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An extreme function for a certain class of analytic functions

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Abstract

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$. Furthermore, the subclass $\mathcal{B}$ of $\mathcal{A}$ concerned with the class of uniformly convex functions or the class $\mathcal{S}_p$ is defined. By virtue of some properties of uniformly convex functions and the class $\mathcal{S}_p$, an extreme function of the class $\mathcal{B}$ and its power series are considered.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. A function $f(z)\in\mathcal{A}$ is said to be in the class of uniformly convex (or starlike) functions denoted by $\mathcal{UCV}$ (or $\mathcal{UST}$) if $f(z)$ is convex (or starlike) in $\mathbb{U}$ and maps every circle or circular arc in $\mathbb{U}$ with center at ζ in $\mathbb{U}$ onto the convex arc (or the starlike arc) with respect to $f(\zeta)$. These classes are introduced by Goodman [1] (see also [2]). For the class $\mathcal{UCV}$, it is defined as the one variable characterization by Rønning [4] and [5], that is, a function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{UCV}$ if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\left|\frac{zf''(z)}{f'(z)}\right| \qquad (z\in\mathbb{U}).$$

It is independently studied by Ma and Minda [3]. Further, a function $f(z)\in\mathcal{A}$ is said to be the corresponding class denoted by $\mathcal{S}_p$ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\left|\frac{zf'(z)}{f(z)}-1\right| \qquad (z\in\mathbb{U}).$$

This class $\mathcal{S}_p$ was introduced by Rønning [4]. We easily know that the relation $f(z)\in\mathcal{UCV}$ if and only if $zf'(z)\in\mathcal{S}_p$. In view of these classes, we introduce the subclass $\mathcal{B}$ of $\mathcal{A}$ consisting

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of all functions $f(z)$ which satisfy

$$\operatorname{Re}\left(\frac{z}{f(z)}\right) > \left|\frac{z}{f(z)} - 1\right| \qquad (z \in \mathbb{U}).$$

We try to derive some properties of functions $f(z)$ belonging to the class $\mathcal{B}$.

Remark 1.1. For $f(z) \in \mathcal{B}$, we write $w(z) = \dfrac{f(z)}{z} = u + iv$, then w lies in the domain which is the part of the complex plane which contains $w = 1$ and is bounded by a kind of teardrop-shape domain such that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 < 0.$$

2 An extreme function for the class $\mathcal{B}$

In this section, we would like to exhibit an extreme function of the class $\mathcal{B}$ and its power series. For our results, we need to recall here some properties of the class $\mathcal{S}_p$.

Lemma 2.1. (Rønning [4]). *The extremal function $f(z)$ for the class $\mathcal{S}_p$ is given by*

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

By using the expansion of logarithmic part of $\dfrac{zf'(z)}{f(z)}$ in Lemma 2.1, we get

Lemma 2.2. (Rønning [4]). *The power series of $\dfrac{zf'(z)}{f(z)}$ is following*

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \\ &= 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\,\frac{1}{2n+1-2k}\right)z^n.\end{aligned}$$

From Remark 1.1 and Lemma 2.1, we have the first result for the class $\mathcal{B}$.

Theorem 2.1. *The extreme function $f(z)$ for the class $\mathcal{B}$ is given by*

$$f(z) = \frac{z}{1 + \dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}.$$

Proof. Let us consider the function $\frac{f(z)}{z}$ as given by

$$\frac{f(z)}{z}=\frac{1}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}.$$

It sufficies to show that $\frac{f(z)}{z}$ maps U onto the interior of the domain such that

$$u^4-2u^3+2u^2v^2-2uv^2+v^4+v^2<0,$$

implying that $\frac{f(z)}{z}$ maps the unit circle onto the boundary of the domain. Taking $z=e^{i\theta}$, we obtain that

$$\begin{aligned}\frac{1}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2} &= \frac{1}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+e^{i\frac{\theta}{2}}}{1-e^{i\frac{\theta}{2}}}\right)\right)^2}\\ &= \frac{1}{1+\frac{2}{\pi^2}\left(\log i-\log\left(\tan\frac{\theta}{4}\right)\right)^2}\\ &= \frac{1}{\frac{1}{2}+\frac{2}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2-i\frac{2}{\pi}\log\left(\tan\frac{\theta}{4}\right)}\\ &= \frac{\frac{1}{2}+\frac{2}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2}{\frac{1}{4}+\frac{6}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2+\frac{4}{\pi^4}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^4}\\ &\quad +i\frac{\frac{2}{\pi}\log\left(\tan\frac{\theta}{4}\right)}{\frac{1}{4}+\frac{6}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2+\frac{4}{\pi^4}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^4}.\end{aligned}$$

Writing $\frac{f(z)}{z}=u+iv$, we see that

$$\log\left(\tan\frac{\theta}{4}\right)=\frac{\pi(u\pm\sqrt{u^2-v^2})}{2v}.$$

Thus we have

$$v=\frac{\frac{2}{\pi}\log\left(\tan\frac{\theta}{4}\right)}{\frac{1}{4}+\frac{6}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2+\frac{4}{\pi^4}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^4}$$

$$= \frac{\frac{2}{\pi}\frac{\pi(u \pm \sqrt{u^2-v^2})}{2v}}{\frac{1}{4}+\frac{6}{\pi^2}\left(\frac{\pi(u \pm \sqrt{u^2-v^2})}{2v}\right)^2+\frac{4}{\pi^4}\left(\frac{\pi(u \pm \sqrt{u^2-v^2})}{2v}\right)^4}.$$

Therefore, we arrive that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 = 0.$$

This completes the proof of the theorem. □

Considering the power series of the function $f(z)$ in Theorem 2.1, we derive

Theorem 2.2. *The power series of the extreme function for the class $\mathcal{B}$ is given by*

$$f(z) = \frac{z}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

$$= z + \sum_{n=2}^{\infty}\sum_{p=1}^{n-1}(-1)^p\left(\frac{8}{\pi^2}\right)^p\sum_{m_1=1}^{n-p}\left(\sum_{k=1}^{m_1}\frac{1}{2k-1}\frac{1}{2m_1+1-2k}\right)$$

$$\times\sum_{m_2=1}^{n+1-p-m_1}\left(\sum_{k=1}^{m_2}\frac{1}{2k-1}\frac{1}{2m_2+1-2k}\right)\times\cdots$$

$$\times\sum_{m_{p-1}=1}^{n-2-A_{p-2}}\left(\sum_{k=1}^{m_{p-1}}\frac{1}{2k-1}\frac{1}{2m_{p-1}+1-2k}\right)\left(\sum_{k=1}^{n-1-A_{p-1}}\frac{1}{2k-1}\frac{1}{2(n-A_{p-1})-1-2k}\right)z^n,$$

where $A_p = \sum_{l=1}^{p} m_l$.

Proof. Let us suppose that

$$\frac{f(z)}{z} = \frac{1}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

as the proof of Theorem 2.1. Then from Lemma 2.2, we have

$$\frac{f(z)}{z} = \frac{1}{1+\frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\frac{1}{2n+1-2k}\right)z^n}$$

$$= 1 - \frac{8}{\pi^2}\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\frac{1}{2n+1-2k}\right)z^n\right)$$

$$+\left(\frac{8}{\pi^2}\right)^2\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\,\frac{1}{2n+1-2k}\right)z^n\right)^2$$

$$-\left(\frac{8}{\pi^2}\right)^3\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\,\frac{1}{2n+1-2k}\right)z^n\right)^3+\cdots$$

$$+(-1)^n\left(\frac{8}{\pi^2}\right)^n\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\,\frac{1}{2n+1-2k}\right)z^n\right)^n+\cdots$$

$$=1-\frac{8}{\pi^2}\sum_{k=1}^{1}\frac{1}{2k-1}\,\frac{1}{3-2k}z$$

$$+\left\{-\frac{8}{\pi^2}\sum_{k=1}^{2}\frac{1}{2k-1}\,\frac{1}{5-2k}+\left(\frac{8}{\pi^2}\right)^2\left(\sum_{k=1}^{1}\frac{1}{2k-1}\,\frac{1}{3-2k}\right)\left(\sum_{k=1}^{1}\frac{1}{2k-1}\,\frac{1}{3-2k}\right)\right\}z^2$$

$$+\left[-\frac{8}{\pi^2}\sum_{k=1}^{3}\frac{1}{2k-1}\,\frac{1}{7-2k}+\left\{\left(\frac{8}{\pi^2}\right)^2\left(\sum_{k=1}^{1}\frac{1}{2k-1}\,\frac{1}{3-2k}\right)\left(\sum_{k=1}^{2}\frac{1}{2k-1}\,\frac{1}{5-2k}\right)\right.\right.$$

$$\left.\left.+\left(\sum_{k=1}^{2}\frac{1}{2k-1}\,\frac{1}{5-2k}\right)\left(\sum_{k=1}^{1}\frac{1}{2k-1}\,\frac{1}{3-2k}\right)\right\}-\left(\frac{8}{\pi^2}\right)^3\left(\sum_{k=1}^{1}\frac{1}{2k-1}\,\frac{1}{3-2k}\right)^3\right]z^3$$

$$+\cdots$$

$$+\left\{-\frac{8}{\pi^2}\sum_{k=1}^{n}\frac{1}{2k-1}\,\frac{1}{2n+1-2k}\right.$$

$$+\left(\frac{8}{\pi^2}\right)^2\sum_{m_1=1}^{n-1}\left(\sum_{k=1}^{m_1}\frac{1}{2k-1}\,\frac{1}{2m_1+1-2k}\right)\left(\sum_{k=1}^{n-m_1}\frac{1}{2k-1}\,\frac{1}{2n-2m_1+1-2k}\right)$$

$$-\left(\frac{8}{\pi^2}\right)^3\sum_{m_1=1}^{n-2}\left(\sum_{k=1}^{m_1}\frac{1}{2k-1}\frac{1}{2m_1+1-2k}\right)\sum_{m_2=1}^{n-1-m_1}\left(\sum_{k=1}^{m_2}\frac{1}{2k-1}\,\frac{1}{2m_2+1-2k}\right)$$

$$\times\left(\sum_{k=1}^{n-m_1-m_2}\frac{1}{2k-1}\,\frac{1}{2n-2m_1-2m_2+1-2k}\right)$$

$$+\cdots$$

$$+(-1)^p\left(\frac{8}{\pi^2}\right)^p\sum_{m_1=1}^{n+1-p}\left(\sum_{k=1}^{m_1}\frac{1}{2k-1}\,\frac{1}{2m_1+1-2k}\right)\sum_{m_2=1}^{n+2-p-m_1}\left(\sum_{k=1}^{m_2}\frac{1}{2k-1}\,\frac{1}{2m_2+1-2k}\right)$$

$$\times\cdots\times\sum_{m_{p-1}=1}^{n-1-A_{p-2}}\left(\sum_{k=1}^{m_{p-1}}\frac{1}{2k-1}\,\frac{1}{2m_{p-1}+1-2k}\right)\left(\sum_{k=1}^{n-A_{p-1}}\frac{1}{2k-1}\,\frac{1}{2(n-A_{p-1})+1-2k}\right)$$

$$\left.+\cdots+\left(\sum_{k=1}^{1}\frac{1}{2k-1}\,\frac{1}{3-2k}\right)^n\right\}z^n$$

$$+\cdots\qquad\left(A_p=\sum_{l=1}^{p}m_l\right)$$

$$= 1 + \sum_{n=2}^{\infty} \sum_{p=1}^{n} (-1)^p \left(\frac{8}{\pi^2}\right)^p \sum_{m_1=1}^{n+1-p} \left(\sum_{k=1}^{m_1} \frac{1}{2k-1} \frac{1}{2m_1+1-2k} \right)$$

$$\times \sum_{m_2=1}^{n+2-p-m_1} \left(\sum_{k=1}^{m_2} \frac{1}{2k-1} \frac{1}{2m_2+1-2k} \right) \times \cdots$$

$$\times \sum_{m_{p-1}=1}^{n-1-A_{p-2}} \left(\sum_{k=1}^{m_{p-1}} \frac{1}{2k-1} \frac{1}{2m_{p-1}+1-2k} \right) \left(\sum_{k=1}^{n-A_{p-1}} \frac{1}{2k-1} \frac{1}{2(n-A_{p-1})+1-2k} \right) z^n.$$

This completes the proof of the theorem. □

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