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ON THE ORDER OF STRONGLY CLOSE-TO-CONVEXITY OF STRONGLY CONVEX FUNCTIONS

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ABSTRACT. In this work the order of strongly close-to-convexity of strongly convex functions is discussed. The sufficient conditions for function to be Bazilevič function are also considered.

1. INTRODUCTION

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{U}=\{z: |z|<1\}$ on the complex plane $\mathbb{C}$. For $a\in\mathbb{C}$ and $n\in\mathbb{N}$ we denote by

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}: f(z)=a+a_nz^n+\cdots\}$$

and

$$\mathcal{A}_n=\left\{f\in\mathcal{H}: f(z)=z+a_{n+1}z^{n+1}+\cdots\right\},$$

so $\mathcal{A}=\mathcal{A}_1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{U}$.

The class $\mathcal{S}^*_\alpha$ of starlike functions of order $\alpha<1$ may be defined as

$$\mathcal{S}^*_\alpha=\left\{f\in\mathcal{A}: \mathfrak{Re}\frac{zf'(z)}{f(z)}>\alpha,\ z\in\mathbb{U}\right\}.$$

The class $\mathcal{S}^*_\alpha$ and the class $\mathcal{K}_\alpha$ of convex functions of order $\alpha<1$

$$\begin{aligned}\mathcal{K}_\alpha:&=\left\{f\in\mathcal{A}: \mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\ z\in\mathbb{U}\right\}\\ &=\{f\in\mathcal{A}: zf'\in\mathcal{S}^*_\alpha\}\end{aligned}$$

introduced Robertson in [13]. If $\alpha\in[0;1)$, then a function in either of these sets is univalent, if $\alpha<0$ it may fail to be univalent. In particular we denote $\mathcal{S}^*_0=\mathcal{S}^*, \mathcal{K}_0=\mathcal{K}$, the classes of starlike and convex functions, respectively.

Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β, $0<\beta\le 1$,

$$\mathcal{SS}^*(\beta):=\left\{f\in\mathcal{A}: \left|\mathrm{Arg}\frac{zf'(z)}{f(z)}\right|<\frac{\beta\pi}{2},\quad z\in\mathbb{U}\right\},$$

which was introduced in [14] and [1]. Furthermore, $\mathcal{SK}(\beta)=\{f\in\mathcal{A}: zf'\in\mathcal{SS}^*(\beta)\}$ denote the class of strongly convex functions of order β. The class $\mathcal{S}^*[A,B]$

$$\mathcal{S}^*[A,B]:=\left\{f\in\mathcal{A}: \frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz},\quad z\in\mathbb{U}\right\}$$

was investigated in [2] for $-1\le B<A\le 1$. Recall, that we write $f\prec g$ and say that the $f\in\mathcal{H}$ is subordinate to $g\in\mathcal{H}$ in the unit disc $\mathbb{U}$, if and only if there exists an analytic

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function $w \in \mathcal{H}$ such that $|w(z)| < |z|$ and $f(z) = g[w(z)]$ for $z \in \mathbb{U}$. Therefore, $f \prec g$ in $\mathbb{U}$ implies $f(\mathbb{U}) \subset g(\mathbb{U})$. In particular if g is univalent in $\mathbb{U}$, then

$$f \prec g \quad \Leftrightarrow \quad [f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U})].$$

2. Preliminaries

To prove the main results, we need the following Nunokawa's Lemma.

Lemma 2.1. [8], [9] *Let p be analytic function in $|z| < 1$ with $p(0) = 1$, $p(z) \neq 0$. If there exists a point z_0, $|z_0| < 1$, such that*

$$|\arg p(z)| < \frac{\pi\alpha}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\alpha}{2}$$

for some $\alpha > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi\alpha}{2}$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi\alpha}{2},$$

where

$$\{p(z_0)\}^{1/\alpha} = \pm ia, \quad \text{and} \quad a > 0.$$

We need also the following four authors lemma [10].

Lemma 2.2. [10] *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic function in $|z| < 1$. If there exists a point z_0, $|z_0| < 1$, such that*

$$\mathfrak{Re}\, p(z) > c \quad \text{for} \quad |z| < |z_0|$$

and

$$\mathfrak{Re}\, p(z_0) = c, \quad p(z_0) \neq c$$

for some $0 < c < 1$, then we have

$$\mathfrak{Re}\left\{\frac{z_0 p'(z_0)}{p(z_0)}\right\} \leq \gamma(c) = \begin{cases} \frac{-c}{2(1-c)} & \text{when} \quad c \in \left(0, \frac{1}{2}\right], \\ \frac{c-1}{2c} & \text{when} \quad c \in \left(\frac{1}{2}, 1\right). \end{cases}$$

3. Main Result

Theorem 3.1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $|z| < 1$ *and suppose that in* $|z| < 1$

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \tan^{-1}\frac{\beta}{1-\alpha}, \tag{3.1}$$

where $0 < \alpha < 1$ *and* $0 < \beta < 1$. *Then we have*

$$\left|\arg\frac{f'(z)}{g'(z)}\right| < \frac{\pi\beta}{2} \quad \text{in} \quad |z| < 1, \tag{3.2}$$

for some $g \in \mathcal{K}_{1-\alpha}$.

Proof. Let us put $g'(z) = (f'(z))^{\alpha}$. By (3.1) $\mathfrak{Re}\{1 + f''(z)/f'(z)\} > 0$ so

$$\begin{aligned} &\mathfrak{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} \\ = \ &\mathfrak{Re}\left\{1 - \alpha + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 1 - \alpha > 0, \end{aligned}$$

hence

$$g \in \mathcal{K}_{1-\alpha}. \tag{3.3}$$

Next, let us put

$$p(z) = f'(z), \quad p(0) = 1.$$

Then it follows that

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zp'(z)}{p(z)}.$$

If there exists a point z_0, $|z_0| < 1$, such that

$$|\arg p(z)| < \frac{\pi\beta}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\beta}{2},$$

then by Nunokawa's Lemma 2.1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\beta k,$$

where

$$k \geq 1 \quad \text{when} \quad \arg p(z_0) = \frac{\pi\beta}{2}$$

and

$$k \leq -1 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi\beta}{2}.$$

For the case $\arg p(z_0) = \pi\beta/2$, we have

$$\begin{aligned} &\arg\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right\} = \arg\left\{1 + \frac{i\beta k}{1-\alpha}\right\} \\ \geq \ &\arg\left\{1 + \frac{i\beta}{1-\alpha}\right\} = \tan^{-1}\frac{\beta}{1-\alpha}. \end{aligned}$$

This contradicts hypothesis of the Theorem 3.1 and for the case $\arg p(z_0) = -\pi\beta/2$, applying the same method as the above, we have

$$\arg\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right\} \leq -\tan^{-1}\frac{\beta}{1-\alpha}.$$

This is also the contradiction and therefore, it completes the proof. □

Recall that $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}_\alpha(\beta)$, [3], of close-to-convex functions of order β, $0 \leq \beta < 1$, if and only if there exist $g \in \mathcal{K}_\alpha$, $\varphi \in \mathbb{R}$, such that

$$\mathfrak{Re}\left\{e^{i\varphi}\frac{f'(z)}{g'(z)}\right\} > \beta, \quad z \in \mathbb{U}. \tag{3.4}$$

Reade [12] introduced the class of strongly close-to-convex functions of order $\beta < 1$ defined by $|\arg\{e^{i\varphi}f'(z)/g'(z)\}| < \pi\beta/2$ instead of (3.4). Therefore, the conditions (3.2) and (3.3) mean that f is strongly close-to-convex functions of order β with respect convex functions of order $1-\alpha$. Functions defined by (3.4) with $\varphi = 0$ where considered earlier by Ozaki [11], see also Umezawa [16, 17]. Moreover, Lewandowski [4, 5] defined the class of functions $f \in \mathcal{A}$ for which the complement of $f(\mathbb{U})$ with respect to the complex plane is a linearly accessible domain in the large sense. The Lewandowski's class is identical with the Kaplan's class $\mathcal{C}_0(0)$, see [3]. If we put $g'(z) = (f'(z))^\alpha$ in Theorem 3.1 and if we denote $\lambda = \beta/(1-\alpha)$, $\lambda \in (0,\infty)$, then we obtain the following corollary.

Corollary 3.2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and suppose that*

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \tan^{-1}\lambda \quad \text{in} \quad |z| < 1,$$

where $0 < \lambda < \infty$. Then we have

$$|\arg\{f'(z)\}| < \frac{\pi\lambda}{2} \quad \text{in} \quad |z| < 1,$$

Remark 3.3. *For the case $0 < \beta < 1$, it is trivial that there exists α, $0 < \alpha < 1$, which satisfies*

$$\begin{aligned}\frac{\beta}{1-\alpha} &> \tan\left(\frac{\pi}{2}\gamma(\beta)\right)\\ &= \tan\left\{\frac{\pi\beta}{2} + \tan^{-1}\frac{\beta\varrho(\beta)\sin\left(\frac{\pi(1-\beta)}{2}\right)}{\rho(\beta) + \beta\varrho(\beta)\cos\left(\frac{\pi(1-\beta)}{2}\right)}\right\},\end{aligned}$$

where

$$\rho(\beta) = (1+\beta)^{(1+\beta)/2} \quad \text{and} \quad \varrho(\beta) = (1-\beta)^{(\beta-1)/2}$$

and

$$\frac{\beta}{1-\alpha} > \tan\frac{\pi\beta}{2} + \frac{\beta\left(\frac{1-\beta}{1+\beta}\right)^{(1+\beta)/2}}{(1-\beta)\cos(\pi\beta/2)}.$$

The right hand sides of the above estimate are Nunokawa's and Mocanu's estimate of the order of strongly starlikeness in the class of strongly convex functions $\mathcal{SK}(\beta)$, for details see [9] and [7] or [6, p. 266].

Theorem 3.4. *Assume that* $1/2 \le \alpha < 1$, $\beta \ge 1$ *and* $0 < c < 1$. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $|z| < 1$ *and suppose that*

$$\Re\mathfrak{e}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad for \quad |z| < 1. \tag{3.5}$$

Furthermore, let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *be analytic in* $|z| < 1$ *such that*

$$\Re\mathfrak{e}\left\{\frac{zg'(z)}{g(z)}\right\} \le \frac{\alpha - \gamma(c) + (\beta - 1)\delta(\alpha)}{\beta} \quad for \quad |z| < 1, \tag{3.6}$$

where $\gamma(c)$ *is given in Lemma 2.2, and where*

$$\delta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}-2} & for \quad \alpha \neq \frac{1}{2}, \\ \frac{1}{2\log 2} & for \quad \alpha = \frac{1}{2}. \end{cases}$$

Then we have

$$\Re\mathfrak{e}\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)} > c \quad for \quad |z| < 1.$$

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}, \quad p(0) = 1.$$

Then it follows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} + (1-\beta)\frac{zf'(z)}{f(z)} + \beta\frac{zg'(z)}{g(z)}. \tag{3.7}$$

If there exists a point z_0, $|z_0| < 1$, such that

$$\Re\mathfrak{e}\, p(z) > c \quad \text{for} \quad |z| < |z_0|$$

and

$$\Re\mathfrak{e}\, p(z_0) = c, \quad p(z_0) \neq c,$$

then by Lemma 2.2, we have

$$\Re\mathfrak{e}\left\{\frac{z_0 p'(z_0)}{p(z_0)}\right\} \le \gamma(c) = \begin{cases} -\frac{c}{2(1-c)} & when \quad c \in (0, 1/2], \\ -\frac{1-c}{2c} & when \quad c \in (1/2, 1). \end{cases} \tag{3.8}$$

Furthermore, by (3.5) $f \in \mathcal{K}_{\alpha}$, thus $f \in \mathcal{S}^{*}_{\delta(\alpha)}$, see [18]. Because $\beta \ge 1$, then in $|z| < 1$

$$\Re\mathfrak{e}\left\{(1-\beta)\frac{zf'(z)}{f(z)}\right\} \le (1-\beta)\delta(\alpha). \tag{3.9}$$

Substituting (3.6), (3.8) and (3.9) in (3.7) we get

$$\begin{aligned} &1+\mathfrak{Re}\frac{z_0 f''(z_0)}{f'(z_0)} \\ &= \mathfrak{Re}\left\{\frac{z_0 p'(z_0)}{p(z_0)} + (1-\beta)\frac{z_0 f'(z_0)}{f(z_0)} + \beta\frac{z_0 g'(z_0)}{g(z_0)}\right\} \\ &\leq \gamma(c) + (1-\beta)\delta(\alpha) + \beta\frac{\alpha-\gamma(c)+(\beta-1)\delta(\alpha)}{\beta} \\ &= \alpha. \end{aligned}$$

This contradicts hypothesis of the Theorem 3.5 and therefore, it completes the proof. □

Remark 3.5. *For the case $1 < \beta$, if α, β and f satisfy the conditions of Theorem 3.4, then f is a Bazilevič function of order c, $0 < c < 1$, see* [15, p. 353].

Applying the same method as in the proof of Theorem 3.4, we have the following theorem.

Theorem 3.6. *Assume that $1/2 \leq \alpha < 1$, $\beta > 1$ and $0 < c < 1$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and suppose that*

$$\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad for \quad |z| < 1.$$

Furthermore, let $g \in \mathcal{S}^*[A, B]$ and let

$$\frac{1-A}{1-B} \leq \frac{\alpha-\gamma(c)+(\beta-1)\delta(\alpha)}{\beta} \quad for \quad |z| < 1,$$

where $\gamma(c)$ is given in Lemma 2.2, and where

$$\delta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}-2} & for \quad \alpha \neq \frac{1}{2}, \\ \frac{1}{2\log 2} & for \quad \alpha = \frac{1}{2}. \end{cases}$$

Then we have

$$\mathfrak{Re}\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)} > c \quad for \quad |z| < 1.$$

Remark 3.7. *If f satisfies the conditions of Theorem 3.6, then f is a Bazilevič function.*

For $\beta = 1$ Theorem 3.6 gives the following corollary.

Corollary 3.8. *Assume that $1/2 \leq \alpha < 1$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and suppose that*

$$\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad for \quad |z| < 1.$$

Furthermore, let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in $|z| < 1$ such that

$$\mathfrak{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \leq \alpha - \gamma(c) \quad for \quad |z| < 1,$$

where $c \in (0, 1)$ is such that $\alpha - \gamma(c) > 1$. Then we have

$$\mathfrak{Re}\frac{zf'(z)}{g(z)} > c \quad for \quad |z| < 1.$$

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