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Strongly starlikeness criteria for certain analytic functions

Yutaka Shimoda, and Shigeyoshi Owa

Abstract

Let $\mathcal{U}_3(\lambda)$ be the subcless of analytic functions $f(z)$ in the open unit disk $U$ which was introduced by S. Ponnusamy (Appl. Math. Lett. 24(2011), 381 - 385). For $f(z) \in \mathcal{U}_3(\lambda)$, some condition for the domain of $|z|$ such that $f(z)$ is strongly starlike of order $\gamma$ in $U$.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

\begin{equation}
  f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\end{equation}

that are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of $f(z)$ which are univalent in $U$.

Obradović and Ponnusamy [2] define the class $\mathcal{U}(\lambda)$ of $f(z) \in \mathcal{A}$ satisfying the condition

\begin{equation}
  \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 < \lambda (z \in U)
\end{equation}

for some real $\lambda > 0$.

The condition (1.2) is equivalent to

\[ |z^2 \left( \frac{1}{f(z)} - \frac{1}{z} \right)' | < \lambda \quad (z \in U). \]

Ponnusamy [3] introduces the class $\mathcal{U}_3(\lambda)$ of function $f(z) \in \mathcal{U}(\lambda)$ for which $a_3 - a_2^2 = 0$.

For some real $\gamma \in (0,1]$, a function $f(z) \in \mathcal{A}$ is called strongly starlike of order $\gamma$ if

\begin{equation}
  \left| \arg \frac{z f'(z)}{f(z)} \right| < \frac{\pi \gamma}{2} \quad (z \in U).
\end{equation}

We denote by $SS(\gamma)$ the set of all strongly starlike functions of order $\gamma$ in $U$.

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Ponnusamy [3] has shown the following theorem.

**Theorem 1.1** Let $f(z) \in \mathcal{U}_3(\lambda)$, $\gamma \in (0,1]$, and

$$
\lambda_*(\gamma, |a_2|) = \frac{-2(1 + 2\cos\frac{\pi\gamma}{2})|a_2| + 2\sin\frac{\pi\gamma}{2} \sqrt{5 + 4\cos\frac{\pi\gamma}{2} - 4|a_2|^2}}{5 + 4\cos\frac{\pi\gamma}{2}}.
$$

Then $f(z) \in \mathcal{S}\mathcal{S}(\gamma)$ for $0 < \lambda \leq \lambda_*(\gamma, |a_2|)$.

The aim of this paper is to derive a condition for the domain of $f(z) \in \mathcal{U}_3(\lambda)$ to be in the class $\mathcal{S}\mathcal{S}(\gamma)$.

## 2 Main Result

Suppose that $f \in \mathcal{U}_3(\lambda)$. Then a simple calculation shows that

$$
-z \left( \frac{z}{f(z)} \right)' + \left( \frac{z}{f(z)} \right) = \left( \frac{z}{f(z)} \right)^2 f'(z)
= 1 + A_3 z^3 + \cdots = 1 + \lambda w(z), \ w(z) \in \mathcal{B}_3,
$$

where $\mathcal{B}_3$ denotes the set of all analytic functions $w(z)$ in $\mathbb{U}$ such that $w(0) = w(0)' = w(0)'' = 0$, $w'''(0) \neq 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$. From (2.1), we easily have the following representation for $\frac{z}{f(z)}$:

$$
\frac{z}{f(z)} - 1 = -a_2 z - \lambda \int_0^1 \frac{w(tz)}{t^2} dt.
$$

Since $w(z) \in \mathcal{B}_3$, from the Schwarz lemma

$$
|w(z)| \leq |z|^3
$$

holds true. Thus, we have that

$$
\left| \frac{z}{f(z)} - 1 \right| \leq |z| \left( |a_2| + \frac{\lambda}{2} |z|^2 \right), \ z \in \mathbb{U}.
$$

We have the following result.

**Theorem 2.1**

If $f(z) \in \mathcal{U}_3(\lambda)$, then $f(z) \in \mathcal{S}\mathcal{S}(\gamma)$ for $|z| < \min\{1, r_0\}$, where $r_0 = \sqrt{R_0}$ for the positive root $R_0$ of the equation

$$
\lambda^2 \left( \frac{5}{4} + \cos\frac{\pi\gamma}{2} \right) X^3 + \lambda |a_2| \left( 1 + 2\cos\frac{\pi\gamma}{2} \right) X^2 + |a_2|^2 X + \cos^2\frac{\pi\gamma}{2} - 1 = 0.
$$
Proof

Suppose that \( f(z) \in U_3(\lambda) \). Then we can see from (2.1) and (2.3) that

\[
(2.6) \quad \left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| = \lambda |w(z)| \leq \lambda |z|^3.
\]

Therefore, it follows from (2.4) and (2.6) that

\[
(2.7) \quad \left| \arg \frac{zf'(z)}{f(z)} \right| \leq \left| \arg \left( \left( \frac{z}{f(z)} \right)^2 f'(z) \right) \right| + \left| \arg \frac{z}{f(z)} \right| \leq \arcsin(\lambda |z|^3) + \arcsin \left( |z| \left( |a_2| + \frac{\lambda}{2} |z|^2 \right) \right).
\]

Now, we have to find the range of \( |z| \) for \( f(z) \in \mathcal{S}\mathcal{S}(\gamma) \) such that

\[
(2.8) \quad \arcsin \left( \lambda |z|^3 \sqrt{1 - |z|^2 \left( |a_2| + \frac{\lambda}{2} |z|^2 \right)^2 + \sqrt{1 - \lambda^2 |z|^6}} \right) \left( |a_2| + \frac{\lambda}{2} |z|^2 \right) < \frac{\pi \gamma}{2},
\]

which is equivalent to

\[
(2.9) \quad \lambda |z|^3 \sqrt{1 - |z|^2 \left( |a_2| + \frac{\lambda}{2} |z|^2 \right)^2 + \sqrt{1 - \lambda^2 |z|^6}} \left( |a_2| + \frac{\lambda}{2} |z|^2 \right) < \sin \frac{\pi \gamma}{2}.
\]

Putting

\[
(2.10) \quad F(X) = \lambda^2 \left( \frac{5}{4} + \cos \frac{\pi \gamma}{2} \right) X^3 + \lambda |a_2| \left( 1 + 2 \cos \frac{\pi \gamma}{2} \right) X^2 + |a_2|^2 X - \sin^2 \frac{\pi \gamma}{2},
\]

and

\[
(2.11) \quad G(X) = \lambda^2 \left( \frac{5}{4} - \cos \frac{\pi \gamma}{2} \right) X^3 + \lambda |a_2| \left( 1 - 2 \cos \frac{\pi \gamma}{2} \right) X^2 + |a_2|^2 X - \sin^2 \frac{\pi \gamma}{2},
\]

(2.9) can be written as \( F(X)G(X) > 0 \) with \( X = |z|^2 \). Since \( F(0) < 0 \) and \( G(0) < 0 \), \( F(X)G(X) > 0 \) is equivalent to \( F(X) < 0 \) and \( G(X) < 0 \). Comparing the coefficients of \( F(X) \) and \( G(X) \), we easily find the inequality \( G(X) < F(X) \).

In order to find the condition of \( |z| \) such that \( f(z) \in U_3(\lambda) \) to be in \( \mathcal{S}\mathcal{S}(\gamma) \), we consider the condition for \( F(X) < 0 \).

Since

\[
F'(X) = 3\lambda^2 \left( \frac{5}{4} + \cos \frac{\pi \gamma}{2} \right) X^2 + 2\lambda |a_2| \left( 1 + 2 \cos \frac{\pi \gamma}{2} \right) X + |a_2|^2 > 0
\]

and \( \lim_{X \to \infty} F'(X) = \infty \),
$F(X)$ is an increasing function for $X$. Thus $F(X)$ has a positive root $R_0 > 0$. Therefore, for $|z| < \min\{1, R_0\}$, inequality (2.9) holds.

**Remark**
Substituting $X = 1$ and solving the equation (2.5) as the equation of $\lambda$, we have $\lambda_*(\gamma, |a_2|)$ of Theorem 1.1.

3  **Example**

We give an example which shows the existence of $r_0$ satisfying Theorem 2.1. Since $F(X)$ has a unique solution for $0 < X < 1$ if $F(1) > 0$, we consider a condition of $|a_2|$ for $F(1) > 0$. From

$$F(1) = \lambda^2 \left( \frac{5}{4} + \cos \frac{\pi \gamma}{2} \right) + \lambda |a_2| \left( 1 + 2 \cos \frac{\pi \gamma}{2} \right) + |a_2|^2 - \sin^2 \frac{\pi \gamma}{2} > 0,$$

we have

$$\left( |a_2| + \frac{\lambda \left( 1 + 2 \cos \frac{\pi \gamma}{2} \right)}{2} \right)^2 > \sin^2 \frac{\pi \gamma}{2} (1 - \lambda^2),$$

This gives us that

$$(3.1) \quad |a_2| > \sin \frac{\pi \gamma}{2} \sqrt{1 - \lambda^2} - \frac{\lambda \left( 1 + 2 \cos \frac{\pi \gamma}{2} \right)}{2} \quad (0 < \lambda < 1).$$

If $|a_2|$ satisfies the condition (3.1), $F(X)$ has a positive root for $0 < X < 1$.

Let us take $\lambda = \frac{1}{2}$ and $\gamma = \frac{2}{3}$. Then $|a_2| > \frac{1}{4}$ from (3.1). Thus, we may take $|a_2| = 1$ and

$$(3.2) \quad F(X) = 7X^3 + 16X^2 + 16X - 12.$$ 

Since $F(0) = -12 < 0$, and $F(1) = 27 > 0$, $F(X)$ has a real positive root $0 < X < 1$. Actually, the root $r_0$ of (3.2) satisfies $0.47605 < r_0 < 0.47615$.

**References**


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