An extreme function for a certain class of analytic functions

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Abstract
Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$. Furthermore, the subclass $\mathcal{B}$ of $\mathcal{A}$ concerned with the class of uniformly convex functions or the class $\mathcal{S}_p$ is defined. By virtue of some properties of uniformly convex functions and the class $\mathcal{S}_p$, an extreme function of the class $\mathcal{B}$ and its power series are considered.

1 Introduction
Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be in the class of uniformly convex (or starlike) functions denoted by $\mathcal{UCV}$ (or $\mathcal{UST}$) if $f(z)$ is convex (or starlike) in $\mathbb{U}$ and maps every circle or circular arc in $\mathbb{U}$ with center at $\zeta$ in $\mathbb{U}$ onto the convex arc (or the starlike arc) with respect to $f(\zeta)$. These classes are introduced by Goodman [1] (see also [2]). For the class $\mathcal{UCV}$, it is defined as the one variable characterization by Renning [4] and [5], that is, a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{UCV}$ if it satisfies

$$\text{Re} \left\{1 + \frac{zf'''(z)}{f''(z)}\right\} > \left|\frac{zf'''(z)}{f''(z)}\right| (z \in \mathbb{U}).$$

It is independently studied by Ma and Minda [3]. Further, a function $f(z) \in \mathcal{A}$ is said to be the corresponding class denoted by $\mathcal{S}_p$ if it satisfies

$$\text{Re} \left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right| (z \in \mathbb{U}).$$

This class $\mathcal{S}_p$ was introduced by Renning [4]. We easily know that the relation $f(z) \in \mathcal{UCV}$ if and only if $zf'(z) \in \mathcal{S}_p$. In view of these classes, we introduce the subclass $\mathcal{B}$ of $\mathcal{A}$ consisting

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of all functions $f(z)$ which satisfy

$$\text{Re} \left( \frac{z}{f(z)} \right) > \left| \frac{z}{f(z)} - 1 \right| \quad (z \in \mathbb{U}).$$

We try to derive some properties of functions $f(z)$ belonging to the class $B$.

**Remark 1.1.** For $f(z) \in B$, we write $w(z) = \frac{f(z)}{z} = u + iv$, then $w$ lies in the domain which is the part of the complex plane which contains $w = 1$ and is bounded by a kind of teardrop-shape domain such that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 < 0.$$

2 An extreme function for the class $B$

In this section, we would like to exhibit an extreme function of the class $B$ and its power series. For our results, we need to recall here some properties of the class $S_p$.

**Lemma 2.1.** (Rønning [4]). The extremal function $f(z)$ for the class $S_p$ is given by

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2.$$

By using the expansion of logarithmic part of $\frac{zf'(z)}{f(z)}$ in Lemma 2.1, we get

**Lemma 2.2.** (Rønning [4]). The power series of $\frac{zf'(z)}{f(z)}$ is following

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2$$

$$= 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n.$$

From Remark 1.1 and Lemma 2.1, we have the first result for the class $B$.

**Theorem 2.1.** The extreme function $f(z)$ for the class $B$ is given by

$$f(z) = \frac{z}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2}.$$
Proof. Let us consider the function \( \frac{f(z)}{z} \) as given by

\[
\frac{f(z)}{z} = \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2}.
\]

It suffices to show that \( \frac{f(z)}{z} \) maps \( \mathbb{U} \) onto the interior of the domain such that

\[
u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 < 0,
\]

implying that \( \frac{f(z)}{z} \) maps the unit circle onto the boundary of the domain. Taking \( z = e^{i\theta} \), we obtain that

\[
\frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right) \right)^2} = \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right) \right)^2}
\]

\[
= \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right) \right)^2} = \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right) \right)^2}
\]

\[
= \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right) \right)^2} = \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right) \right)^2}
\]

Writing \( \frac{f(z)}{z} = u + iv \), we see that

\[
\log \left( \frac{\tan \theta}{4} \right) = \frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v}.
\]

Thus we have

\[
v = \frac{\frac{2}{\pi} \log \left( \frac{\tan \theta}{4} \right)}{\frac{1}{4} + \frac{6}{\pi^2} \left( \log \left( \frac{\tan \theta}{4} \right) \right)^2 + \frac{4}{\pi^4} \left( \log \left( \frac{\tan \theta}{4} \right) \right)^4}.
\]
\[
\frac{2 \pi (u \pm \sqrt{u^2 - v^2})}{\pi} \cdot \frac{6}{2v} \left( \frac{\pi (u \pm \sqrt{u^2 - v^2})}{2v} \right)^2 + \frac{4}{\pi^4} \left( \frac{\pi (u \pm \sqrt{u^2 - v^2})}{2v} \right)^4.
\]

Therefore, we arrive that
\[u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 = 0.\]

This completes the proof of the theorem. \(\square\)

Considering the power series of the function \(f(z)\) in Theorem 2.1, we derive

**Theorem 2.2.** The power series of the extreme function for the class \(\mathcal{B}\) is given by

\[
f(z) = \frac{z}{1 + \frac{8}{\pi^2} \left( \sum_{n=1}^\infty \left( \sum_{k=1}^{n} \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n \right)}
\]

\[
= z + \sum_{n=2}^{\infty} \sum_{p=1}^{n-1} (-1)^p \left( \frac{8}{\pi^2} \right)^p \sum_{m_1=1}^{n-p} \left( \sum_{k=1}^{m_1} \frac{1}{2k-1} \frac{1}{2m_1+1-2k} \right) \times \cdots \times \sum_{m_{p-1}=1}^{n-2-A_{p-2}} \left( \sum_{k=1}^{m_{p-1}} \frac{1}{2k-1} \frac{1}{2m_{p-1}+1-2k} \right) \sum_{k=1}^{n-A_{p-1}} \frac{1}{2k-1} \frac{1}{2(n-A_{p-i})-1-2k} z^n,
\]

where \(A_p = \sum_{i=1}^{p} m_i\).

**Proof.** Let us suppose that
\[
f(z) = \frac{1}{z} \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1\pm \sqrt{z}}{1-\sqrt{z}} \right) \right)^2}
\]
as the proof of Theorem 2.1. Then from Lemma 2.2, we have
\[
f(z) = \frac{1}{z} - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n
\]

\[
= 1 - \frac{8}{\pi^2} \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n \right)
\]
\[
\left( \frac{8}{\pi^2} \right)^2 \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n \right)^2 \\
- \left( \frac{8}{\pi^2} \right)^3 \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n \right)^3 + \cdots \\
+ (-1)^n \left( \frac{8}{\pi^2} \right)^n \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) z^n \right)^n + \cdots
\]

\[
= 1 - \frac{8}{\pi^2} \sum_{k=1}^{1} \frac{1}{2k-1} \frac{1}{3-2k} z^1 \\
+ \left\{ -\frac{8}{\pi^2} \sum_{k=1}^{2} \frac{1}{2k-1} \frac{1}{5-2k} + \left( \frac{8}{\pi^2} \right)^2 \left( \sum_{k=1}^{1} \frac{1}{2k-1} \frac{1}{3-2k} \right) \left( \sum_{k=1}^{1} \frac{1}{2k-1} \frac{1}{3-2k} \right) \right\} z^2 \\
+ \left\{ -\frac{8}{\pi^2} \sum_{k=1}^{3} \frac{1}{2k-1} \frac{1}{7-2k} + \left\{ \left( \frac{8}{\pi^2} \right)^2 \left( \sum_{k=1}^{1} \frac{1}{2k-1} \frac{1}{3-2k} \right) \left( \sum_{k=1}^{1} \frac{1}{2k-1} \frac{1}{3-2k} \right) \right\} z^3 \\
+ \cdots \\
+ \left\{ -\frac{8}{\pi^2} \sum_{k=1}^{n} \frac{1}{2k-1} \frac{1}{2n+1-2k} \right\} + \left\{ \left( \frac{8}{\pi^2} \right)^2 \left( \sum_{k=1}^{m_1} \frac{1}{2k-1} \frac{1}{2m_1+1-2k} \right) \left( \sum_{n-m_1}^{n} \frac{1}{2k-1} \frac{1}{2n+1-2k} \right) \right\} z^n + \cdots \\
+ (-1)^p \left( \frac{8}{\pi^2} \right)^{p+1} \left( \sum_{m_1=1}^{m_1} \frac{1}{2k-1} \frac{1}{2m_1+1-2k} \right) \left( \sum_{m_2=1}^{n-m_1} \frac{1}{2k-1} \frac{1}{2m_2+1-2k} \right) + \cdots \\
+ \left( \frac{8}{\pi^2} \right)^p \left( \sum_{m_1=1}^{m_1} \frac{1}{2k-1} \frac{1}{2m_1+1-2k} \right) \left( \sum_{m_2=1}^{n-m_1} \frac{1}{2k-1} \frac{1}{2m_2+1-2k} \right) \left( \sum_{m_3=1}^{n-m_1-m_2} \frac{1}{2k-1} \frac{1}{2m_3+1-2k} \right) \left( \sum_{m_4=1}^{n-m_1-m_2-m_3} \frac{1}{2k-1} \frac{1}{2m_4+1-2k} \right) \cdots \cdots \cdots + \left( \frac{8}{\pi^2} \right)^n \left( \sum_{k=1}^{1} \frac{1}{2k-1} \frac{1}{3-2k} \right)^n z^n \\
+ \cdots
\]

\[A_p = \sum_{i=1}^{p} m_i\]
\[= 1 + \sum_{n=2}^{\infty} \sum_{p=1}^{n} (-1)^{p} \left( \frac{8}{\pi^2} \right)^{p} \sum_{m_1=1}^{n+1-p} \left( \sum_{k=1}^{m_1} \frac{1}{2k-1} \frac{1}{2m_1+1-2k} \right) \times \sum_{m_1=1}^{n+2-p-m_1} \left( \sum_{k=1}^{m_2} \frac{1}{2k-1} \frac{1}{2m_2+1-2k} \right) \times \cdots \times \sum_{m_{p-1}=1}^{n-A_{p-2}} \left( \sum_{k=1}^{m_{p-1}} \frac{1}{2k-1} \frac{1}{2m_{p-1}+1-2k} \right) \left( \sum_{k=1}^{n-A_{p-1}} \frac{1}{2k-1} \frac{1}{2(n-A_{p-1})+1-2k} \right) z^n. \]

This completes the proof of the theorem. \(\square\)

References


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