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ON THE REPRESENTATION AND THE RESIDUE OF CONCAVE FUNCTIONS

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ABSTRACT. In [2] we introduced several integral representation formulas for concave functions. Using those, we gave a general formula to describe the residue of concave functions with a pole at \( p \in (0, 1) \). In the present article we will present alternate versions of the formulas, as well as a shortcut for the calculation to obtain the range of the residue.

Key words: concave univalent functions, integral representations

1. INTRODUCTION

Let \( \mathbb{C} \) be the complex plane, \( \hat{\mathbb{C}} \) the Riemann sphere and \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) be the unit disk. A univalent function \( f : \mathbb{D} \rightarrow \hat{\mathbb{C}} \) is said to be concave, if \( f(\mathbb{D}) \) is concave, i.e. \( \mathbb{C} \setminus f(\mathbb{D}) \) is convex. Commonly there are several types of concave functions, which map \( \mathbb{D} \) conformally onto a simply connected, concave domain in \( \hat{\mathbb{C}} \):

1. meromorphic, univalent functions \( f \) with a simple pole at the origin and the normalization \( f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \), said to belong to the class \( \mathcal{C}0_0 \),
2. meromorphic, univalent functions \( f \) with a simple pole at the point \( p \in (0, 1) \) and the normalization \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), said to belong to the class \( \mathcal{C}0_p \) and
3. analytic, univalent functions \( f \) satisfying \( f(1) = \infty \) with the normalizations \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and an opening angle of \( f(\mathbb{D}) \) at \( \infty \) less or equal to \( \alpha \pi \) with \( \alpha \in (1, 2] \), said to belong to the class \( \mathcal{C}0(\alpha) \).

A detailed discussion of these classes has already been done in [2]. We therefore concentrate on the class \( \mathcal{C}0_p \) for the present article.

2. ALTERNATIVE FORMULAS

In [2] we introduced the following integral representation formula for functions of \( \mathcal{C}0_p \).

**Theorem 1.** [2] Let \( p \in (0, 1) \). For a meromorphic function \( f : \mathbb{D} \rightarrow \hat{\mathbb{C}} \) of class \( \mathcal{C}0_p \), there exists a function \( \varphi : \mathbb{D} \rightarrow \mathbb{D} \), holomorphic in \( \mathbb{D} \) with \( \varphi(p) = p \), such that the concave function can be represented as

\[
(1) \quad f'(z) = \frac{p^2}{(z-p)^2(1-zp)^2} \exp \int_0^z \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)} d\zeta
\]

for \( z \in \mathbb{D} \). Conversely, for any holomorphic function \( \varphi \) mapping \( \mathbb{D} \rightarrow \mathbb{D} \) with \( \varphi(p) = p \), there exists a concave function of class \( \mathcal{C}0_p \) described by (1).

However, a fixed point of the function \( \varphi \) at \( p \) is not very useful for further discussions. Using several transformations we obtain an alternate version of Theorem 1.

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Key words and phrases. Concave functions; Integral representations.
Corollary 2. Let \( p \in (0, 1) \). For a meromorphic function \( f : \mathbb{D} \to \hat{\mathbb{C}} \) of class \( Co_p \), there exists a function \( \Psi : \mathbb{D} \to \mathbb{D} \), holomorphic in \( \mathbb{D} \) with \( \Psi(0) = 0 \) such that the concave function can be represented as

\[
f'(z) = \frac{p^2}{(z - p)^2(1 - zp)^2} \exp \left( 2 \int_p^{1 - \frac{z}{px}} \frac{\Psi(\zeta)}{1 - \zeta \Psi(\zeta)} d\zeta \right)
\]

for \( z \in \mathbb{D} \). Conversely, for any holomorphic function \( \Psi \) mapping \( \mathbb{D} \to \mathbb{D} \) with \( \Psi(0) = 0 \), there exists a concave function of class \( Co_p \) described by (2).

Proof. Let \( p \in (0, 1) \) and \( z \in \mathbb{D} \). Applying the transformation \( \zeta = \frac{z - x}{px} \) and \( \Phi(x) = \varphi(\zeta) \) we obtain

\[
\int_0^z \frac{-2 \varphi(\zeta)}{1 - \zeta \varphi(\zeta)} d\zeta = \int_p^{1 - \frac{z}{px}} \frac{-2 \Phi(x)}{1 - \frac{x}{px} \Phi(x)} \cdot \frac{p^2 - 1}{(1 - px)^2} dx
\]

Here the function \( \Phi \) is holomorphic in \( \mathbb{D} \) with \( \Phi(0) = p \). Therefore there exists a function \( \Psi : \mathbb{D} \to \mathbb{D} \) holomorphic in \( \mathbb{D} \) with \( \Psi(0) = 0 \), such that \( \Phi(x) = \frac{p - \Psi(x)}{1 - p \Psi(x)} \). Then

\[
\int_0^z \frac{-2 \varphi(\zeta)}{1 - \zeta \varphi(\zeta)} d\zeta = \int_p^{1 - \frac{z}{px}} \frac{-2 (p - \Psi(\zeta))(p^2 - 1)}{(1 - px)^2 - (px - z)(1 - px)} dx
\]

Changing the variable inside the integration and replacing the integral in (1) leads to the statement.

The formula for the residue derived from the integral representation in [2] was given as follows.

Theorem 3. [2] Let \( f(z) \in Co_p \) be a concave function with a simple pole at some point \( p \in (0, 1) \). Then the residue of this function \( f \) can be described by some function \( \varphi : \mathbb{D} \to \mathbb{D} \), holomorphic in \( \mathbb{D} \) and \( \varphi(p) = p \), such that

\[
\text{Res}_p f = \frac{-p^2}{(1 - p^2)^2} \exp \int_0^p \frac{-2 \varphi(x)}{1 - x \varphi(x)} dx.
\]

Applying the alternative representation from Corollary 2, we obtain

Corollary 4. Let \( f(z) \in Co_p \) be a concave function with a simple pole at some point \( p \in (0, 1) \). Then the residue of this function \( f \) can be described by some function \( \Psi : \mathbb{D} \to \mathbb{D} \), holomorphic in \( \mathbb{D} \) and \( \Psi(0) = 0 \), such that

\[
\text{Res}_p f = \frac{-p^2}{(1 - p^2)^2} \exp 2 \int_0^p \frac{\Psi(x)}{1 - x \Psi(x)} - \frac{p}{1 - px} dx.
\]
The advantage of Corollary 4 over the original presentation is the fixed point of $\Psi$ at the origin. This provides much easier means for the construction, than a fixed point at $p$. Furthermore, the Schwarz Lemma can be applied directly without any complicated analysis, giving a way for the estimate of special values. We will show an application in the next section.

3. RANGE OF THE RESIDUE

Wirths proved the following statement in [3] using the inequality

$$\left| \frac{1}{f(z)} - \frac{1}{z} + \frac{1+p^2}{p} \right| < 1$$

provided by Miller in [1].

**Theorem 5.** [3] Let $p \in (0, 1)$. For $a \in \mathbb{C}$ there exists a function $f \in C_0$ such that $a = \text{Res}_p f$ if and only if

$$|a + \frac{p^2}{1-p^4}| \leq \frac{p^4}{1-p^4}.$$  

Let $\theta \in [0, 2\pi)$. A function $f \in C_0$ has the residue

$$a = -\frac{p^2}{1-p^4} + e^{i\theta} \frac{p^4}{1-p^4}$$

if and only if

$$f_\theta(z) = \frac{z - \frac{p}{1+p^4} (1 + e^{i\theta}) z^2}{(1 - \frac{z}{p}) (1 - pz)}.$$  

The established representation formula for the residue can be used for a different approach of the same statement as described in [2]. For the present discussion we will use Corollary 4, which provides a shortcut for the proof. We also present some details, omitted in [2].

**Proof.** Let $p \in (0, 1)$ and $\Psi : \mathbb{D} \to \mathbb{D}$ be holomorphic in $\mathbb{D}$ with fixed point at the origin. For $a = \text{Res}_p f$ with $f \in C_0$, we obtain with the use of Corollary 4

$$|a + \frac{p^2}{1-p^4}| = |\frac{p^2}{1-p^4}| \exp \left( \frac{2}{1-p^4} \exp \left( 2 \int_0^p \frac{\Psi(x)}{1-x\Psi(x)} - \frac{p}{1-px} dx \right) - 1 \right).$$

Some basic calculations yield

$$\frac{1+p^2}{1-p^2} = \exp \left( \frac{1}{2} \log \frac{1+p^2}{1-p^2} \right) = \exp \int_0^p \frac{2p}{1-p^2x^2} dx$$

and therefore

$$|a + \frac{p^2}{1-p^4}| = \frac{p^2}{1-p^4} \exp \int_0^p \frac{2}{1-p^2x^2} \left( \frac{p}{1-p^2x^2} - \frac{p}{1-px} + \frac{\Psi(x)}{1-x\Psi(x)} \right) dx - 1 \right|$$

$$= \frac{p^2}{1-p^4} \exp \int_0^p \frac{2}{1-p^2x^2} \left( \frac{\Psi(x) - p^2x}{(1-x\Psi(x))(1-p^2x^2)} dx - 1 \right).$$

From the triangle inequality, we know that

$$|e^w - 1| = \left| \sum_{n=1}^\infty \frac{w^n}{n!} \right| \leq \sum_{n=1}^\infty \frac{|w|^n}{n!} = e^{|w|} - 1.$$
Hence
\[ |a + \frac{p^2}{1-p^4}| \leq \frac{p^2}{1-p^4} \left( \exp \int_0^p \frac{\Psi(x) - p^2 x}{(1-x\Psi(x))(1-p^2 x^2)} \, dx - 1 \right). \]

Due to the fixed point at the origin, we can apply the Schwarz Lemma and have $|\Psi(x)| \leq x$ for $0 < x < p$. Furthermore, since $|\frac{w - p^2 x}{1-xw}| \leq \frac{(1-p^2)x}{1-x^2}$ for $|w| \leq x$, we have

\[ |\frac{\Psi(x) - p^2 x}{1-x\Psi(x)}| \leq \frac{(1-p^2)x}{1-x^2}. \tag{7} \]

Using the above, we finally obtain
\[ |a + \frac{p^2}{1-p^4}| \leq \frac{p^2}{1-p^4} \left( \exp \int_0^p \frac{(1-p^2)x}{(1-x^2)(1-p^2 x^2)} \, dx - 1 \right) \]
\[ = \frac{p^2}{1-p^4} \left( \exp(\log(1+p^2)) - 1 \right) \]
\[ = \frac{p^4}{1-p^4}. \]

The rest of the proof goes according to the way described in [2].

\[ \square \]

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