density and the Stokes law (1.12) holds, then, by (1.16), we have that
\[
\partial_t u + (u \cdot \nabla)u = -\frac{1}{\rho} \nabla \Pi + \frac{\mu}{\rho} \Delta u, \tag{1.20}
\]
where the constant \( \nu \) is the kinematic viscosity.

2 The mathematical framework in the case of non-random forcing term

From here on, we assume that the container of the fluid is the \( d \)-dimensional torus:
\[
\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0, 1]^d.
\]

This is a part of idealization. The unknown functions of the Navier-Stokes equation (NS) are

- **velocity of fluid** \( u = u_t(x) \in \mathbb{R}^d, \) \((t, x) \in [0, \infty) \times \mathbb{T}^d\) with suitable regularity, say \( C^2 \) in \((t, x)\).

- **pressure** \( \Pi = \Pi_t(x) \in \mathbb{R}, \) \((t, x) \in [0, \infty) \times \mathbb{T}^d\) with suitable regularity, say \( C^1 \) in \((t, x)\).

Given an initial velocity \( u_0 : \mathbb{T}^d \to \mathbb{R}^d \),
\[
\text{div} u = 0, \tag{2.1}
\]
\[
\partial_t u + (u \cdot \nabla)u = -\nabla \Pi + \nu \Delta u + F, \tag{2.2}
\]

where \( \nu > 0 \) is a constant, called *kinematic viscosity* and \( F = F_t(x), \) \((t, x) \in [0, \infty) \times \mathbb{T}^d\) is a given external force. Physical interpretation of (2.1) and (2.2) were explained in section 1.

2.1 A weak formulation

Let \( \mathcal{V} \) be the set of \( \mathbb{R}^d \)-valued divergence free, mean-zero trigonometric polynomials, i.e., the set of \( v : \mathbb{T}^d \to \mathbb{R}^d \) of the following form:
\[
v(x) = \sum_{z \in \mathbb{Z}^d} \bar{v}_z \psi_z(x), \quad x \in \mathbb{T}^d, \tag{2.3}
\]
where $\psi_z(x) = \exp(2\pi iz \cdot x)$ and the coefficients $\hat{v}_z \in \mathbb{R}^d$ satisfy
\begin{align*}
\hat{v}_z &= 0 \text{ for } z = 0 \text{ and except for finitely many } z \neq 0, \quad (2.4) \\
\overline{\hat{v}_z} &= \hat{v}_{-z} \text{ for all } z, \quad (2.5) \\
\hat{v}_z \cdot \psi_z &= 0 \text{ for all } z. \quad (2.6)
\end{align*}

Note that (2.6) implies that: \[ \text{div} v = 0 \text{ for all } v \in \mathcal{V}. \]

We equip the torus $\mathbb{T}^d$ with the Lebesgue measure and denote by $\|f\|_p$ the usual $L_p$-norm of $f \in L_p(\mathbb{T}^d)$. For $\alpha \in \mathbb{R}$ and $v \in \mathcal{V}$ we define
\[ (1 - \Delta)^{\alpha/2} v = \sum_{z \in \mathbb{Z}^d} (1 + 4\pi^2|z|^2)^{\alpha/2} \hat{v}_z \psi_z. \]

We then introduce:
\[ V_{2,\alpha} = \text{the completion of } \mathcal{V} \text{ with respect to the norm } \| \cdot \|_{2,\alpha}, \quad \alpha \in \mathbb{R}, \quad (2.7) \]
where
\[ \|v\|^2_{2,\alpha} = \int_{\mathbb{T}^d} |(1 - \Delta)^{\alpha/2} v|^2 = \sum_{z \in \mathbb{Z}^d} (1 + 4\pi^2|z|^2)^{\alpha} |\hat{v}_z|^2. \quad (2.8) \]

Here are some basic properties of the space $V_{2,\alpha}$:

- Any $v \in V_{2,\alpha}$ is identified with a summation of the form (2.3) with (2.4) replaced by the condition that the last summation in (2.8) converges.

- $V_{2,-\alpha}$ is identified with the set of continuous linear functional on $V_{2,\alpha}$.

\[ V_{2,\alpha+\beta} \hookrightarrow V_{2,\alpha}, \quad \text{for } \alpha \in \mathbb{R} \text{ and } \beta > 0. \quad (2.9) \]

cf. Definition 2.1.1 and Exercise 2.1.1 below.

**Definition 2.1.1** Let $E_0, E_1$ be normed vector spaces.

- $E_0 \hookrightarrow E_1$ means that $E_0$ is continuously imbeded into $E_1$, i.e., $E_0 \subset E_1$ with the inclusion map being continuous.

- $E_0 \hookrightarrow \hookrightarrow E_1$ means that $E_0$ is compactly imbeded into $E_1$, i.e., $E_0 \subset E_1$ with the inclusion map being a compact operator.

**Exercise 2.1.1** Recall that any $v \in V_{2,\alpha}$ is identified with a summation of the form (2.3) with (2.4) replaced by the condition that the last summation in (2.8) converges. Let $\alpha \in \mathbb{R}$, $\beta > 0$ and $v \in V_{2,\alpha+\beta}$. Prove that
\[ \|v - I_n v\|_{2,\alpha} \leq (1 + 4\pi^2n^2)^{-\beta/2} \|v\|_{2,\alpha+\beta}, \quad \text{where} \quad I_n v = \sum_{|z| \leq n} \hat{v}_z \psi_z. \]

Then, conclude (2.9) from this.

**Exercise 2.1.2** Prove the following interpolation inequality:
\[ \|v\|_{2,\alpha+(1-\theta)\beta} \leq \|v\|_{2,\alpha}^{\theta} \|v\|_{2,\beta}^{1-\theta} \text{ for } \alpha, \beta \in \mathbb{R} \text{ and } \theta \in [0,1]. \quad (2.10) \]
For $v, w : T^d \to \mathbb{R}^d$, with $w$ supposed to be differentiable (for a moment), we define a vector field:

$$(v \cdot \nabla)w = \sum_{i=1}^{d} v_i \partial_i w,$$

(2.11)

which is bilinear in $(v, w)$. Later on, we will generalize the definition of the above vector field (cf. (2.18)).

**Lemma 2.1.2** For $v \in \mathcal{V}$, $w, \varphi \in C^1(T^d \to \mathbb{R}^d)$,

$$\langle \varphi, (v \cdot \nabla)w \rangle = -\langle w, (v \cdot \nabla)\varphi \rangle,$$

(2.12)

In particular, $\langle w, (v \cdot \nabla)w \rangle = 0$.

Proof: Since $\text{div} v = 0$, we have that

1) $\sum_j \partial_j (\varphi_i v_j) = \sum_j (\partial_j \varphi_i) v_j + \varphi_i \sum_j \partial_j v_j$.

Therefore,

$$\text{LHS (2.12)} = \sum_{i,j} \langle \varphi_i, v_j \partial_j w_i \rangle = -\sum_{i,j} \langle \partial_j (\varphi_i v_j), w_i \rangle \overset{1)}{=} -\sum_{i,j} \langle (\partial_j \varphi_i) v_j, w_i \rangle = \text{RHS (2.12)}.$$ 

\[\square\]

Suppose that $u, \Pi, F$ in (NS) ((2.1)–(2.2)) have suitable regularity. Then, for a test function $\varphi \in \mathcal{V}$,

*$$\partial_t \langle \varphi, u \rangle = -\langle \varphi, (u \cdot \nabla)u \rangle + \nu \langle \varphi, \Delta u \rangle - \langle \varphi, \nabla \Pi \rangle + \langle \varphi, F \rangle.$$*

(2.12)

Thus, *) becomes

$$\partial_t \langle \varphi, u \rangle = \langle u, (u \cdot \nabla)\varphi \rangle + \nu\langle \Delta \varphi, u \rangle + \langle \varphi, F \rangle.$$ 

By integration, we arrive at:

$$\langle \varphi, u_t \rangle = \langle \varphi, u_0 \rangle + \int_0^t \left( \langle u_s, (u_s \cdot \nabla)\varphi \rangle + \nu\langle \Delta \varphi, u_s \rangle + \langle \varphi, F_s \rangle \right) ds.$$ 

(2.13)

This is a standard weak formulation of (NS) ((2.1)–(2.2)).
2.2 Bounds on the non-linear term

Lemma 2.2.1 Suppose $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with at least two of them being non-zero, and that $\alpha_1 + \alpha_2 + \alpha_3 \geq \frac{d}{2}$. Then, there exists $C \in (0, \infty)$ such that:

$$|\langle w, (v \cdot \nabla)\varphi \rangle| \leq C \Vert v \Vert_{2,\alpha_1} \Vert w \Vert_{2,\alpha_2} \Vert \varphi \Vert_{2,1+\alpha_3},$$

(2.14)

for $v, w, \varphi \in C^\infty(T^d \to \mathbb{R}^d)$.

Proof: Since the norm $\Vert \cdot \Vert_{2,\alpha}$ is increasing in $\alpha$, it is enough to prove (2.16) with $\alpha_i$ replaced by $\tilde{\alpha}_i = \frac{(d/2)\alpha_i}{\alpha_1 + \alpha_2 + \alpha_3}$. Therefore, we may assume without loss of generality that

$$(\alpha_1, \alpha_2, \alpha_3) \in [0, \frac{d}{2})^3 \text{ and } \alpha_1 + \alpha_2 + \alpha_3 = \frac{d}{2}.$$ 

Let $q_i \in [2, \infty)$, $i=1,2,3$ be defined by $\frac{1}{q_i} = \frac{1}{2} - \frac{\alpha}{d} > 0$. Since

$$\sum_{i,j} \vert w_i v_j \partial_j \varphi_i \vert \leq \vert w \vert \vert v \vert \vert \nabla \varphi \vert,$$

we have

$$\left| \langle w, (v \cdot \nabla)\varphi \rangle \right| \leq \left\| v \right\|_{q_1} \left\| w \right\|_{q_2} \left\| \nabla \varphi \right\|_{q_3}.$$ 

We then use the following Sobolev imbedding theorem (e.g. [Ta96, p.4, (2.11)]):

$$V_{2,\alpha} \hookrightarrow L_q(T^d \to \mathbb{R}^d), \text{ if } \frac{1}{q} = \frac{1}{2} - \frac{\alpha}{d} \text{ def} > 0.$$ 

(2.15)\hfill\square

We have the following variant of Lemma 2.2.1, which is applicable even when $\alpha_2 = \alpha_3 = 0$:

Lemma 2.2.2 Let $\alpha_1, \alpha_2, \alpha_3 \geq 0$ such that $\alpha_1 + \alpha_2 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 \geq \frac{d}{2}$. Then, there exists $C \in (0, \infty)$ such that:

$$|\langle w, (v \cdot \nabla)\varphi \rangle| \leq C \Vert \varphi \Vert_{2,1+\alpha_3} \sqrt{\Vert v \Vert_{2,\alpha_1} \Vert v \Vert_{2,\alpha_2} \Vert w \Vert_{2,\alpha_1} \Vert w \Vert_{2,\alpha_2}},$$

(2.16)

for $v, w, \varphi \in C^\infty(T^d \to \mathbb{R}^d)$.

Proof: Note that

1) $\left\| u \right\|_{2,\alpha_1+\alpha_2} \leq \sqrt{\left\| u \right\|_{2,\alpha_1} \left\| u \right\|_{2,\alpha_2}} \text{ for } u \in V_{2,\alpha_1} \cap V_{2,\alpha_2}.$

On the other hand, by (2.14) with $\left( \frac{\alpha_1+\alpha_2}{2}, \frac{\alpha_1+\alpha_2}{2}, \alpha_3 \right)$ in place of $(\alpha_1, \alpha_2, \alpha_3)$, we have

$$\left| \langle w, (v \cdot \nabla)\varphi \rangle \right| \leq C \left\| v \right\|_{2,\alpha_1+\alpha_2} \left\| w \right\|_{2,\alpha_1+\alpha_2} \left\| \varphi \right\|_{2,1+\alpha_3} \text{ for } \left( \frac{\alpha_1+\alpha_2}{2} \right) \leq \text{RHS} \ (2.16).$$ 

\hfill\square

Remark: (2.16) gives a generalization of [Te79, p.292, Lemma 3.4]

Let

$$\alpha_1, \alpha_2 \geq 0, \ \alpha_1 + \alpha_2 > 0, \text{ and } \alpha_3 \defeq \left( \frac{d}{2} - \alpha_1 - \alpha_2 \right)^+. \quad (2.17)$$
Then, $\alpha_i$'s ($i = 1, 2, 3$) satisfy conditions for Lemma 2.2.2. Let also $v, w \in V_{2,\alpha_1 \wedge \alpha_2}$. In view of (2.12), we think of $(v \cdot \nabla)w$ as the following linear functional on $\mathcal{V}$:

$$\varphi \mapsto \langle \varphi, (v \cdot \nabla)w \rangle^{def} = -\langle w, (v \cdot \nabla)\varphi \rangle,$$

which, by (2.16), extends continuously on $V_{2,1+\alpha_3}$. This way, we regard

$$(v \cdot \nabla)w \in V_{2,-1-\alpha_3},$$

with

$$\| (v \cdot \nabla)w \|_{2,-1-\alpha_3} \leq C \sqrt{\| v \|_{2,\alpha_1} \| v \|_{2,\alpha_2} \| w \|_{2,\alpha_1} \| w \|_{2,\alpha_2}}. \quad (2.18)$$

Let us consider the case $v = w$ and $\alpha_1 \geq \alpha_2$ (Although $v$ and $w$ are identical, it is convenient to take $\alpha_1 > \alpha_2$, as we will see later on). Note that:

$$\Delta v \in V_{2,\alpha_1-2} \text{ with } \| \Delta v \|_{2,\alpha_1-2} \leq \| v \|_{2,\alpha_1},$$

By this and (2.18), we have that:

$$b(v) = \nu \Delta v - (v \cdot \nabla)v \in V_{2,-\beta(\alpha_1, \alpha_2)},$$

with

$$\| b(v) \|_{2,-\beta(\alpha_1, \alpha_2)} \leq \nu \| v \|_{2,\alpha_1} + C \| v \|_{2,\alpha_1} \| v \|_{2,\alpha_2}, \quad (2.19)$$

where

$$\beta(\alpha_1, \alpha_2) = (1 + (\frac{d}{2} - \alpha_1 - \alpha_2)^+) \vee (2 - \alpha_1). \quad (2.20)$$

With this notation, (2.13) takes the form:

$$\langle \varphi, u_t \rangle = \langle \varphi, u_0 \rangle + \int_0^t \langle \varphi, b(u_s) \rangle ds + \int_0^t \langle \varphi, F_s \rangle ds.$$

i.e.,

$$u_t = u_0 + \int_0^t b(u_s)ds + \int_0^t F_sds \quad (2.21)$$

as linear functionals on $\mathcal{V}$.

**Lemma 2.2.3** Let $\alpha_1 > 0$ and $\alpha_1 \geq \alpha_2 \geq 0$ for which $\beta(\alpha_1, \alpha_2)$ is defined by (2.20). Then, there exists $C \in (0, \infty)$ such that:

$$\int_0^T \| b(v_t) \|^q_{2,-\beta(\alpha_1, \alpha_2)} dt \leq \int_0^T (\nu + C \| v_t \|_{2,\alpha_2})^q \| v_t \|_{2,\alpha_1}^q dt \quad (2.22)$$

for any measurable $v : [0, T] \to V_{2,\alpha_1}$ and $q \in [1, \infty)$. Moreover, for $\alpha > 0$, the following map is continuous:

$$v \mapsto \int_0^t b(v_s)ds; \quad L_2([0, T] \to V_{2,\alpha}) \to C([0, T] \to V_{2,-\beta(\alpha, \alpha)}).$$

**Proof:** (2.22) is a direct consequence of (2.19). For the rest of this proof, we write $\beta = \beta(\alpha, \alpha)$ for simplicity. Let $v, w \in L_2([0, T] \to V_{2,\alpha})$. Then,

$$\sup_{0 \leq t \leq T} \left\| \int_0^t (b(v_s) - b(w_s))ds \right\|_{2,-\beta} \leq \int_0^T \| b(v_s) - b(w_s) \|_{2,-\beta} ds.$$
On the other hand, for $\varphi \in V_{2,\beta}$,

$$
\langle \varphi, b(v_s) - b(w_s) \rangle \overset{(2.19)}{=} \nu \left( \Delta \varphi, v_s - w_s \right) - \left( v_s, (v_s \cdot \nabla) \varphi \right) + \left( w_s, (w_s \cdot \nabla) \varphi \right),
$$

$$
|(2)| \leq \|\varphi\|_{2,2-\alpha} \|v_s - w_s\|_{2,\alpha},
$$

$$
|(3)| \leq |\langle v_s - w_s, (v_s \cdot \nabla) \varphi \rangle| + |\langle w_s, ((v_s - w_s) \cdot \nabla) \varphi \rangle| \overset{(2.14)}{\leq} C \|v_s - w_s\|_{2,\alpha} \|v_s\|_{2,\alpha} \|\varphi\|_{2,\beta} + C \|v_s - w_s\|_{2,\alpha} \|w_s\|_{2,\alpha} \|\varphi\|_{2,\beta},
$$

which implies that:

$$
\|b(v_s) - b(w_s)\|_{2,-\beta} \leq (\nu + C \|v_s\|_{2,\alpha} + C \|w_s\|_{2,\alpha}) \|v_s - w_s\|_{2,\alpha}.
$$

Plugging this into (1), we arrive at:

$$
\sup_{0 \leq t \leq T} \left\| \int_0^t (b(v_s) - b(w_s)) ds \right\|_{2,-\beta} \leq \sqrt{3} \left( \int_0^T (\nu^2 + C^2 \|v_s\|_{2,\alpha}^2 + C^2 \|w_s\|_{2,\alpha}^2) ds \right)^{1/2} \left( \int_0^T \|v_s - w_s\|_{2,\alpha}^2 ds \right)^{1/2},
$$

which implies the desired continuity.

\[\square\]

**Remark:** By (2.22) for $q = 1$ and $(\alpha_1, \alpha_2) = (1, 1)$, we see that

$$
v \in L_2([0, T] \rightarrow V_{2,1}) \implies b(v) \in L_1([0, T] \rightarrow V_{2,-\beta(1,1)})
$$

(2.23)

On the other hand, by (2.22) for $q = 2$ and $(\alpha_1, \alpha_2) = (1, 0)$, we see that

$$
v \in L_2([0, T] \rightarrow V_{2,1}) \cap L_\infty([0, T] \rightarrow V_{2,0}) \implies b(v) \in L_2([0, T] \rightarrow V_{2,-\beta(1,0)})
$$

(2.24)

Note also that:

$$
\beta(1,1) = \begin{cases} 
1 & \text{if } d \leq 4, \\
\frac{d}{2} - 1 & \text{if } d \geq 5
\end{cases} \quad \beta(1,0) = \begin{cases} 
1 & \text{if } d = 2, \\
\frac{d}{2} & \text{if } d \geq 3.
\end{cases}
$$

(2.25)

### 3 The stochastic Navier-Stokes equation

The construction of a weak solution to the Navier-Stokes equation (2.1)–(2.2) goes back to classical results by J. Leray [Le33, Le34a, Le34b] and E. Hopf [Ho50]. Here, following [Fl08], we consider the case in which the external force is given by a colored noise.

#### 3.1 Introduction of the noise

Throughout this subsection, let $H$ be a separable Hilbert space, and $\Gamma : H \rightarrow H$ be a bounded self-adjoint, non-negative definite operator. We suppose in addition that $\Gamma$ is of **trace class**, that is, the following summation converges for any CONS $\{\varphi_n\}_{n \geq 1}$ of $H$:

$$
\operatorname{tr}(\Gamma) \overset{\text{def}}{=} \sum_{n \geq 1} \langle \varphi_n, \Gamma \varphi_n \rangle.
$$

(3.1)

The number defined above is called the **trace** of $\Gamma$ and is independent of the choice of the CONS [RS72, p.206, Theorem VI.18].
Definition 3.1.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(a) A r.v. $B = (B_t)_{t \geq 0}$ with values in $C([0, \infty) \to \mathbb{R}^d)$ is called a **standard $d$-dimensional Brownian motion** (abbreviated by $BM^d$ below) if, for each $\theta \in \mathbb{R}^d$ and $0 \leq s < t$,

$$E \left[ \exp \left( i \theta \cdot (B_t - B_s) \right) \right] = \exp \left( -\frac{t-s}{2} |\theta|^2 \right) \text{, a.s.}$$

(3.2)

where $\mathcal{G}_s^B$ denotes the $\sigma$-field generated by $(B_u)_{u \leq s}$. (cf. the complement at the end of this subsection for a definition of the conditional expectation.)

(b) A r.v. $W = (W_t)_{t \geq 0}$ with values in $C([0, \infty) \to H)$ is called a $H$-valued Brownian motion with the covariance operator $\Gamma$ (abbreviated by $BM(H, \Gamma)$ below) if, for each $\varphi \in H$ and $0 \leq s < t$,

$$E \left[ \exp \left( i \varphi \cdot (W_t - W_s) \right) \right] = \exp \left( -\frac{t-s}{2} \langle \varphi, \Gamma \varphi \rangle \right), \text{ a.s.}$$

(3.3)

where $\mathcal{G}_s^W$ denotes the $\sigma$-field generated by $(W_u)_{u \leq s}$.

**Remark:** The distributional time derivative $\partial_t W_t$ of a $BM(H, \Gamma)$ is sometimes called the colored noise.

**Exercise 3.1.1** Let $W_t$ be as in Definition 3.1.1 b) and $H_0 \subset H$ be a $d$-dimensional subspace of $H$ such that $\Gamma H_0 \subset H_0$ with the orthogonal projection $\pi_0$. Then, conclude from (3.3) that

$$(\pi_0 W_t)_{t \geq 0} \text{ and } (\sigma B_t)_{t \geq 0} \text{ have the same law,}$$

where $(B_t)_{t \geq 0}$ is $BM^d$ on $H_0$ (identified with $\mathbb{R}^d$) and $\sigma : H_0 \to H_0$ is a square root of $\Gamma|_{H_0}$. In particular, for each $\varphi \in H$, the process $\langle \varphi, W_t \rangle$, $t \geq 0$ is of the following form:

$$\langle \varphi, W_t \rangle = \sqrt{\langle \varphi, \Gamma \varphi \rangle} B_t, \ t \geq 0,$$

where $B_t$ is a BM$^1$.

**Complement:** Let $X \in L_1(P)$ and $\mathcal{G}$ be a sub $\sigma$-field of $\mathcal{F}$. We define the **conditional expectation** $E[X \mid \mathcal{G}]$ of $X$, given $\mathcal{G}$. An implicit definition is given by declaring that $Y = E[X \mid \mathcal{G}]$ is the unique $\mathcal{G}$-measurable r.v. in $L^1(P)$ such that:

1) $E[Y \cdot 1_G] = E[X \cdot 1_G]$ for any $G \in \mathcal{G}$.

Another definition is given by explicitly writing down $E[X \mid \mathcal{G}]$ as a certain Radon Nikodym derivative, which proves that the r.v. $Y$ as referred to above does exist. To do so, we introduce the following signed measure:

$$E^X(F) \overset{def}{=} E[X \cdot 1_F], \ F \in \mathcal{F}.$$ 

Since $E^X|_{\mathcal{G}}$ is absolutely continuous with respect to $P|_{\mathcal{G}}$, we can define:

$$E[X \mid \mathcal{G}] = \frac{dE^X|_{\mathcal{G}}}{dP|_{\mathcal{G}}},$$

where the RHS stands for the Radon Nikodym derivative. Then, it is clear that $Y = E[X \mid \mathcal{G}]$ satisfies 1).
Let us relate the above abstract definition with the elementary conditional expectation of $X \in L_1(P)$, given an event $A \in \mathcal{F}$ with $0 < P(A) < 1$:

$$E[X|A] = \frac{E[X1_A]}{P(A)}.$$ 

For the $\sigma$-field $\mathcal{G} = \{A, A^c, \emptyset, \Omega\}$, it is clear that

$$E[X|\mathcal{G}] = E[X|A]1_A + E[X|A^c]1_{A^c}.$$ 

3.2 The existence theorem for the stochastic Navier-Stokes equation

We recall (2.19)–(2.21).

**Theorem 3.2.1** Let

- $\Gamma : V_{2,0} \rightarrow V_{2,0}$ be a self-adjoint, non-negative definite operator of trace class, $\Delta \Gamma = \Gamma \Delta$ and;
- $\mu_0$ be a Borel probability measure on $V_{2,0}$ such that $m_0 \overset{\text{def}}{=} \int \|v\|^2_2 d\mu_0(v) < \infty$.

Then, there exist a process $(X, Y) = ((X_t, Y_t))_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, P)$, where

- $X = (X_t)_{t \geq 0}$ takes values in
  $$L_{2,loc}([0, \infty) \rightarrow V_{2,1}) \cap L_{\infty,1oc}([0, \infty) \rightarrow V_{2,0}) \cap C([0, \infty) \rightarrow V_{2,-\beta(1,1)}),$$  
  (3.4)

  with $\beta(1,1) = 1$ for $d \leq 4$ and $\beta(1,1) = \frac{d}{2} - 1$ for $d \geq 5$. cf. (2.25);

- $Y = (Y_t)_{t \geq 0}$ is a $BM(V_{2,0}, \Gamma)$ (cf. Definition 3.1.1).

The couple $(X, Y)$ is a weak solution to the Navier-Stokes equation with the initial law $\mu_0$ in the sense that:

$$P(X_0 \in \cdot) = \mu_0;$$  

(3.5)

$$Y_{t+} - Y_t \text{ and } \{\langle \varphi, X_s \rangle ; s \leq t, \varphi \in \mathcal{V}\} \text{ are independent for any } t \geq 0;$$  

(3.6)

$$\langle \varphi, X_t \rangle = \langle \varphi, X_0 \rangle + \int_0^t \langle \varphi, b(X_s) \rangle ds + \langle \varphi, Y_t \rangle, \text{ for all } \varphi \in \mathcal{V} \text{ and } t \geq 0. \quad (3.7)$$

Moreover, the following a priori bounds hold true: for any $T > 0$,

$$E\left[\|X_T\|^2_2 + 2\nu\int_0^T \|X_t\|^2_{2,1}dt\right] \leq m_0 + \text{tr}(\Gamma)T, \quad (3.8)$$

$$E\left[\sup_{t \leq T} \|X_t\|^2_2\right] \leq (1 + T)C < \infty, \quad (3.9)$$

with $C \in (0, \infty)$ depending only on $\text{tr}(\Gamma)$, and $m_0$.

**Remark:** 1) The integral $\int_0^t \langle \varphi, b(X_s) \rangle ds$ in (3.7) is well defined because of (2.23) (or (2.24)) and (3.4).

2) The bound (3.8) is sometimes referred to as the *energy balance inequality*. The interpretation is that

$$\frac{1}{2}\|X_T\|^2_2 = \text{the kinetic energy},$$

$$\nu\int_0^T \|X_t\|^2_{2,1}dt = \text{the energy dissipated by the friction},$$

$$\frac{1}{2}\text{tr}(\Gamma)T = \text{the energy injected from outside (by the colored noise)}.\quad (3.10)$$
Although the validity of the equality is not known in general, the equality does hold at the level of finite dimensional approximation (see (5.10) below).

**Theorem 3.2.2** For $d = 2$, the weak solution in Theorem 3.2.1 is pathwise unique in the sense: if $(X,Y)$ and $(\tilde{X},Y)$ are two solutions on a common probability space $(\Omega, \mathcal{F}, P)$ with a common $BM(V_{2,0}, \Gamma)$ $Y$ such that $X_0 = \tilde{X}_0$ a.s., then,

$$P(X_t = \tilde{X}_t \text{ for all } t \geq 0) = 1.$$ 

### 4 The Itô theory for beginners

In this section, we will explain elements in Itô’s stochastic calculus without going much into proofs. In what follows, $(\Omega, \mathcal{F}, P)$ is a probability space and $B = (B_t)_{t \geq 0}$ denotes a $BM^r$.

#### 4.1 Stochastic integrals with respect to the Brownian motion

We fix some notation and terminology:

- A family $X = (X_t)_{t \geq 0}$ of r.v.’s indexed by $t \geq 0$ (most commonly interpreted as “time”) is called a process. A process $X$ is said to be continuous if $t \mapsto X_t$ is continuous a.s.

- Let $(\mathcal{F}_t)_{t \geq 0}$ be a family of sub $\sigma$-fields which are increasing in $t \geq 0$, as such a filtration. We assume that it is right-continuous in the sense that:

$$\bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} = \mathcal{F}_t, \ t \geq 0. \quad (4.1)$$

- In general, a process $X = (X_t)_{t \geq 0}$ is said to be $(\mathcal{F}_t)$-adapted, if $X_t$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

- We assume that $B = (B_t)_{t \geq 0}$ is a $BM^r$ with respect to $(\mathcal{F}_t)$, that is, $B$ is $(\mathcal{F}_t)$-adapted and

$$E[\exp(i\theta \cdot (B_t - B_s)) | \mathcal{F}_s] = \exp\left(-\frac{t-s}{2} |\theta|^2\right), \ \text{a.s.} \quad (4.2)$$

for each $\theta \in \mathbb{R}^r$ and $0 \leq s < t$. We also assume that

$$\mathcal{N}^B \subset \mathcal{F}_t, \ t \geq 0, \quad (4.3)$$

where $\mathcal{N}^B$ is the null-set with respect to $B$ define as follows:

$$\mathcal{G}^B_t = \sigma(B_s, s \leq t), \ 0 \leq t < \infty, \ G^B_{\infty} = \sigma(\cup_{t \geq 0} \mathcal{G}^B_t),$$

$$\mathcal{N}^B = \{N \subset \Omega; \ \exists \tilde{N} \in \mathcal{G}^B_{\infty}, N \subset \tilde{N}, P(\tilde{N}) = 0\},$$

An example of such $(\mathcal{F}_t)_{t \geq 0}$ is given by the argumented filtration defined by:

$$\mathcal{F}_t = \sigma(\mathcal{G}^B_t \cup \mathcal{N}^B), \ 0 \leq t < \infty. \quad (4.4)$$

See [KS91,pp.90–91] for the proof the properties (4.1)–(4.2) of the argumented filtration. On the other hand, $\mathcal{G}^B_t$ is not right-continuous [KS91,p.89, Problem 7.1].
Definition 4.1.1 (Stopping times) A r.v. \( \tau : \Omega \to [0, \infty] \) is called a stopping time if

\[ \{ \tau \leq t \} \in \mathcal{F}_t \text{ for all } t \geq 0. \tag{4.5} \]

Example 4.1.2 Let \( \Gamma \subset \mathbb{R}^r \) and define

\[ \tau(\Gamma) = \inf\{ t > 0 ; B_t \in \Gamma \}. \]

It is known that \( \tau(\Gamma) \) is a stopping time if \( \Gamma \subset \mathbb{R}^r \) is a Borel set. This is not difficult to prove if \( \Gamma \) is either open or closed. Here, in the proof, one sees how the right continuity of \( \mathcal{F}_t \) is used.

Consider the following condition\(^2\) for a r.v. \( \tau : \Omega \to [0, \infty] \):

\[ \{ \tau < t \} \in \mathcal{F}_t \text{ for all } t \geq 0. \tag{4.6} \]

Then, this is equivalent to (4.5). In fact, we have

1) \[ \{ \tau < t \} = \bigcup_{n \geq 1} \{ \tau \leq t - \frac{1}{n} \}, \]

2) \[ \{ \tau > t \} = \cap_{m \geq 1} \bigcup_{n \geq m} \{ \tau \geq t - \frac{1}{n} \}. \]

We see from 1) that (4.5) implies (4.6), while the converse can be seen from 2) and the right continuity of \( \mathcal{F}_t \).

The observation above can be used to prove that \( \tau(\Gamma) \) defined in Example 4.1.2 is a stopping time for an open set \( \Gamma \). We prove that \( \tau(\Gamma) \) satisfies (4.6) as follows:

\[ \{ \tau(\Gamma) < t\} = \bigcup_{s \in (0,t)} \{ B_s \in \Gamma \} = \bigcup_{s \in \mathbb{Q} \cap (0,t)} \{ B_s \in \Gamma \} \in \mathcal{F}_t, \]

where, to get the second equality, we have used that \( \Gamma \) is open and that \( s \mapsto B_s \) is continuous.

Exercise 4.1.1 Prove that \( \tau(\Gamma) \) defined in Example 4.1.2 is a stopping time if \( \Gamma \) is closed. Hint: There is a sequence of open sets \( G_1 \supset G_2 \supset \ldots \) such that \( \Gamma = \cap_{m \geq 1} G_m \).

We now define some classes of integrands for the stochastic integral.

Definition 4.1.3 (Integrands for stochastic integral) We define a function space \( \Phi \) as the totality of \( \varphi : [0, \infty) \times \Omega \to \mathbb{R} \((s, \omega) \mapsto \varphi_s(\omega)\) \) such that\(^3\):

\[ \varphi|_{[0,t] \times \Omega} \text{ is } \mathcal{B}([0, t]) \otimes \mathcal{F}_t \text{ measurable for all } t \geq 0. \]

We also define

\[ \Phi_2 = \{ \varphi \in \Phi ; \ E \int_0^t |\varphi_s|^2 ds < \infty \text{ for all } t > 0 \}, \tag{4.7} \]

\[ \Phi_2^{loc} = \{ \varphi \in \Phi ; \ \int_0^t |\varphi_s|^2 ds < \infty, \ P\text{-a.s. for all } t > 0 \}. \tag{4.8} \]

Clearly, \( \Phi_2 \subset \Phi_2^{loc} \subset \Phi \).

\(^2\)A r.v. \( \tau \) with this condition is called an optional time. We see from the argument of this remark that a stopping time is always an optional time, and that the converse is true when the filtration is right continuous.

\(^3\)This property is called progressive measurability.
Example 4.1.4 Let $g : \mathbb{R}^r \to \mathbb{R}$ be Borel measurable and
\[ \varphi_s(\omega) = g(B_s(\omega)). \]
Then,
\begin{itemize}
  \item If $g$ is bounded, then $\varphi \in \Phi_2$.
  \item If $\sup_K |g| < \infty$ for any bounded set $K \subset \mathbb{R}^r$ (in particular, if $g \in C(\mathbb{R}^r)$), then $\varphi \in \Phi_2^{\text{loc}}$.
\end{itemize}

Theorem 4.1.5 For $\varphi \in \Phi_2^{\text{loc}}$, there are continuous processes (called the stochastic integral with respect to the Brownian motion)
\[ \left( \int_0^t \varphi_s dB_s^i \right)_{t \geq 0}, \quad i = 1, \ldots, r \tag{4.9} \]
with the following properties;

a) If
\[ \varphi_s(\omega) = \xi_a(\omega)1_{(a,b]}(s) \tag{4.10} \]
where $0 \leq a < b$ and $\xi_a$ is a bounded, $\mathcal{F}_a$-measurable r.v., then
\[ \int_0^t \varphi_s dB_s^i = \xi_a(\omega)(B_{t \wedge b}^i - B_{t \wedge a}^i). \tag{4.11} \]

b) For $t \geq 0$, $\alpha, \beta \in \mathbb{R}$ and $\varphi, \psi \in \Phi_2^{\text{loc}}$,
\[ \int_0^t (\alpha \varphi_s + \beta \psi_s) dB_s^i = \alpha \int_0^t \varphi_s dB_s^i + \beta \int_0^t \psi_s dB_s^i, \tag{4.12} \]

c) If $\varphi, \psi \in \Phi_2$ and $t \geq 0$, then,
\[ E \left[ \left( \int_0^t \varphi_s dB_s^i \right) \left( \int_0^t \psi_s dB_s^j \right) \right] = \delta_{ij} \int_0^t \varphi_s \psi_s ds < \infty, \tag{4.13} \]
\[ E \left[ \int_0^t \varphi_u dB_u^i \bigg| \mathcal{F}_s \right] = \int_s^t \varphi_u dB_u^i \text{ whenever } 0 \leq s \leq t. \tag{4.14} \]

We now indicate how the construction of the integrals (4.9) goes (See [KS91, Section 3.2] for details).

**Step 1:** Let $\Phi_0$ be the set of linear combinations of r.v.'s of the form (4.10). We proceed as follows:

1) For $\varphi \in \Phi_0$, define the integral (4.9) by (4.11) and (4.12).

2) Properties (4.13)–(4.14) hold for $\varphi, \psi \in \Phi_0$ (not difficult to see).

**Step 2:** We define the integral (4.9) for $\varphi \in \Phi_2$. To do so, we note that $\Phi_2$ is a Fréchet space generated by the semi-norms:
\[ \left( E \int_0^T |\varphi_s|^2 ds \right)^{1/2}, \quad T = 1, 2, \ldots \]

We also introduce:
Definition 4.1.6 A process $M = (M_t)_{t \geq 0}$ is said to be a martingale, if:

\begin{align*}
(F_t)\text{-adapted, } M_t \in L_1(P) \text{ for all } t \geq 0; \\
E[M_t | F_s] = M_s \text{ whenever } 0 \leq s < t.
\end{align*}

(4.15)

A martingale $M$ is said to be square integrable, if $E[M_T^2] < \infty$ for all $T > 0$.

Let $\mathcal{M}_2$ = the set of continuous, square-integrable martingales.

Then, $\mathcal{M}_2$ is a a Fréchet space generated by the semi-norms:

$$E \left[ \sup_{s \leq T} M_s^2 \right]^{1/2}, \quad T = 1, 2, \ldots$$

(cf. (4.16) below). We define:

$$I(\varphi)_t = \int_0^t \varphi_s dB^i_s, \quad \varphi \in \Phi_0, \quad t \geq 0.$$

We make the following observations:

1) From what we saw in Step 1.2,

$$E[I(\varphi)_T^2] = E \int_0^T |\varphi_s|^2 ds, \quad I(\varphi) \in \mathcal{M}_2, \quad \text{for } \varphi \in \Phi_0$$

2) $\Phi_0$ is dense in $\Phi_2$ (cf. [IW89, p.46, Lemma 1.1]). Thus, by 1) above, $I$ extends uniquely to a uniformly continuous mapping $I : \Phi_2 \rightarrow \mathcal{M}_2$. This justifies the definition of the integral (4.9) for $\varphi \in \Phi_2$:

$$\int_0^t \varphi_s dB^i_s \overset{\text{def}}{=} I(\varphi)_t, \quad t \geq 0.$$

Properties (4.12)-(4.14) for $\varphi \in \Phi_2$ is then automatic from the construction.

Step 3: We define the integral (4.9) for $\varphi \in \Phi_2^{loc}$. For $\varphi \in \Phi_2^{loc}$, we consider

$$\tau^{(n)} = n \wedge \inf \left\{ t > 0; \int_0^t |\varphi_s|^2 ds \geq n \right\}$$

$$\varphi^{(n)}_s(\omega) = \varphi_s(\omega)1_{[0, \tau^{(n)}]}(s).$$

Then, $\tau^{(n)} \nearrow \infty$ and $\varphi^{(n)} \in \Phi_2$. We then define the integrals (4.9) by

$$\int_0^t \varphi_s dB^i_s = \int_0^t \varphi^{(n)}_s dB^i_s, \quad \text{for } t \leq \tau^{(n)}.$$

This finishes the construction.

Finally, we mention the following useful inequality:
Theorem 4.1.7 (Doob's $L^2$-maximal inequality) For a square-integrable martingale $M$,

$$E\left[\sup_{0\leq s\leq t} M_s^2\right] \leq 4E[M_t^2].$$

(4.16)

In particular, if $\varphi \in \Phi_2$, then

$$E\left[\sup_{0\leq s\leq t} \left|\int_0^s \varphi_u dB_u^i\right|^2\right] \leq 4E\int_0^t |\varphi_s|^2 ds.$$

(4.17)

For a proof, see e.g. [IW89, p.33, Theorem 6.10], [KS91, p.13, 3.8 Theorem].

4.2 Itô’s formula for semi-martingales

Definition 4.2.1 Let $(\mathcal{F}_t)$ be a right-continuous filtration and $B = (B_t)_{t\geq 0}$ be a BM with respect to $(\mathcal{F}_t)$ (cf. (4.1)-(4.3)).

An $\mathbb{R}^d$-valued process $X = (X_t)_{t\geq 0}$ is said to be a semi-martingale if it is of the following form:

$$X_t = X_0 + \int_0^t \sigma_s dB_s + \int_0^t b_s ds,$$

(4.18)

or more precisely,

$$X_t^i = X_0^i + \sum_{j=1}^r \int_0^t \sigma_s^{ij} dB_s^j + \int_0^t b_s^i ds, \quad i = 1, \ldots, d.$$

where

- $X_0$ is a $\mathcal{F}_0$-measurable r.v.;
- $\sigma = (\sigma^{ij})$ is a matrix with $\sigma^{ij} \in \Phi_2^{loc}$ (cf. (4.8));
- $b = (b_t)_{t\geq 0}$ is an $(\mathcal{F}_t)$-adapted process such that $t \mapsto b_t$ is continuous.

For the semi-martingale (4.18) and a process $(\varphi_t)_{t\geq 0}$, we define:

$$\int_0^t \varphi_s dX_s^i = \sum_{j=1}^r \int_0^t \varphi_s \sigma_s^{ij} dB_s^j + \int_0^t \varphi_s b_s^i ds, \quad i = 1, \ldots, d,$$

(4.19)

if each integral on the RHS is well defined, i.e.,

$$\varphi\sigma^{ij} \in \Phi_2^{loc} \quad \text{and} \quad \int_0^t |\varphi_s b_s^i| ds < \infty \text{ a.s. } i, j = 1, \ldots, d.$$

The integral (4.19) is called the stochastic integral with respect to the semi-martingale (4.18).

For a semi-martingale (4.18), we define the bracket processes by:

$$\langle X^i, X^j \rangle_t = \sum_{k=1}^r \int_0^t \sigma_s^{ik} \sigma_s^{jk} ds, \quad i, j = 1, \ldots, d.$$

(4.20)
Theorem 4.2.2 (Itô’s formula for semi-martingales) Suppose that $X$ is a semi-martingale given by (4.18) and $f \in C^2(\mathbb{R}^d)$. Then, $P$-a.s.,

$$f(X_t) - f(X_0) = \sum_{i=1}^{d} \int_{0}^{t} \partial_i f(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_{0}^{t} \partial_i \partial_j f(X_s) d\langle X^i, Y^j \rangle_s, \quad \text{for all } t \geq 0. \quad (4.21)$$

The proof goes along the following line (e.g. [IW89, pp.67-71], [KS91, pp.150-153]). Let $d = r = 1$ for simplicity, and $0 = t_0 < t_1 < \ldots < t_n = t$ be the division for which $\delta_n \overset{\text{def}}{=} \max_{1 \leq k \leq n}(t_k - t_{k-1}) \to 0 \ (n \to \infty)$. For the indices to be read easily, we write $\tilde{X}_k = X_{t_k}$. Then, by Taylor expanding $f$ around $\tilde{X}_{k-1}$, we have:

$$f(\tilde{X}_k) - f(\tilde{X}_{k-1}) = f'(\tilde{X}_{k-1}) \Delta_k + \frac{1}{2} f''(\tilde{X}_{k-1} + \theta_k \Delta_k) \Delta_k^2$$

where $\Delta_k = \tilde{X}_k - \tilde{X}_{k-1}$ and $\theta_k \in (0, 1)$. This implies that:

$$f(X_t) - f(X_0) = \sum_{k=1}^{n} f'(\tilde{X}_{k-1}) \Delta_k + \frac{1}{2} \sum_{k=1}^{n} f''(\tilde{X}_{k-1} + \theta_k \Delta_k) \Delta_k^2.$$  

By verifying

$$\lim_{n \to \infty} I_n = \int_{0}^{t} f'(X_s) dX_s \quad \text{and} \quad \lim_{n \to \infty} J_n = \int_{0}^{t} f''(X_s) d\langle X, X \rangle_s,$$

in an appropriate sense, one obtains (4.21) for $d = r = 1$. The extension to general $d, r$ is straightforward.

Example 4.2.3 For the semi-martingale (4.18), we have:

$$|X_t|^2 - |X_0|^2 = 2M_t + \int_{0}^{t} (2X_s \cdot b_s + |\sigma_s|^2) ds, \quad \text{with} \quad M_t = \sum_{1 \leq i \leq d} \int_{0}^{t} X_s^i \sigma_s^i dW_s^i. \quad (4.22)$$

Here, and in what follows, $|\sigma|^2 = \sum_{1 \leq i \leq d} \sum_{1 \leq j \leq r} (\sigma_s^{ij})^2$. Suppose in particular that

$$E[|X_0|^2] \leq m_0 < \infty, \quad X_t \cdot b_t \leq C, \quad |\sigma|^2 \leq C,$$

where $m_0$ and $C$ is a non-random constant. Then, for any $t > 0$,

$$E[|X_t|^2] = E[|X_0|^2] + E \int_{0}^{t} (2X_s \cdot b_s + |\sigma_s|^2) ds, \quad (4.24)$$

$$E \left[ \sup_{s \leq t} |X_s|^2 \right] \leq E[|X_0|^2] + C' t, \quad (4.25)$$

where the constant $C'$ depends only on $m_0$ and $C$. 

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Proof: Note that
\[ \partial_i|x|^2 = 2x^i, \quad \partial_i\partial_j|x|^2 = 2\delta_{i,j}. \]
Thus, we see from Itô's formula that:
\[
|X_t|^2 - |X_0|^2 = \sum_{j=1}^{d} \int_{0}^{t} 2X_j^i \cdot dX_j^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_{0}^{t} 2\delta_{i,j} d\langle X^i, X^j \rangle_s,
\]
with
\[
I = 2M_t + 2 \int_{0}^{t} X_s \cdot b(X_s) ds,
\]
\[
J = \sum_{1 \leq i \leq d} \langle X^i, X^i \rangle_t \quad \text{for} \quad \sum_{i,k=1}^{d} (\sigma_{s}^{ik})^2 ds.
\]
This proves (4.22). We next assume (4.23) to show (4.24)–(4.25). This will be straightforward, once we know that \( M \) is a square-integrable martingale. However, we have to settle this technical point first. We start by showing that:

1) \[ E[|X_t|^{2}] \leq m_0 + 3Ct, \]

Since \( X \) is continuous and \( |X_0| < \infty \) a.s., we have that:
\[ e_n \overset{\text{def}}{=} \inf \{ t \mid |X_t| \geq n \} \nearrow \infty, \quad \text{as} \quad n \nearrow \infty. \]

Note also that:
\[
M_{t \wedge e_n} = \sum_{\frac{i \leq s \leq r}{1 \leq j \leq d}} \int_{0}^{t \wedge e_n} X_s^i \sigma_{s}^{ij} dB_s^j = \sum_{\frac{i \leq s \leq r}{1 \leq j \leq d}} \int_{0}^{t} 1_{\{s \leq e_n \}} X_s^i \sigma_{s}^{ij} dB_s^j
\]
and that \( 1_{\{s \leq e_n \}} X_s^i \sigma_{s}^{ij} \in \Phi_2 \). These and (4.14) imply that \( E[M_{t \wedge e_n}] = 0 \). Combining this with:

2) \[ |X_t|^{2} \overset{(4.22),(4.23)}{\leq} |X_0|^{2} + 2M_t + 3Ct, \]
we have that:
\[ E[X_{t \wedge e_n}^2] \leq m_0 + 3Ct. \]

Thus, 1) follows from Fatou's lemma. 1) and (4.23) imply that:
\[ X_s^i \sigma_{s}^{ij} \in \Phi_2. \]

Then, \( E[M_t] = 0 \) by (4.14). Thus, (4.24) follows from (4.22) taking expectation. We next show that:

3) \[ E \left[ \sup_{s \leq t} |M_s|^2 \right] \leq C_1 (t + t^2). \]

To do so, we start by noting that:
4) \[ \sum_{j} \left( \sum_{i} X^{i}_{S} \sigma^{ij}_{s} \right)^{2} = |\sigma^{*}_{s} X_{S}|^{2} \leq |\sigma_{s}|^{2} |X_{S}|^{2}. \]

Then,

\[
E \left[ \sup_{s \leq t} |M_{s}|^{2} \right] \leq 4 E \left[ |M_{t}|^{2} \right] = 4 \sum_{j} E \int_{0}^{t} \left( \sum_{i} X^{i}_{S} \sigma^{ij}_{s} \right)^{2} ds \\
\leq 4E \int_{0}^{t} |\sigma_{s}|^{2} |X_{s}|^{2} ds \leq 4C(m_{0}t + \frac{3C}{2}t^{2}).
\]

we then get (4.22) as follows:

\[
E \left[ \sup_{s \leq t} |X_{s}|^{2} \right] \leq m_{0} + 2E \left[ \sup_{s \leq t} |M_{s}|^{2} \right]^{1/2} + 3Ct \leq m_{0} + C_{2}t.
\]

Example 4.2.4 (Itô's formula for the Brownian motion) Suppose that \( f \in C^{2}(\mathbb{R}^{r}) \). Then, \( P \)-a.s.,

\[
f(B_{t}) - f(0) = \sum_{1 \leq i \leq r} \int_{0}^{t} \partial_{i} f(B_{s}) dB_{s}^{i} + \frac{1}{2} \int_{0}^{t} \Delta f(B_{s}) ds, \quad \text{for all } t \geq 0. \quad (4.26)
\]

Proof: A special case of (4.21) with \( d = r, \sigma^{ij} = \delta^{ij}, \) and \( b \equiv 0. \)

\[ \square \]

4.3 Stochastic differential equations: an existence and uniqueness theorem

Let \( \sigma \in C(\mathbb{R}^{d} \rightarrow \mathbb{R}^{d} \otimes \mathbb{R}^{r}), \) \( b \in C(\mathbb{R}^{d} \rightarrow \mathbb{R}^{d}) \) and \( \xi \) be an \( \mathbb{R}^{d} \)-valued r.v. We consider a stochastic differential equation (SDE):

\[
X_{t} = \xi + \int_{0}^{t} \sigma(X_{s}) dB_{s} + \int_{0}^{t} b(X_{s}) ds, \quad \text{or more precisely,} \quad X_{t}^{i} = \xi^{i} + \sum_{j=1}^{r} \int_{0}^{t} \sigma^{ij}(X_{s}) dB_{s}^{j} + \int_{0}^{t} b^{i}(X_{s}), \quad i = 1, \ldots, d. \quad (4.27)
\]

We define:

\[
\mathcal{G}_{t}^{\xi,B} = \sigma(\xi, B_{s}, s \leq t), \quad 0 \leq t < \infty, \quad \mathcal{G}_{\infty}^{\xi,B} = \sigma \left( \bigcup_{t \geq 0} \mathcal{G}_{t}^{\xi,B} \right), \quad \mathcal{N}^{\xi,B} = \{ N \subset \Omega, \exists \tilde{N} \in \mathcal{G}_{\infty}^{\xi,B}, N \subset \tilde{N}, P(\tilde{N}) = 0 \}, \quad \mathcal{F}_{t}^{\xi,B} = \sigma(\mathcal{G}_{t}^{\xi,B} \cup \mathcal{N}^{\xi,B}), \quad 0 \leq t < \infty. \quad (4.28)
\]

We now state the following existence and uniqueness theorem:
Theorem 4.3.1 Referring to (4.27), suppose that

\[ m_0 \overset{\text{def}}{=} E[|\xi|^2] < \infty \]

and that there exist \( K, L_n \in (0, \infty), n = 1, 2, \ldots \) such that:

\[ |\sigma(x) - \sigma(y)|^2 + |b(x) - b(y)|^2 \leq L_n |x - y|^2 \quad \text{if } |x|, |y| \leq n, \tag{4.29} \]

\[ |\sigma(x)|^2 + 2x \cdot b(x) \leq K(1 + |x|^2), \quad x \in \mathbb{R}^d. \tag{4.30} \]

Then, there exists a unique process \( X \) such that:

a) \( X_t \) is \( \mathcal{F}_t^{\xi,B} \)-measurable for all \( t \geq 0 \) (cf. (4.28));

b) the SDE (4.27) is satisfied.

Proof: By [IW89, p.178, Theorem 3.1], the condition (4.29) ensures existence of the unique solution admitting the possibility of explosion at finite time:

\[ \lim_{t \nearrow \tau} |X_t| = \infty, \quad \text{for some } \tau < \infty. \]

However, such possibility is excluded by the condition (4.30) [IW89, p.177, Theorem 2.4]. \( \square \)

5 The Galerkin approximation

5.1 The approximating SDE

For each \( z \in \mathbb{Z}^d \setminus \{0\} \), let \( \{e_{z,j}\}_{j=1}^{d-1} \subset \mathbb{R}^d \) be an orthonormal basis of the hyperplane:

\[ \{x \in \mathbb{R}^d; z \cdot x = 0\} \]

and let:

\[ \psi_{z,j}(x) = \begin{cases} \sqrt{2}e_{z,j} \cos(2\pi z \cdot x), & j = 1, \ldots, d-1, \\ \sqrt{2}e_{z,|j|} \sin(2\pi z \cdot x), & j = -1, \ldots, -(d-1) \end{cases}, \quad x \in \mathbb{T}^d. \tag{5.1} \]

Then,

\[ \{\psi_{z,j}; z \in \mathbb{Z}^d \setminus \{0\}, j = \pm 1, \ldots, \pm(d-1)\} \]

is an orthonormal basis of \( V_{2,0} \). We also introduce:

\[ \mathcal{V}_n = \text{the linear span of } \{\psi_{z,j}; (z,j) \text{ with } z \in [-n,n]^d\}, \]

\[ \mathcal{P}_n = \text{the orthogonal projection} : L^2(\mathbb{T}^d \to \mathbb{R}^d) \to \mathcal{V}_n. \tag{5.2} \]

Using the orthonormal basis (5.1), we identify \( \mathcal{V}_n \) with \( \mathbb{R}^N \), \( N = \dim \mathcal{V}_n \). Let \( \mu_0 \) and \( \Gamma : V_{2,0} \to V_{2,0} \) be as in Theorem 3.2.1. Let also \( \xi \) be a r.v. such that \( P(\xi \in \cdot) = \mu_0 \). Finally, let \( W_t \) be a BM(\( V_0, \Gamma \)) defined on a probability space \( (\Omega, \mathcal{F}, P) \). Then, \( \mathcal{P}_nW_t \) is identified with an \( N \)-dimensional Brownian motion with covariance matrix \( \Gamma \mathcal{P}_n \). Then, we consider the following approximation of (3.7):

\[ X^n_t = X^n_0 + \int_0^t \mathcal{P}_n b(X^n_s)ds + \mathcal{P}_nW_t \quad t \geq 0, \tag{5.3} \]
where $X_0^n = \mathcal{P}_n \xi$. Let:

$$X_t^{n,z,j} = \langle \psi_{z,j}, X_t^n \rangle + \int_0^t b_t^{z,j}(X_s^n) \, ds + W_t^{z,j}, \tag{5.5}$$

where

$$b_t^{z,j}(v) = \langle v, (v \cdot \nabla) \psi_{z,j} \rangle + v \langle v, \Delta \psi_{z,j} \rangle, \quad v \in \mathcal{V}_n. \tag{5.6}$$

Let $\gamma_{z,j} \geq 0$ be such that $\Gamma \psi_{z,j} = \gamma_{z,j} \psi_{z,j}$ and $I_n = \{(z, j); |z| \leq n, \gamma_{z,j} > 0\}$. Then,

$$B_t^{z,j} = \frac{W_t^{z,j}}{\sqrt{\gamma_{z,j}}}, \quad (z, j) \in I_n$$

are independent BM's and

$$\mathcal{P}_n W_t = \sum_{(z,j) \in I_n} W_t^{z,j} \psi_{z,j} = \sum_{(z,j) \in I_n} \sqrt{\gamma_{z,j}} B_t^{z,j} \psi_{z,j}.$$ 

Thus, the SDE (5.3) can be thought of as a special case of (4.27), where

$$\sigma(\cdot) \text{ is a constant diagonal matrix with } |\sigma(\cdot)|^2 = \text{tr}(\Gamma \mathcal{P}_n). \tag{5.7}$$

Also by (5.6),

the drift $\mathcal{P}_n b(v)$ is a polynomial in $v \in \mathcal{V}_n$ of degree two. \hfill (5.8)

Moreover, for $v \in \mathcal{V}_n$,

$$\langle v, \mathcal{P}_n b(v) \rangle = \langle v, \nu \Delta v + (v \cdot \nabla) v \rangle \text{ Lemma } 2.1.2 \nu(\cdot, \Delta \cdot) = -\nu \Vert \nabla v \Vert_2^2 \leq 0. \tag{5.9}$$

We see from (5.7)–(5.9) above that the SDE (5.3) satisfies the assumptions (4.29)–(4.30) of Theorem 4.3.1, and hence admits a unique solution. The solution is then a semi-martingale of the form (4.18) for which the assumption (4.23) of Example 4.2.3 is valid. Therefore, for any $T > 0$,

$$E \left[ \left\| X_T^n \right\|_2^2 + 2\nu \int_0^T \left\| X_t^n \right\|_2^2 \, dt \right] = E[\left\| X_0^n \right\|_2^2] + \text{tr}(\Gamma \mathcal{P}_n)T, \tag{5.10}$$

$$E \left[ \sup_{t \leq T} \left\| X_t^n \right\|_2^2 \right] \leq (1 + T^2)C < \infty, \tag{5.11}$$

where $C = C(\Gamma, m_0) \in (0, \infty)$.

We will summarize the above considerations as Theorem 5.1.1 below. To do so, we define:

$$\mathcal{G}_t^\xi W = \sigma(\xi, W_s, \ s \leq t), \quad 0 \leq t < \infty, \quad \mathcal{G}_\infty^\xi W = \sigma \left( \bigcup_{t \geq 0} \mathcal{G}_t^\xi W \right),$$

$$\mathcal{N}_t^\xi W = \{ N \subset \Omega; \ \exists \tilde{N} \in \mathcal{G}_\infty^\xi W, \ N \subset \tilde{N}, \ P(\tilde{N}) = 0 \},$$

and

$$\mathcal{F}_t^\xi W = \sigma \left( \mathcal{G}_t^\xi W \cup \mathcal{N}_t^\xi W \right), \quad 0 \leq t < \infty. \tag{5.12}$$

**Theorem 5.1.1** Let $W, \xi$, and $\mathcal{F}_t^\xi W$ as above. Then, for each $n$, there exists a unique process $X^n$ such that:

a) $X_t^n$ is $\mathcal{F}_t^\xi W$-measurable for all $t \geq 0$;

b) (5.3), (5.10) and (5.11) are satisfied;
5.2 Compact imbedding lemmas

We will need some compact imbedding lemmas from [FG95]. We first introduce:

**Definition 5.2.1** Let $p \in [1, \infty), T \in (0, \infty)$, and $E$ be a Banach space.

a) We let $L_{p,1}([0, T] \rightarrow E)$ denote the Sobolev space of all $u \in L_p([0, T] \rightarrow E)$ such that:

$$u(t) = u(0) + \int_0^t u'(s) \, ds,$$

for almost all $t \in [0, T]$ with some $u(0) \in E$ and $u' \in L_p([0, T] \rightarrow E)$. We endow the space $L_{p,1}([0, T] \rightarrow E)$ with the norm $\|u\|_{L_{p,1}([0,T] \rightarrow E)}$ defined by

$$\|u\|_{L_{p,1}([0,T] \rightarrow E)}^p = \int_0^T (|u(t)|_E^p + |u'(t)|_E^p) \, dt.$$

b) For $\alpha \in (0, 1)$, we let $L_{p,\alpha}([0, T] \rightarrow E)$ denote the Sobolev space of all $u \in L_p([0, T] \rightarrow E)$ such that:

$$\int_{0<s<t<T} \frac{|u(t) - u(s)|_E^p}{|t-s|^{1+\alpha p}} \, ds \, dt < \infty.$$

We endow the space $L_{p,\alpha}([0, T] \rightarrow E)$ with the norm $\|u\|_{L_{p,\alpha}([0,T] \rightarrow E)}$ defined by

$$\|u\|_{L_{p,\alpha}([0,T] \rightarrow E)}^p = \int_0^T |u(t)|^p dt + \int_{0<s<t<T} \frac{|u(t) - u(s)|_E^p}{|t-s|^{1+\alpha p}} \, ds \, dt.$$

**Remark:** Note that:

$$\int_{0<s<t<T} \frac{ds \, dt}{|t-s|^{1+\lambda}} = \begin{cases} \infty & \text{if } \lambda \geq 0, \\ \frac{T^{1+|\lambda|}}{(1+|\lambda|)|\lambda|} & \text{if } \lambda < 0 \end{cases} \quad (5.13)$$

Therefore, roughly speaking, a function in $L_{p,\alpha}([0, T] \rightarrow E)$ is, "Hölder continuous with the exponent bigger than $\alpha"."

**Exercise 5.2.1** Prove that $L_{p,\beta}([0, T] \rightarrow E) \hookrightarrow L_{p,\alpha}([0, T] \rightarrow E)$ if $0 < \alpha < \beta \leq 1$.

**Lemma 5.2.2** [FG95, p.370, Theorem 2.1] Let:

- $E_1, ..., E_n$ and $E$ be Banach spaces such that each $E_i \hookrightarrow \hookrightarrow E$, $i = 1, ..., n$.
- $p_1, ..., p_n \in (1, \infty)$, $\alpha_1, ..., \alpha_n \in (0, 1)$ are such that $p_i \alpha_i > 1$, $i = 1, ..., n$.

Then, for any $T > 0$,

$$L_{p_1,\alpha_1}([0, T] \rightarrow E_1) + ... + L_{p_n,\alpha_n}([0, T] \rightarrow E_n) \hookrightarrow \hookrightarrow C([0, T] \rightarrow E).$$

**Lemma 5.2.3** [FG95, p.372, Theorem 2.2] Let:

$$E_0 \hookrightarrow \hookrightarrow E \hookrightarrow E_1$$

be Banach spaces such that the first imbedding is compact, and $E_0, E_1$ are reflexible. Then, for any $p \in (1, \infty)$, $\alpha \in (0, 1)$ and $T > 0$,

$$L_p([0, T] \rightarrow E_0) \cap L_{p,\alpha}([0, T] \rightarrow E_1) \hookrightarrow \hookrightarrow L_p([0, T] \rightarrow E).$$
5.3 Regularity of the noise

Let $H$ be a separable Hilbert space, and $\Gamma : H \to H$ be a non-negative self-adjoint operator of trace class, as in section 3.1. By the Hilbert-Schmidt theorem [RS72, p.203, Theorem VI.16], there exist a CONS $(\varphi_n)_{n \geq 1}$ of $H$ and numbers $\gamma_n \geq 0$ such that:

$$\Gamma \varphi_n = \gamma_n \varphi_n, \quad n \geq 1. \quad (5.14)$$

Let $W$ be a BM$(H, \Gamma)$. Then, the processes:

$$B^k = \langle W_t, \varphi_k \rangle / \sqrt{\gamma_k}, \quad k \in I = \{ k \in \mathbb{N} ; \gamma_k > 0 \}$$

are independent BM's. Let $(B_k^n)_{k \in I}$ be independent BM's which are independent of $(B_k)_{k \in I}$. Then, $\langle W_t, \varphi_k \rangle = \sqrt{\gamma_k}B^k_t$ for all $k \in \mathbb{N}$, and thus,

$$W_t = \sum_{k=0}^{\infty} \langle W_t, \varphi_k \rangle \varphi_k = \sum_{k=0}^{\infty} \sqrt{\gamma_k}B^k_t \varphi_k, \quad t \geq 0.$$ 

Let us consider the finite summation:

$$W^n_t = \sum_{k=0}^{n} \langle W_t, \varphi_k \rangle \varphi_k = \sum_{k=0}^{n} \sqrt{\gamma_k}B^k_t \varphi_k, \quad t \geq 0, \quad (5.15)$$

**Lemma 5.3.1** Referring to (5.15), for any $p \in [1, \infty)$, $\alpha \in [0, 1/2)$ and $T > 0$, there exists $C = C_{\alpha, p, T} \in (0, \infty)$ such that:

$$\sup_{n \geq 0} E[\| W^n_t \|_{L^p_{\alpha}([0,T] \to H)}^p] \leq C \text{tr}(\Gamma)^{p/2}. \quad (5.16)$$

Proof: We first prepare an exponential moment bound. Let $\varepsilon \in (0, 1)$, $\lambda, t \geq 0$ be such that $0 \leq \lambda t \gamma_k \leq 1 - \varepsilon$ for all $k \in \mathbb{N}$. Then,

1) $E \left[ \exp \left( \frac{\lambda}{2} \| W^n_t \|^2 \right) \right] = \prod_{k=0}^{n} \frac{1}{\sqrt{1 - \lambda t \gamma_k}} \leq \exp \left( \frac{\lambda t}{2\varepsilon} \text{tr}(\Gamma) \right).$

Since $\| W^n_t \|^2 = \sum_{k=0}^{n} \gamma_k |B^k_t|^2$,

$$E \left[ \exp \left( \frac{\lambda}{2} \| W^n_t \|^2 \right) \right] = \prod_{k=0}^{n} E \left[ \exp \left( \frac{\lambda \gamma_k}{2} |B^k_t|^2 \right) \right]$$

$$= \prod_{k=0}^{n} \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \exp \left( -\frac{1}{2} \frac{(x^2)}{t - \lambda \gamma_k} \right) \frac{1}{\sqrt{1 - \lambda t \gamma_k}} = \sqrt{\frac{2\pi}{1 - \lambda t \gamma_k}}.$$

We next observe for any $\delta \in [0, 1 - \varepsilon]$ that

$$\frac{1}{1 - \delta} = 1 + \frac{\delta}{1 - \delta} \leq 1 + \frac{\delta}{\varepsilon} \leq e^\varepsilon.$$

Hence, considering $\delta = \lambda t \gamma_k$ and taking the square root, and then the product over $k = 0, \ldots, n$, we have

$$\prod_{k=0}^{n} \frac{1}{\sqrt{1 - \lambda t \gamma_k}} \leq \exp \left( \frac{\lambda t}{2\varepsilon} \text{tr}(\Gamma) \right).$$

Thus, we get 1). Then, it is not difficult (Exercise 5.3.1 below) to see from 1) that
2) \( E[\|W_t^n - W_s^n\|^p] \leq C_p (\text{tr} (\Gamma) t)^{p/2} \) for any \( p \in (0, \infty) \),

with \( C_p \in (0, \infty) \) depending only on \( p \). Noting that

\[
E[\|W_t^n - W_s^n\|^p] = E[\|W_{t-s}^n\|^p] \leq C_p (\text{tr} (\Gamma) (t-s))^{p/2}, \quad s < t,
\]

we get

\[
E \int_{0<s<t<T} \frac{\|W_t^n - W_s^n\|^p}{(t-s)^{1+\alpha p}} ds dt \leq C_p \text{tr} (\Gamma)^{p/2} \int_{0<s<t<T} \frac{ds dt}{(t-s)^{1+(\alpha-\frac{1}{2})p}} \leq C_{p,\alpha} \text{tr} (\Gamma)^{p/2} T^{1+(\frac{1}{2}-\alpha)p}.
\]

This and 2) imply (5.16).

\( \square \)

Exercise 5.3.1 Conclude 2) from 1) in the proof of Lemma 5.3.1. Hint: Take \( \lambda = \frac{1}{2\text{tr} (\Gamma) t} \) in 1).

5.4 A digression on tightness

Let \( X^n = (X^n_t)_{t \geq 0} \in \mathcal{V} \) be the unique solution of (5.3) for the Galerkin approximation. In section 5.5, we will find a "convergent subsequence", the limit of which eventually solves (3.7). This can be done by showing that the laws of \( X^n, n \in \mathbb{N} \) are tight (see Definition 5.4.1). This subsection serves as a collection of notions and facts regarding the tightness, which we will use in section 5.5.

Throughout this subsection, let \( S = (S, \rho) \) be a separable metric space and \( (\Omega, \mathcal{F}, P) \) be a probability space.

Definition 5.4.1 A sequence \( \{X_n : \Omega \rightarrow S\}_{n \in \mathbb{N}} \) of r.v.'s (or more precisely, the laws of these r.v.'s) are said to be tight, if, for any \( \varepsilon \in (0, 1) \), there exists a relatively compact set \( K \subset S \) such that:

\[
\inf_{n \in \mathbb{N}} P(X_n \in K) \geq 1 - \varepsilon.
\]

Here is a common way to check the tightness:

Lemma 5.4.2 Let \( \{X_n : \Omega \rightarrow S\}_{n \in \mathbb{N}} \) be r.v.'s. Suppose that there exists a function \( F : S \rightarrow [0, \infty) \) such that:

- the set \( K_R \overset{\text{def}}{=} \{x \in S ; F(x) \leq R\} \) is relatively compact for all \( R > 0 \);
- \( \sup_{n \in \mathbb{N}} E[F(X_n)] \leq C < \infty \).

Then, \( \{X_n\}_{n \in \mathbb{N}} \) are tight.

Proof: We then have that:

\[
\sup_{n \in \mathbb{N}} P(X_n \not\in K_R) = \sup_{n \in \mathbb{N}} P(F(X_n) > R) \leq \sup_{n \in \mathbb{N}} \frac{E[F(X_n)]}{R} \leq \frac{C}{R} \rightarrow 0.
\]

This proves the tightness. \( \square \)

Once we are able to check that a sequence of r.v.'s is tight, we have the following consequence:
Lemma 5.4.3 Suppose that $S$ is complete and that a sequence $\{X_n : \Omega \rightarrow S\}_{n \in \mathbb{N}}$ of r.v.’s are tight. Then, there exist a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, a sequence $n(k) \nearrow \infty$ of integers, and a sequence

$$\{\bar{X}_k : \bar{\Omega} \rightarrow S\}_{k \in \mathbb{N} \cup \{\infty\}}$$

of r.v.’s such that:

$$\bar{P}(\bar{X}_k \in \cdot) = P(X_{n(k)} \in \cdot) \quad \text{for all } k \in \mathbb{N};$$

$$\lim_{k \rightarrow \infty} \bar{X}_k = \bar{X}_\infty, \; \bar{P} \text{-a.s.}$$

Proof: This is a consequence of Prohorov’s theorem [IW89, p.7, Theorem 2.6] and Skorohod’s representation theorem [IW89, p.9, Theorem 2.7].

Lemma 5.4.4 Suppose that $(S_j, \rho_j) (j = 1, \ldots, m)$ are complete separable metric spaces such that all of $S_j (j = 1, \ldots, m)$ are subsets of a common set. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables with values in $S \overset{\text{def}}{=} \bigcap_{j=1}^{m} S_j$ which is tight in each of $(S_j, \rho_j), \; j = 1, \ldots, m$ separately. Then, there exist a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, a sequence $n(k) \nearrow \infty$ of integers, and a sequence

$$\{\bar{X}_k : \bar{\Omega} \rightarrow S\}_{k \in \mathbb{N} \cup \{\infty\}}$$

of r.v.’s such that:

$$\bar{P}(\bar{X}_k \in \cdot) = P(X_{n(k)} \in \cdot) \quad \text{for all } k \in \mathbb{N};$$

$$\lim_{k \rightarrow \infty} \sum_{j=1}^{m} \rho_j(X, \bar{X}_k) = 0 \text{ a.s.}$$

Proof: By induction, it is enough to consider the case of $m = 2$. Let $\varepsilon > 0$ be arbitrary. Then, for $j = 1, 2$, there exists a compact subset $K_j$ of $S_j$ such that:

$$P(X_n \in K_j) \geq 1 - \varepsilon, \quad \text{for all } j = 1, 2 \text{ and } n = 1, 2, \ldots$$

Now, a very simple, but crucial observation is that $K_1 \cap K_2$ is compact in $S_1 \cap S_2$ with respect to the metric $\rho_1 + \rho_2$. Also,

$$P(X_n \in \overline{K_1 \cap K_2}) \geq 1 - 2\varepsilon, \quad \text{for all } j = 1, 2 \text{ and } n = 1, 2, \ldots$$

These imply that $(X_n)$ is tight in $S_1 \cap S_2$ with respect to the metric $\rho_1 + \rho_2$. Thus, the lemma follows from Lemma 5.4.3. \qed

5.5 Convergence of the approximation along a subsequence

Let $X^n = (X^n_t)_{t \geq 0} \in \mathcal{V}$ be the unique solution of (5.3) for the Galerkin approximation. Recall the notation from (2.25):

$$\beta(1, 0) = \begin{cases} 
1 & \text{if } d = 2, \\
\frac{d}{2} & \text{if } d \geq 3
\end{cases}$$

Proposition 5.5.1 For $\alpha \in [0, 1)$ and $\beta > \beta(1, 0)$ (cf. (2.25)), Then, there exist a process $X$ and a sequence $(\bar{X}_k)_{k \geq 1}$ of processes defined on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ such that the following properties are satisfied:
a) The process $X$ takes values in

$$C([0, \infty) \to V_{2-\beta}) \cap L_{2,loc}([0, \infty) \to V_{2,\alpha}).$$

(5.17)

b) For some sequence $n(k) \nrightarrow \infty$, $\tilde{X}^k$ has the same law as $X^n(k)$ and

$$\lim_{k \rightarrow \infty} \tilde{X}^k = X \text{ in the metric space (5.17), } P\text{-a.s.}$$

(5.18)

We divide the proof of Proposition 5.5.1 into the series of lemmas: To prepare the proof of these lemmas, we write (5.3) as:

$$X_t^n = X_0^n + J_t^n + W_t^n$$

with

$$J_t^n = \int_0^t \mathcal{P}_n b(X_{s}^n) ds.$$  

(5.19)

**Lemma 5.5.2** Let $\beta(1,0)$ and $J_t^n$ be as in (2.25) and (5.19). Then, there exists $C_T \in (0, \infty)$ such that:

$$\sup_{n \geq 1} E\left[\|\sqrt{\iota}\|_{L_{2,1}([0,T] \to V_{2,-\beta(1,0)})}\right] \leq C_T < \infty.$$  

(5.20)

Proof: It is not difficult to see that:

1) $$\|J^n\|_{L_{2,1}([0,T] \to V_{2,-\beta(1,0)})}^2 \leq C_T \int_0^T \|\mathcal{P}_n b(X_s^n)\|_{V_{2,-\beta(1,0)}}^2 ds.$$ (cf. Exercise 5.5.1)

By (2.22) for $q = 2$ and $(\alpha_1, \alpha_2) = (1,0)$, we see that

2) $$\int_0^T \|b(X_s^n)\|_{2,-\beta(1,0)}^2 ds \leq (\nu + C \sup_{s \leq T} \|X_s^n\|_2)^2 \int_0^T \|X_s^n\|_{2,1}^2 ds.$$  

Since $\mathcal{P}_n$ is contraction on $V_{2,\alpha}$ for any $\alpha \in \mathbb{R}$, we can combine the above bounds and (5.10)–(5.11) to obtain n (5.20) as follows:

$$E\left[\|J^n\|_{L_{2,1}([0,T] \to V_{2,-\beta(1,0)})}\right] \leq C_T E\left[ (\nu + C \sup_{s \leq T} \|X_s^n\|_2)^2 \left( \int_0^T \|X_s^n\|_{2,1}^2 ds \right)^{1/2} \right] \leq C_T \left[ (\nu + C \sup_{s \leq T} \|X_s^n\|_2)^2 \right]^{1/2} E\left[ \int_0^T \|X_s^n\|_{2,1}^2 ds \right]^{1/2} 

(5.10)-(5.11) \leq C_T \leq C_T < \infty.$$

\square

**Exercise 5.5.1** Let everything be as in Definition 5.2.1 a) and suppose that $u(0) = 0$. Prove then that

$$\|u\|_{L_{p,1}([0,T] \to E)}^p \leq C_T \int_0^T \|u'(s)\|_E^p ds.$$  

**Lemma 5.5.3** Let $\beta > \beta(1,0)$. Then, $\{X^n\}_{n=1}^\infty$ are tight on $C([0, \infty) \to V_{2-\beta}).$
Proof: It is enough to prove the following for each fixed $T > 0$:

1) $(X_{t}^{n})_{t \leq T}, \ n = 1, 2, \ldots$ are tight on $C([0, T] \rightarrow V_{2,-\beta})$.

To see this, we set:

$$ S = L_{2,1}([0, T] \rightarrow V_{2,-\beta(1,0)}) + L_{p,\alpha}([0, T] \rightarrow V_{2,0}), \text{ with } \alpha \in (0, 1/2), p > 1/\alpha. $$

The idea is to take $\| \cdot \|_{S}$ as the function $F$ in Lemma 5.4.2. We have that

2) $\sup_n E[\|X_{0}^{n} + J^{n}\|_{L_{2,1}([0,T] \rightarrow V_{2,-\beta(1,0)})}] \leq C_{T} < \infty$ (5.20)

On the other hand,

3) $\sup E[\|W^{n}\|_{L_{p,\alpha}([0,T] \rightarrow V_{2,0})}] \leq C_{T} < \infty$. (5.16)

We conclude from 2)–3) and the decomposition (5.19) that

$$ \sup_n E[\|X_{n}\|_{S}] \leq C_{T} < \infty $$

On the other hand, we see from Lemma 5.2.2 that

$$ \mathcal{S} \hookrightarrow C([0, T] \rightarrow V_{2,-\beta}) $$

hence that the set:

$$ \{X; \|X_{n}\|_{S} \leq R\} $$

is relatively compact in $C([0, T] \rightarrow V_{2,-\beta})$. Thus, we have the tightness 1) by Lemma 5.4.2. $\square$

Lemma 5.5.4 Suppose that $\alpha \in [0, 1)$. Then, $(X_{n})_{n=1}^{\infty}$ are tight on $L_{2,\text{loc}}([0, \infty) \rightarrow V_{2,\alpha})$.

Proof: It is enough to prove the following for each fixed $T > 0$:

1) $(X_{t}^{n})_{t \leq T}, \ n = 1, 2, \ldots$ are tight on $L_{2}([0, T] \rightarrow V_{2,\alpha})$.

To see this, we set:

$$ \mathcal{I} = L_{2}([0, T] \rightarrow V_{2,1}) \cap L_{2,\gamma}([0, T] \rightarrow V_{2,-\beta(1,0)}), \text{ with } \gamma \in (0, 1/2). $$

The idea is to take $\| \cdot \|_{\mathcal{I}}$ as the function $F$ in Lemma 5.4.2. We have that

2) $\sup_n E[\|X_{n}\|_{L_{2,\gamma}([0,T] \rightarrow V_{2,1})}^{2}] = \sup_n E[\int_{0}^{T} \|X_{t}^{n}\|_{2,1}^{2} dt] \leq C_{T} < \infty$ (5.10)

On the other hand,

$$ \sup_n E[\|X_{n}\|_{L_{2,\gamma}([0,T] \rightarrow V_{2,-\beta(1,0)})}^{2}] \leq \sup_n E[\|X_{0}^{n} + J^{n}\|_{L_{2,\gamma}([0,T] \rightarrow V_{2,-\beta(1,0)})}^{2}] + \sup E[\|W_{n}\|_{L_{2,\gamma}([0,T] \rightarrow V_{2,0})}] \leq C_{T} < \infty. $$
We conclude from this and 2) that
\[ \sup_n E[\|X^n\|_\mathcal{I}] \leq C_T < \infty. \]

On the other hand, we will see from Lemma 5.2.3 that
\[ I \hookrightarrow L_2([0, T] \rightarrow V_{2,\alpha}), \]

hence that the set:
\[ \{X_n; \|X^n\|_\mathcal{I} \leq R\} \]
is relatively compact in \( L_2([0, T] \rightarrow V_{2,\alpha}) \). Thus, we have the tightness 1) by Lemma 5.4.2. \( \square \)

Finally, Proposition 5.5.1 follows from Lemma 5.5.3–Lemma 5.5.4 and Lemma 5.4.4.

6 Proof of Theorem 3.2.1 and Theorem 3.2.2

6.1 Proof of Theorem 3.2.1

Let \( X \) and \( \tilde{X}^k \) be as in Proposition 5.5.1. We will verify that \( X \) takes values in the metric space \((3.4)\) as well as properties \((3.5)-(3.9)\) for \( X \). \( (3.5) \) can easily be seen. In fact,
\[
\tilde{X}_0^k \rightarrow X_0 \text{ a.s. in } V_{2,-\beta},
\]
\[
\tilde{X}_0^k \equiv X_0^{n(k)} = \mathcal{P}_{n(k)} \xi \rightarrow \xi \text{ a.s. in } V_{2,0}.
\]

Thus the laws of \( X_0 \) and \( \xi \) are identical. To see \((3.8)-(3.9)\), note that: \[
\|v_T\|_2^2, \sup_{t \leq T} \|v_t\|_2^2, \int_0^T \|v_t\|_{2,1}^2 dt
\]
are lower semi-continuous functions of \( v \) on the metric space \((5.17)\). Thus, \((3.8)-(3.9)\) follow from \((5.10)-(5.11)\) and Proposition 5.5.1 via Fatou's lemma.

To show \((3.6)-(3.7)\), we prepare the following:

**Lemma 6.1.1** Let \( \varphi \in \mathcal{V} \) and \( T > 0 \). Then,
\[
\lim_{k \rightarrow \infty} \int_0^T \langle \varphi, (\tilde{X}_t^k \cdot \nabla) \tilde{X}_t^k \rangle dt = \int_0^T \langle \varphi, (X_t \cdot \nabla) X_t \rangle dt \text{ in probability}, \quad (6.1)
\]
\[
\lim_{k \rightarrow \infty} \int_0^T \langle \Delta \varphi, \tilde{X}_t^k \rangle dt = \int_0^T \langle \Delta \varphi, X_t \rangle dt \text{ a.s.}, \quad (6.2)
\]
\[
\lim_{k \rightarrow \infty} \int_0^T \langle \varphi, \mathcal{P}_{n(k)} b(\tilde{X}_t^k) \rangle dt = \int_0^T \langle \varphi, b(X_t) \rangle dt \text{ in probability}. \quad (6.3)
\]

Proof: (6.1): Since,
\[
\tilde{X}_t^k \cdot \nabla \tilde{X}_t^k - X_t \cdot \nabla X_t = (\tilde{X}_t^k - X_t) \cdot \nabla \tilde{X}_t^k + X_t \cdot \nabla (\tilde{X}_t^k - X_t),
\]
we have
\[
\int_0^T |\langle \varphi, (\tilde{X}_t^k \cdot \nabla \tilde{X}_t^k - X_t \cdot \nabla X_t) \rangle dt | \leq I_1 + I_2,
\]

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where
\[ I_1 = \int_0^T |\langle \varphi, (\tilde{X}_t^k - X_t) \cdot \nabla \tilde{X}_t^k \rangle| \, dt \]
and
\[ I_2 = \int_0^T |\langle \varphi, X_t \cdot \nabla (\tilde{X}_t^k - X_t) \rangle| \, dt. \]

To bound $I_1$, we take
\[ \alpha_1 = \alpha \in (0, 1 \wedge \frac{d}{2}), \quad \alpha_2 = 0, \quad \alpha_3 = \frac{d}{2} - \alpha \in (0, \frac{d}{2}). \]

in Lemma 2.2.1. Then, by (2.14), we have that
\[ |\langle \varphi, (\tilde{X}_t^k - X_t) \cdot \nabla \tilde{X}_t^k \rangle| \leq C \|X_t\|_2 \|\tilde{X}_t^k - X_t\|_{2, \alpha} \|\varphi\|_{2, 1 + \alpha_3} \]
and hence that,
\[ I_1 \leq C \|\varphi\|_{2, 1 + \alpha_3} \sup_{t \leq T} \|X_t\|_2 \int_0^T \|\tilde{X}_t^k - X_t\|_{2, \alpha} \, dt. \]

By (5.11) and Proposition 5.5.1,
\[ \sup_{k \geq 1} E[\sup_{t \leq T} \|\tilde{X}_t^k\|_2^2] < \infty \]
and
\[ \lim_{k \to \infty} \int_0^T \|\tilde{X}_t^k - X_t\|_{2, \alpha} \, dt = 0 \quad \text{P-a.s.} \]

Then, it is easy to conclude from these that $\lim_{k \to \infty} I_1 = 0$ in probability (Exercise 6.1.1 below). To bound $I_2$, we take
\[ \alpha_1 = 0, \quad \alpha_2 = \alpha \in (0, 1 \wedge \frac{d}{2}), \quad \alpha_3 = \frac{d}{2} - \alpha \in (0, \frac{d}{2}) \]

in Lemma 2.2.1. On the other hand, we have by (2.14) that
\[ |\langle \varphi, X_t \cdot \nabla (\tilde{X}_t^k - X_t) \rangle| \leq C \|X_t\|_2 \|\tilde{X}_t^k - X_t\|_{2, \alpha} \|\varphi\|_{2, 1 + \alpha_3} \]
and hence that,
\[ I_2 \leq C \|\varphi\|_{2, 1 + \alpha_3} \sup_{t \leq T} \|X_t\|_2 \int_0^T \|\tilde{X}_t^k - X_t\|_{2, \alpha} \, dt. \]

By (3.9) and Proposition 5.5.1,
\[ E[\sup_{t \leq T} \|X_t\|_2^2] < \infty \]
and
\[ \lim_{k \to \infty} \int_0^T \|\tilde{X}_t^k - X_t\|_{2, \alpha} \, dt = 0 \quad \text{P-a.s.} \]

Then, it is easy to conclude from these that $\lim_{k \to \infty} I_2 = 0$ in probability (Exercise 6.1.1 below).

(6.2): This is an easy consequence of Proposition 5.5.1.
(6.3) follows from (6.1) and (6.2). Since $\varphi \in \mathcal{V}$ is fixed and $k$ is tending to $\infty$, we do not have to care about $\mathcal{P}_{n(k)}$ here. $\square$

**Exercise 6.1.1** Let $X_n, Y_n$ be r.v.’s such that $\{X_n\}_{n \geq 1}$ are tight and $Y_n \to 0$ in probability. Prove then that $X_n Y_n \to 0$ in probability.

We see (3.6)–(3.7) from the following:
Lemma 6.1.2 Let:

$$Y_t = Y_t(X) = X_t - X_0 - \int_0^t b(X_s)ds, \quad t \geq 0.$$  \hspace{1cm} (6.4)

Then, $Y$ is a BM($V_{2,0}, \Gamma$). Moreover, $Y_{t^+} - Y_t$ and $\{(\varphi, X_s) ; s \leq t, \varphi \in \mathcal{V}\}$ are independent for any $t \geq 0$.

It is enough to prove that for each $\varphi \in \mathcal{V}$ and $0 \leq s < t$,

1) \hspace{1cm} $E\left[ \exp\left( i \langle \varphi, Y_t - Y_s \rangle \right) | \mathcal{G}_s \right] = \exp\left( -\frac{t-s}{2} \langle \varphi, \Gamma \varphi \rangle \right), \quad \text{a.s.}$

where $\mathcal{G}_s = \sigma(\langle \varphi, X_u \rangle ; u \leq s, \varphi \in \mathcal{V})$. We set

$$F(X) = f(\langle \varphi_1, X_{u_1} \rangle, \ldots, \langle \varphi_n, X_{u_n} \rangle),$$

where $f \in C_b(\mathbb{R}^n), \ 0 \leq u_1 < \ldots < u_n \leq s$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{V}$ are chosen arbitrary in advance. Then, 1) can be verified by showing that

2) \hspace{1cm} $E\left[ \exp\left( i \langle \varphi, Y_t - Y_s \rangle \right) F(X) \right] = \exp\left( -\frac{t-s}{2} \langle \varphi, \Gamma \varphi \rangle \right) E[F(X)].$

Let:

$$Y_t^k = \tilde{X}_t^k - \tilde{X}_0^k - \int_0^t \mathcal{P}_{n(k)}b(\tilde{X}_s^k)ds, \quad t \geq 0.$$  \hspace{1cm} (6.4)

We then see from Theorem 5.1.1 that

3) \hspace{1cm} $E\left[ \exp\left( i \langle \varphi, Y_t^k - Y_s^k \rangle \right) F(\tilde{X}^k) \right] = \exp\left( -\frac{t-s}{2} \langle \varphi, \Gamma \mathcal{P}_{n(k)} \varphi \rangle \right) E[F(\tilde{X}^k)],$

Moreover, we have for any $\varphi \in \mathcal{V}$,

$$\lim_{k \rightarrow \infty} \langle \varphi, Y_t^k - Y_s^k \rangle \overset{(5.18),(6.3)}{=} \langle \varphi, Y_t - Y_s \rangle \quad \text{in probability},$$

and hence

$$\lim_{k \rightarrow \infty} \text{LHS of 3)} = \text{LHS of 2)}.$$  \hspace{1cm} (5.18)

On the other hand,

$$\lim_{k \rightarrow \infty} \text{RHS of 3)} \overset{(5.18)}{=} \text{RHS of 2)}.$$  \hspace{1cm} (6.3)

These prove 2). \hspace{1cm} \square

Finally, we prove that $X$ takes values in the metric space (3.4). It follows from (3.9) that

$$X \in L_{2, \text{loc}}([0, \infty) \rightarrow V_{2,1}) \cap L_{\infty, \text{loc}}([0, \infty) \rightarrow V_{2,0}).$$

Thus, it remains to show that $X \in C([0, \infty) \rightarrow V_{2,-\beta(1,1)})$. We see from Lemma 2.2.3 that:

$$\int_0^t b(X_s)ds \in C([0, \infty) \rightarrow V_{2,-\beta(1,1)}) \quad \text{if} \quad X \in L_2([0, \infty) \rightarrow V_{2,1}).$$

On the other hand, $Y \in C([0, \infty) \rightarrow V_{2,0})$. These show that $X \in C([0, \infty) \rightarrow V_{2,-\beta(1,1)})$. \hspace{1cm} \square
6.2 Proof of Theorem 3.2.2

Here, we can follow the argument of [Te79, p. 294, Theorem 3.2] almost verbatim. We will present it for the convenience of the readers.

We need technical lemmas:

**Lemma 6.2.1** [Te79, pp. 60-61, Lemma 1.2] Let $H$ and $V$ be a Hilbert spaces such that:

$$V \hookrightarrow H \hookrightarrow V^*.$$

Suppose that $f \in L_2([0, T] \rightarrow V)$ has derivative $f'$ in $L_2([0, T] \rightarrow V^*)$. Then,

$$\frac{d}{dt} |f|_H^2 = 2V \langle f, f' \rangle_{V^*},$$

in the distributional sense on $(0, T)$.

**Lemma 6.2.2** For any $T > 0$, there exists $C_T \in (0, \infty)$ such that:

$$E \left[ \int_0^T \| b(X_t) \|_{2, \beta(1,0)} \right] \leq C_T < \infty. \tag{6.6}$$

Proof: Using (3.9), the lemma can be shown in the same way as Lemma 5.5.2. \qed

Let $X$ and $\tilde{X}$ be as in the assumptions of Theorem 3.2.2 and

$$Z_t = X_t - \tilde{X}_t = \int_0^t (b(X_s) - b(\tilde{X}_s)) ds.$$

Then,

1) $Z \in L_{2,loc}([0, \infty) \rightarrow V_{2,1})$

and by Lemma 6.2.2,

2) $\partial_t Z = b(X) - b(\tilde{X}) \in L_{2,loc}([0, \infty) \rightarrow V_{2,\beta(1,0)})$

Since $\beta(1,0) = 1$, we see from 2) and Lemma 6.2.1 (applied to $f = Z$ and $V = V_{2,1}$) that

3) $\frac{1}{2} \frac{d}{dt} \| Z_t \|_2^2 \overset{(6.5)}{=} \langle Z_t, b(X_t) - b(\tilde{X}_t) \rangle = -I_t - J_t$

in the distributional sense, where

$$I_t = \langle Z_t, (X_t \cdot \nabla)X_t - (\tilde{X}_t \cdot \nabla)\tilde{X}_t \rangle,$$

$$J_t = \nu \langle \nabla Z_t, \nabla X_t - \nabla \tilde{X}_t \rangle = \nu \| \nabla Z_t \|_2^2.$$

On the other hand, since $\tilde{X}_t = X_t - Z_t$, we see that

$$\langle Z_t, (\tilde{X}_t \cdot \nabla)\tilde{X}_t \rangle \overset{\text{Lemma 2.2.2}}{=} \langle Z_t, (\tilde{X}_t \cdot \nabla)X_t \rangle = \langle Z_t, ((X_t - Z_t) \cdot \nabla)X_t \rangle,$$

and hence that

$$I_t = \langle Z_t, (Z_t \cdot \nabla)X_t \rangle.$$

We now apply Lemma 2.2.2 with $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, 0)$. Note that these $\alpha_i$ satisfy the assumption of Lemma 2.2.2 only when $d = 2$. 33
4) \(|I_t| \leq C_3 \|Z_t\|_{2,1} \|Z_t\|_2 \|X_t\|_{2,1} \leq \nu \|Z_t\|_{2,1}^2 + C_4 \|X_t\|_{2,1}^2 \|Z_t\|_2^2\).

We see from 3)–4) that
\[
\frac{1}{2} \frac{d}{dt} \|Z_t\|_2^2 \leq C_4 \|X_t\|_{2,1}^2 \|Z_t\|_2^2.
\]

This implies, via Gronwall's lemma (We need an appropriate generalization, since the derivative above is in the distributional sense.) that
\[
\|Z_t\|_2^2 \leq \|Z_0\|_2^2 \exp(C_4 \int_0^t \|X_s\|_{2,1}^2 ds).
\]

This proves that \(\|Z_t\|_2 \equiv 0\). \(\square\)

7 Appendix

Lemma 7.0.3 Suppose that a CONS \(\{\varphi_n\}_{n \geq 1}\) of \(H\) and numbers \(\gamma_n \geq 0\) satisfy (5.14).

a) Let \(\{B^k\}_{k \in \mathbb{N}}\) be independent standard BM\(^1\)'s. Then, the process

\[
W_t^n = \sum_{k=0}^{n} \sqrt{\gamma_k} B^k_t \varphi_k, \quad t \geq 0,
\]

converges to a BM(\(H, \Gamma\)) \(W\) in the sense that:
\[
\lim_{n \to \infty} E \left[ \sup_{t \leq T} \|W_t^n - W_t\|^2 \right] = 0 \quad \text{for any } T > 0.
\]

b) For any BM(\(H, \Gamma\)) \(W\), there are independent standard BM\(^1\)'s such that (7.2) holds with the process defined by (5.15).

Proof: a): Let us show that

1) \((W_t^n)_{n \in \mathbb{N}}\) is a Cauchy sequence with respect to seminorms:
\[
|||W|||_t = E \left[ \sup_{s \leq t} \|W_s\|^2 \right]^{1/2}, \quad t \in (0, \infty).
\]

In fact, for \(m < n\),
\[
\|W_t^n - W_t^m\|^2 = \sum_{m < k \leq n} \gamma_k |B^k_t|^2.
\]

By this and Doob's \(L^2\)-maximal inequality,
\[
E \left[ \sup_{s \leq t} \|W_s^n - W_s^m\|^2 \right] \leq \sum_{m < k \leq n} \gamma_k E \left[ \sup_{s \leq t} |B^k_s|^2 \right] \overset{(4.16)}{\leq} 4t \sum_{m < k \leq n} \gamma_k \overset{m \to \infty}{\to} 0.
\]

By 1), there exists a random variable \(W\) with values in \(C([0, \infty) \to H)\) such that (7.2) holds. It is easy to see from this that for \(0 \leq s < t\):
\[
\lim_{n \to \infty} \exp(i \langle \varphi, W_t^n - W_s^n \rangle) = \exp(i \langle \varphi, W_t - W_s \rangle) \quad \text{in } L^1(P),
\]

and hence
2) \[ \lim_{n \to \infty} E \left[ \exp \left( i \langle \varphi, W_t^n - W_s^n \rangle \right) | \mathcal{G}_s^W \right] = E \left[ \exp \left( i \langle \varphi, W_t - W_s \rangle \right) | \mathcal{G}_s^W \right] \] in \( L^1(P) \).

On the other hand,
\[
E \left[ \exp \left( i \langle \varphi, W_t^n - W_s^n \rangle \right) | \mathcal{G}_s^W \right] = \prod_{k=0}^{n} E \left[ \exp \left( i \sqrt{\gamma_k} \langle \varphi, \varphi_k \rangle (B_t^k - B_s^k) \right) \right] = \prod_{k=0}^{n} \exp \left( -\frac{t-s}{2} \gamma_k \langle \varphi, \varphi_k \rangle^2 \right) \overset{n \to \infty}{\longrightarrow} \exp \left( -\frac{t-s}{2} \langle \varphi, \Gamma \varphi \rangle \right).
\]

By this and 2), we have (3.3).

b): Processes:
\[ B_k \overset{\text{def}}{=} \langle W, \varphi_k \rangle / \sqrt{\gamma_k}, \quad k \in I \overset{\text{def}}{=} \{ k \in \mathbb{N} ; \gamma_k > 0 \} \]
are independent BM\(^1\)s. Let \( \{ B_k \}_{k \in N^I} \) be independent BM\(^1\)s which are independent of \( \{ B_k \}_{k \in I} \). Then, \( \langle W, \varphi_k \rangle = \sqrt{\gamma_k} B_k \) for all \( k \in \mathbb{N} \), and hence (5.15) holds. \( \square \)

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References


