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An extension of Nunokawa lemma

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Abstract
Let $\mathcal{H}[a_0,n]$ be the class of functions $p(z) = a_0 + a_n z^n + \cdots$ which are analytic in the open unit disk $U$. For functions $f(z)$ which are analytic in $U$ with $f(0) = 1$, M. Nunokawa (Proc. Japan Acad., Ser. A 68 (1992), 152–153) have shown some theorems. The object of the present paper is to discuss Nunokawa lemma for the class $\mathcal{H}[a_0,n]$.

1 Introduction

Let $\mathcal{H}[a_0,n]$ denote the class of functions $p(z)$ of the form

$$p(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ for some $a_0 \in \mathbb{C}$ and a positive integer $n$.

The basic tool in proving our results is the following lemma due to S. S. Miller and P. T. Mocanu [1] (also [2]).

Lemma 1. Let the function $w(z)$ defined by

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots \quad (n = 1, 2, 3, \cdots)$$

be analytic in $U$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in U$, then there exists a real number $m \geq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = m.$$

2 Main result

Applying Lemma 1, we derive the following result.

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Theorem 1. Let \( p(z) \in \mathcal{H}[a_0, n] \) for some real \( a_0 > 0 \) and suppose that there exists a point \( z_0 \in \mathbb{U} \) such that
\[
\operatorname{Re}(p(z)) > 0 \quad \text{for} \quad |z| < |z_0|
\]
and \( p(z_0) = \beta i \) is a pure imaginary number for some real \( \beta \neq 0 \).

Then we have
\[
\frac{z_0 p'(z_0)}{p(z_0)} = il
\]
where
\[
l \geq \frac{n}{2} \left( \frac{a_0}{\beta} + \frac{\beta}{a_0} \right) \geq n
\]
if \( \beta > 0 \) and
\[
l \leq \frac{n}{2} \left( \frac{a_0}{\beta} + \frac{\beta}{a_0} \right) \leq -n
\]
if \( \beta < 0 \).

Proof. Let us put
\[
w(z) = \frac{a_0 - p(z)}{a_0 + p(z)} = c_n z^n + c_{n+1} z^{n+1} + c_{n+2} z^{n+2} + \cdots \quad (z \in \mathbb{U}).
\]

Then, we have that \( w(z) \) is analytic in \( |z| < |z_0| \), \( w(0) = 0 \), \( |w(z)| < 1 \) for \( |z| < |z_0| \) and
\[
|w(z_0)| = \left| \frac{a_0^2 - \beta^2 - 2a_0 \beta i}{a_0^2 + \beta^2} \right| = 1.
\]

From Lemma 1, we obtain
\[
\frac{z_0 w'(z_0)}{w(z_0)} = \frac{-2a_0 z_0 p'(z_0)}{a_0^2 - \{p(z_0)\}^2} = \frac{-2a_0 z_0 p'(z_0)}{a_0^2 + \beta^2} = m \quad (m \geq n).
\]

This shows that
\[
z_0 p'(z_0) = -\frac{m}{2} \left( \frac{a_0 + \frac{\beta^2}{a_0}}{a_0} \right) \quad (m \geq n).
\]

From the fact that \( z_0 p'(z_0) \) is a real number and \( p(z_0) \) is a pure imaginary number, we can put
\[
\frac{z_0 p'(z_0)}{p(z_0)} = il
\]
where \( l \) is a real number.
For the case $\beta > 0$, we have

$$l = \text{Im} \left( \frac{z_0 p'(z_0)}{p(z_0)} \right)$$
$$= \text{Im} \left( -z_0 p'(z_0) \frac{1}{\beta} i \right)$$
$$= \frac{m}{2} \left( a_0 + \frac{\beta^2}{a_0} \right)$$
$$\geq \frac{n}{2} \left( a_0 + \frac{\beta^2}{a_0} \right) \frac{1}{\beta}$$
$$= \frac{n}{2} \left( \frac{a_0}{\beta} + \frac{\beta}{a_0} \right) \geq n$$

and for the case $\beta < 0$, we get

$$l = \text{Im} \left( \frac{z_0 p'(z_0)}{p(z_0)} \right)$$
$$= \text{Im} \left(-z_0 p'(z_0) \frac{1}{\beta} i \right)$$
$$= \frac{m}{2} \left( a_0 + \frac{\beta^2}{a_0} \right)$$
$$\leq \frac{n}{2} \left( a_0 + \frac{\beta^2}{a_0} \right) \frac{1}{\beta}$$
$$= \frac{n}{2} \left( \frac{a_0}{\beta} + \frac{\beta}{a_0} \right) \leq -n.$$

This completes our proof.

Putting $a_0 = 1$ in Theorem 1, we have Corollary 1.

**Corollary 1.** Let $p(z) \in \mathcal{H}[1, n]$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\text{Re}(p(z)) > 0 \quad \text{for} \quad |z| < |z_0|,$$

$$\text{Re}(p(z_0)) = 0 \quad \text{and} \quad p(z_0) \neq 0.$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = il$$

where $l$ is a real and $|l| \geq n$. 
References


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