<table>
<thead>
<tr>
<th>Title</th>
<th>Geometric properties of certain meromorphic functions (On Schwarzian Derivatives and Its Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Saitoh, Hitoshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2013), 1824: 86-90</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/194723">http://hdl.handle.net/2433/194723</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Geometric properties of certain meromorphic functions

Hitoshi Saitoh

Abstract

In this paper, we aim at investigating several geometric properties of the solutions of the following differential equations:

\[ w''(z) + a(z)w'(z) + b(z)w(z) = 0, \]

where the functions \( a(z) \) and \( b(z) \) are meromorphic in the punctured disk \( \mathbb{D} = \{z : 0 < |z| < 1\} \).

1 Introduction

Let \( \Sigma \) be the class of functions of the form

\[ f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \]

which are meromorphic in the punctured disk \( \mathbb{D} = \{z : 0 < |z| < 1\} \).

A function \( f(z) \in \Sigma \) is said to be meromorphic starlike of order \( \alpha \) in \( \mathbb{D} \) if it satisfies

\[ \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < -\alpha \quad (z \in \mathbb{D}) \]

for some \( \alpha \) (0 \leq \alpha < 1). We denoted by \( \Sigma S_0^*(\alpha) \) the subclass of \( \Sigma \) consisting of all such functions.

2 A class of bounded functions

We begin with the definition and lemma.

**Definition 1** Let \( \mathcal{H}_J \) be the class of complex functions \( h(s, t) \) satisfying:

(i) \( h(s, t) \) is continuous in a domain \( \mathbb{D} \subset \mathbb{C} \times \mathbb{C} \),
(ii) \( (0, 0) \in \mathbb{D} \) and \( |h(0, 0)| < J \) (\( J > 0 \)),
(iii) \( |h(Ke^{i\theta}, Ke^{i\theta})| \geq J \) when \( (Ke^{i\theta}, Ke^{i\theta}) \in \mathbb{D} \), \( \theta \) is real and \( K \geq J \).
Definition 2  Let $h \in \mathcal{H}_J$ with corresponding domain $\mathbb{D}$. We denote by $B_J(h)$ the class of functions $u(z) = u_1 z + u_2 z^2 + \cdots$ which are analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and satisfy

(i) $(u(z), zu'(z)) \in \mathbb{D},$

(ii) $|h(u(z), zu'(z))| < J \quad (z \in \Delta).$

Lemma 1 ([3])  Let $h \in \mathcal{H}_J$ and $b(z)$ be an analytic function in $\Delta$ with $|b(z)| < J$. If the differential equation

(2.1) $h(u(z), zu'(z)) = b(z)$

has a solution $u(z)$ analytic in $\Delta$, then $|u(z)| < J$.

Lemma 2 ([1])  If $f(z) \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathbb{D}$ and

(2.2) $-\text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} < 4 - \beta \quad (z \in \Delta),$

then

(2.3) $-\text{Re} \{z^2 f'(z)\} > \frac{1}{5 - 2\beta} \quad (z \in \Delta),$

that is, $f(z)$ is meromorphic close-to-convex of order $\frac{1}{5 - 2\beta}$, where $\frac{3}{2} \leqq \beta < 2$.

3  Main results

First, we prove

Theorem 1  Let $w(z), a(z) \in \Sigma$ and $b(z)$ are meromorphic in $\mathbb{D}$ with

(3.1) $\left| z^2 \left( b(z) - \frac{1}{2} a'(z) - \frac{1}{4} (a(z))^2 \right) \right| < \frac{1}{2} \quad (z \in \mathbb{D})$

and

$\text{Re} \{za(z)\} \geqq 2 + 2\alpha \quad (0 \leqq \alpha < 1).$

Also, let $w(z)$ be the solution of the following second order linear differential equation

(3.2) $w''(z) + a(z)w'(z) + b(z)w(z) = 0.$

Then $w(z)$ is meromorphic starlike of order $\alpha$.

Proof.  Put $w(z) = e^{-\frac{1}{2} \int a(\xi) d\xi} v(z)$. Then (3.2) leads to the normal form

(3.3) $v''(z) + \left( b(z) - \frac{1}{2} a'(z) - \frac{1}{4} (a(z))^2 \right) v(z) = 0.$
If we put

\[(3.4) \quad u(z) = \frac{zv'(z)}{v(z)} - \frac{1}{2} \quad (z \in \mathbb{D}),\]

then \(u(z)\) is analytic in \(\Delta\) and (3.3) becomes

\[(3.5) \quad (u(z))^2 + zu'(z) - \frac{1}{4} = -z^2 \left( b(z) - \frac{1}{2}a'(z) - \frac{1}{4}(a(z))^2 \right),\]

or equivalently

\[(3.6) \quad h(u(z), zu'(z)) = -z^2 \left( b(z) - \frac{1}{2}a'(z) - \frac{1}{4}(a(z))^2 \right),\]

where \(h(s, t) = s^2 + t - \frac{1}{4}\). It is easy to check \(h(s, t) \in \mathcal{H}_{\tau^1}\), that is

(i) \(h(s, t)\) is continuous in \(\mathbb{C} \times \mathbb{C}\),
(ii) \(|h(0, 0)| = \frac{1}{4} < \frac{1}{2}\),
(iii) \(\left|h\left(\frac{1}{2}e^{i\theta}, Ke^{i\theta}\right)\right| \geq \frac{1}{2} \quad (K \geq \frac{1}{2})\).

From assumption, we have

\[\left|-z^2 \left( b(z) - \frac{1}{2}a'(z) - \frac{1}{4}(a(z))^2 \right)\right| < \frac{1}{2} \quad (z \in \mathbb{D}).\]

By using Lemma 1, we have \(|u(z)| < \frac{1}{2} \quad (z \in \Delta)\). Therefore, we obtain

\[\left|\frac{zv'(z)}{v(z)} - \frac{1}{2}\right| < \frac{1}{2} \quad (z \in \Delta).\]

This implies

\[0 < \text{Re}\left\{\frac{zv'(z)}{v(z)}\right\} < 1 \quad (z \in \Delta).\]

From \(w(z) = e^{-\frac{1}{2} \int a(t) \, dt} v(z)\), we have

\[(3.7) \quad \exp\left(\frac{1}{2} \int a(\xi) \, d\xi\right) w(z) = v(z).\]

Logarithmically differentiating of (3.7) leads to

\[(3.8) \quad \frac{zw'(z)}{w(z)} = \frac{zv'(z)}{v(z)} - \frac{1}{2}za(z).\]

Combining (3.8) and \(\text{Re}\{za(z)\} \geq 2 + 2\alpha \quad (0 \leq \alpha < 1)\), we obtain

\[\text{Re}\left\{\frac{zw'(z)}{w(z)}\right\} = \text{Re}\left\{\frac{zv'(z)}{v(z)}\right\} - \frac{1}{2}\text{Re}\{za(z)\} < 1 - \frac{1}{2}(2 + 2\alpha) = -\alpha \quad (z \in \mathbb{D}),\]

that is, \(w(z)\) is meromorphic starlike of order \(\alpha\). \(\square\)
Example 1  In Theorem 1, let \( a(z) = \frac{2}{z} \) and \( b(z) = \frac{1}{2} \). The solution of
\[
(3.9) \quad w''(z) + \frac{2}{z}w'(z) + \frac{1}{2}w(z) = 0
\]
is given by \( w(z) = \frac{\cos \frac{\pi}{3}}{z} \). This solution \( w(z) \) is meromorphic starlike function.

Next, we prove

**Theorem 2**  Let \( w(z), Q(z) \in \Sigma \). We consider the following second order differential equation.
\[
(3.10) \quad w''(z) + Q(z)w(z) = 0 \quad (z \in \mathbb{D}).
\]
If
\[
\text{Re} \left\{ Q(z) \frac{zw(z)}{w'(z)} \right\} < 4 - \beta \quad (z \in \mathbb{D}),
\]
then we have
\[
-\text{Re}\{z^2w'(z)\} > \frac{1}{5 - 2\beta} \quad \left(\frac{3}{2} \leq \beta < 2\right).
\]

**Proof.** From (3.10), we have
\[
(3.11) \quad Q(z) \frac{zw(z)}{w'(z)} = -\frac{zw''(z)}{w'(z)}.
\]
Applying Lemma 2 to (3.11), we can prove Theorem 2. \( \square \)

Example 2  In Theorem 2, let \( Q(z) = -\frac{2}{z^2} \). A solution of
\[
 w''(z) - \frac{2}{z^2}w(z) = 0
\]
is given by \( w(z) = \frac{1}{z} + \frac{3}{50}z^2 \). Then
\[
\text{Re} \left\{ Q(z) \frac{zw(z)}{w'(z)} \right\} < 2.404 \cdots < \frac{5}{2}
\]
and
\[
-\text{Re}\{z^2w'(z)\} > 0.88 > \frac{1}{2}.
\]
Therefore, \( w(z) \) is meromorphic close-to-convex function.

Remark 1  Let \( MC(\alpha) \) be the subclass of \( \Sigma \) consisting of functions \( f(z) \) which satisfy
\[
(3.12) \quad -\text{Re}\{z^2f'(z)\} > \alpha \quad (z \in \Delta)
\]
for some \( \alpha \) \((0 \leq \alpha < 1)\). A function \( f(z) \in MC(\alpha) \) is meromorphic close-to-convex of order \( \alpha \) in \( \mathbb{D} \).
References


Hitoshi Saitoh
Department of Mathematics,
Gunma National College of Technology
Maebashi, Gunma 371-8530,
Japan
E-mail: saitoh@nat-gunma-ct.ac.jp