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Solutions to Some Homogeneous Special Ordinary Differential Equation by Means of N- Fractional Calculus Operator

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Abstract

In this article, the solutions to the homogeneous special ordinary differential equation

$$\varphi_2+\varphi_1\cdot\frac{a}{z}+\varphi\cdot b=0 \qquad (z\neq 0)$$

$$(\varphi_\nu=d^\nu\varphi/dz^\nu \text{ for } \nu>0\,,\ \varphi_0=\varphi=\varphi(z)\ )$$

are discussed by means of N-fractional calculus operator (NFCO- Method).

By our method, some particular solutions to the above equations are given as below for example, in fractional differintegrated forms.

Group I.

(i) $$\varphi=Ke^{i\sqrt{b}z}(e^{-2i\sqrt{b}z}\cdot z^{-\frac{a}{2}})_{\frac{a}{2}-1}\equiv\varphi_{[1](a,b)} \quad \text{(denote)}$$

and

(ii) $$\varphi=Ke^{i\sqrt{b}z}(z^{-\frac{a}{2}}\cdot e^{-2i\sqrt{b}z})_{\frac{a}{2}-1}\equiv\varphi_{[2](a,b)} \quad .$$

And the familiar forms are

$$\varphi_{[1](a,b)}=K(-2i\sqrt{b})^{\frac{a}{2}}z^{-\frac{a}{2}}e^{-i\sqrt{b}z}\,{}_2F_0(1-\tfrac{a}{2},\tfrac{a}{2}\,;\tfrac{i}{2\sqrt{b}z}) \quad (|i/2\sqrt{b}z|<1)$$

and

$$\varphi_{[2](a,b)}=K(-e^{-i\pi\frac{a}{2}}\frac{\Gamma(a-1)}{\Gamma(\frac{a}{2})}z^{1-a}e^{-i\sqrt{b}z}\,{}_1F_1(1-\tfrac{a}{2};2-a\,;2i\sqrt{b}z)$$

$$(|\Gamma(a-1-k)/\Gamma(\tfrac{a}{2})|<\infty,\ |2i\sqrt{b}z|<1)$$

respectively.

Where ${}_pF_q(\cdots)$ is the generalized Gauss hypergeometric function.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D=\{D_-, D_+\}$, $C=\{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_-, D_+ be a domain surrounded by C_+.

(Here D contains the points over the curve C .)

Moreover, let $f=f(z)$ be a regular function in D $(z \in D)$,

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\,d\zeta \qquad (\nu \notin Z^-), \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \qquad (m \in Z^+), \tag{2}$$

where $-\pi \le \arg(\zeta - z) \le \pi$ for C_-, $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+,

$\zeta \ne z$, $z \in C$, $\nu \in R$, Γ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z, of the function f, if $|(f)_\nu| < \infty$.

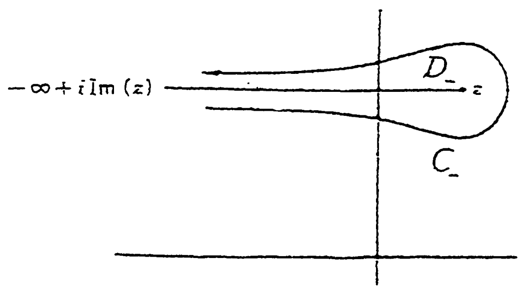

Fig. 1.

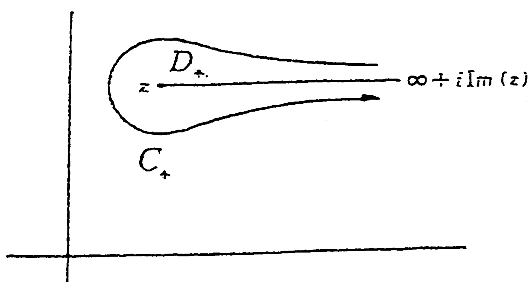

Fig. 2.

Notice that (1) is reduced to Goursat's integral for $\nu = n$ $(\in Z^+)$ and is reduced to the famous Cauchy's integral for $\nu = 0$. That is, (1) is an extention of Cauchy's integral and of Goursat's one, conversely Cauchy's and Goursat's ones are special cases of (1).

Moreover, notice that (1) is the representation which unifies the derivatives and integrations.

(II) On the fractional calculus operator N^{ν} [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^{ν} be*

$$N^{\nu}=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_{C}\frac{d\zeta}{(\zeta-z)^{\nu+1}}\right)\qquad(\nu\notin Z),\quad[\textit{Refer to}(1)]\tag{3}$$

with

$$N^{-m}=\lim_{\nu\to -m}N^{\nu}\qquad(m\in Z^{+}),\tag{4}$$

and define the binary operation $\circ$ *as*

$$N^{\beta}\circ N^{\alpha}f=N^{\beta}N^{\alpha}f=N^{\beta}(N^{\alpha}f)\qquad(\alpha,\beta\in R),\tag{5}$$

then the set

$$\left\{N^{\nu}\right\}=\left\{N^{\nu}\mid \nu\in R\right\}\tag{6}$$

is an Abelian product group (having continuous index ν *) which has the inverse transform operator* $(N^{\nu})^{-1}=N^{-\nu}$ *to the fractional calculus operator* N^{ν} *, for the function* f *such that* $f\in F=\left\{f;\ 0\neq|f_{\nu}|<\infty,\nu\in R\right\}$, *where* $f=f(z)$ *and* $z\in C$. (*vis.* $-\infty<\nu<\infty$).

(For our convenience, we call $N^{\beta}\circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an* " *Action product group which has continuous index* ν " *for the set of* F . (*F.O.G.* ; Fractional calculus operator group) [3]

Theorem C. *Let be*

$$S:=\{\pm N^{\nu}\}\cup\{0\}=\{N^{\nu}\}\cup\{-N^{\nu}\}\cup\{0\}\quad(\nu\in R)\,.\tag{7}$$

Then the set S is a commutative ring for the function $f\in F$ *, when the identity*

$$N^{\alpha}+N^{\beta}=N^{\gamma}\qquad(N^{\alpha},N^{\beta},N^{\gamma}\in S)\,,\tag{8}$$

holds. [5]

(III) Lemma. We have [1]

(i) $$((z-c)^{b})_{\alpha}=e^{-i\pi\alpha}\frac{\Gamma(\alpha-b)}{\Gamma(-b)}(z-c)^{b-\alpha}\qquad\left(\left|\frac{\Gamma(\alpha-b)}{\Gamma(-b)}\right|<\infty\right),$$

(ii) $$(\log(z-c))_{\alpha}=-e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha}\qquad\left(|\Gamma(\alpha)|<\infty\right),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha}=-e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c)\qquad\left(|\Gamma(\alpha)|<\infty\right),$$

where $z-c\neq0$ for (i) and $z-c\neq0,1$ for (ii), (iii) ,

(iv) $$(u\cdot v)_{\alpha}:=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}u_{\alpha-k}v_{k}\qquad\left(\begin{array}{l}u=u(z),\\ v=v(z)\end{array}\right).$$

§ 1. Preliminary

(I) The theorem below is reported by the author already (cf. J.F C, Vol. 27, May (2005), 83 - 88.). [31]

Theorem D. *Let be*

$$P = P(\alpha, \beta, \gamma) := \frac{\sin \pi\alpha \cdot \sin \pi(\gamma - \alpha - \beta)}{\sin \pi(\alpha + \beta) \cdot \sin \pi(\gamma - \alpha)} \quad (|P(\alpha, \beta, \gamma)| = M < \infty) \tag{1}$$

and

$$Q = Q(\alpha, \beta, \gamma) := P(\beta, \alpha, \gamma), \quad (|P(\beta, \alpha, \gamma)| = M < \infty) \tag{2}$$

When $\alpha, \beta, \gamma \notin Z_0^+$, *we have* ;

(i) $$((z-c)^{\alpha} \cdot (z-c)^{\beta})_{\gamma} = e^{-i\pi\gamma} P(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z-c)^{\alpha + \beta - \gamma}, \tag{3}$$

$$(\mathrm{Re}(\alpha + \beta + 1) > 0, \quad (1 + \alpha - \gamma) \notin Z_0^-),$$

(ii) $$((z-c)^{\beta} \cdot (z-c)^{\alpha})_{\gamma} = e^{-i\pi\gamma} Q(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z-c)^{\alpha + \beta - \gamma}, \tag{4}$$

$$(\mathrm{Re}(\alpha + \beta + 1) > 0, \quad (1 + \beta - \gamma) \notin Z_0^-)$$

(iii) $$((z-c)^{\alpha + \beta})_{\gamma} = e^{-i\pi\gamma} \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z-c)^{\alpha + \beta - \gamma}, \tag{5}$$

where

$$z - c \neq 0, \quad \left| \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \right| < \infty .$$

Then the inequalities below are established from this theorem.

Corollary 1. *We have the inequalities*

(i) $$((z-c)^{\alpha} \cdot (z-c)^{\beta})_{\gamma} \neq ((z-c)^{\beta} \cdot (z-c)^{\alpha})_{\gamma}, \tag{6}$$

and

(ii) $$((z-c)^{\alpha} \cdot (z-c)^{\beta})_{\gamma} \neq ((z-c)^{\alpha + \beta})_{\gamma}, \tag{7}$$

where

$$\alpha, \beta, \gamma \notin Z_0^+, \quad \alpha \neq \beta, \quad z - c \neq 0.$$

Corollary 2.

(i) *When* $\alpha, \beta, \gamma \notin Z_0^+$, *and*

$$P(\alpha, \beta, \gamma) = Q(\beta, \alpha, \gamma) = 1, \tag{8}$$

we have

$$((z-c)^{\alpha} \cdot (z-c)^{\beta})_{\gamma} = ((z-c)^{\beta} \cdot (z-c)^{\alpha})_{\gamma} = ((z-c)^{\alpha+\beta})_{\gamma}, \tag{9}$$

$$(\operatorname{Re}(\alpha+\beta+1) > 0, \ (1+\alpha-\gamma) \notin Z_0^-, \ (1+\beta-\gamma) \notin Z_0^-).$$

(ii) *When* $\gamma = m \in Z_0^+$, *we have* ;

$$((z-c)^{\alpha} \cdot (z-c)^{\beta})_m = ((z-c)^{\beta} \cdot (z-c)^{\alpha})_m = ((z-c)^{\alpha+\beta})_m. \tag{10}$$

(II) The Teorem below is reported by the author already (cf. J. Frac. Calc. Vol. 29, May (2006),pp.35 - 44.) . [7]

Theorem E . *We have*

(i)
$$\left(((z-b)^{\beta}-c)^{\alpha}\right)_{\gamma} = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}$$
$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{11}$$
$$\left(\left|\frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)}\right| < \infty\right)$$

and

(ii)
$$\left(((z-b)^{\beta}-c)^{\alpha}\right)_n = (-1)^n (z-b)^{\alpha\beta-n}$$
$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^{\beta}}\right)^k \qquad (n \in Z_0^+) \tag{12}$$

where

$$\left|\frac{c}{(z-b)^{\beta}}\right| < 1,$$

and

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \ \textit{with} \ [\lambda]_0 = 1,$$

(Notation of Pochhammer).

§2. Solutions to some homogeneous special ordinary differential equation by means of N-fractional Calculus Operator

Theorem 1. *Let be* $\varphi = \varphi(z) \in F$ *, then the homogeneous ordinary differential equation*

$$\varphi_2 + \varphi_1 \cdot \frac{a}{z} + \varphi \cdot b = 0 \qquad (z \neq 0) \tag{1}$$

$$(\varphi_\nu = d^\nu \varphi / dz^\nu \text{ for } \nu > 0 \,,\, \varphi_0 = \varphi = \varphi(z) \,)$$

has particular solutions of the forms (fractional differintegrated form)

Group I.

(i) $$\varphi = K e^{i\sqrt{b}z} (e^{-2i\sqrt{b}z} \cdot z^{-\frac{a}{2}})_{\frac{a}{2}-1} \equiv \varphi_{[1](a,b)} \quad \text{(denote)} \tag{2}$$

(ii) $$\varphi = K e^{i\sqrt{b}z} (z^{-\frac{a}{2}} \cdot e^{-2i\sqrt{b}z})_{\frac{a}{2}-1} \equiv \varphi_{[2](a,b)} \tag{3}$$

(iii) $$\varphi = K e^{-i\sqrt{b}z} (e^{2i\sqrt{b}z} \cdot z^{-\frac{a}{2}})_{\frac{a}{2}-1} \equiv \varphi_{[3](a,b)} \tag{4}$$

(iv) $$\varphi = K e^{-i\sqrt{b}z} (z^{-\frac{a}{2}} \cdot e^{2i\sqrt{b}z})_{\frac{a}{2}-1} \equiv \varphi_{[4](a,b)} \tag{5}$$

Group II.

(i) $$\varphi = K z^{1-a} e^{i\sqrt{b}z} (e^{-2i\sqrt{b}z} \cdot z^{\frac{a}{2}-1})_{-\frac{a}{2}} \equiv \varphi^{\circ}_{[1](a,b)} \tag{6}$$

(ii) $$\varphi = K z^{1-a} e^{i\sqrt{b}z} (z^{\frac{a}{2}-1} \cdot e^{-2i\sqrt{b}z})_{-\frac{a}{2}} \equiv \varphi^{\circ}_{[2](a,b)} \tag{7}$$

(iii) $$\varphi = K z^{1-a} e^{-i\sqrt{b}z} (e^{2i\sqrt{b}z} \cdot z^{\frac{a}{2}-1})_{-\frac{a}{2}} \equiv \varphi^{\circ}_{[3](a,b)} \tag{8}$$

(iv) $$\varphi = K z^{1-a} e^{-i\sqrt{b}z} (z^{\frac{a}{2}-1} \cdot e^{2i\sqrt{b}z})_{-\frac{a}{2}} \equiv \varphi^{\circ}_{[4](a,b)} \tag{9}$$

where $K(\neq 0)$ *is an arbitrary constant.*

Note. Notice that we have ;

$$\varphi_{[1](a,b)} = \varphi_{[2](a,b)} \quad \textit{and} \quad \varphi_{[3](a,b)} = \varphi_{[4](a,b)} \quad \textit{for} \quad (\tfrac{a}{2} - 1) \in Z_0^+ \,,$$

and

$$\varphi^{\circ}_{[1](a,b)} = \varphi^{\circ}_{[2](a,b)} \quad \textit{and} \quad \varphi^{\circ}_{[3](a,b)} = \varphi^{\circ}_{[4](a,b)} \quad \textit{for} \quad (-\tfrac{a}{2}) \in Z_0^+ \,.$$

Proof of Group I.

We have

$$\varphi_2 \cdot z + \varphi_1 \cdot a + \varphi \cdot bz = 0 \qquad (z \neq 0) \,. \tag{10}$$

from (1).

Set $$\varphi = e^{\lambda z} \phi \qquad (\phi = \phi(z)) \,. \tag{11}$$

we have then

$$\phi_2 \cdot z + \phi_1 \cdot (2\lambda z + a) + \phi \cdot \{(\lambda^2 + b)z + \lambda a\} = 0 \tag{12}$$

from (10), applying (11).

Here, choose λ such that

$$\lambda^2 + b = 0 \,, \tag{13}$$

then we obtain

$$\lambda = \begin{cases} i\sqrt{b} \equiv c & (i = \sqrt{-1}) \qquad (14) \\ -i\sqrt{b} \equiv -c & \qquad\qquad (15) \end{cases} .$$

I) Case $\lambda = c$;

In this case we have

$$\phi_2 \cdot z + \phi_1 \cdot (2cz + a) + \phi \cdot ca = 0 \tag{16}$$

from (12).

Operate N-fractional calculus (NFC) operator N^ν to the both sides of equation (16), we have then

$$(\phi_2 \cdot z)_\nu + (\phi_1 \cdot (2cz + a))_\nu + (\phi \cdot ca)_\nu = 0 \quad (\nu \notin Z^-) \,. \tag{17}$$

Now we have

$$(\phi_2 \cdot z)_\nu = \sum_{k=0}^{1} \frac{\Gamma(\nu+1)}{k!\,\Gamma(\nu+1-k)} (\phi_2)_{\nu-k} (z)_k \tag{18}$$

$$= \phi_{2+\nu} \cdot z + \phi_{1+\nu} \cdot \nu \quad , \tag{19}$$

$$(\phi_1 \cdot (2cz + a))_\nu = \phi_{1+\nu} \cdot (2cz + a) + \phi_\nu \cdot 2c\nu \tag{20}$$

and

$$(\phi \cdot ca)_\nu = \varphi_\nu \cdot ca \quad , \tag{21}$$

respectively, by Lemmas (i) and (i v).

Therefore, we have

$$\phi_{2+\nu} \cdot z + \phi_{1+\nu} \cdot (2cz + \nu + a) + \phi_\nu \cdot c(2\nu + a) = 0 \tag{22}$$

from (17), applyimg (19) , (20) and (21).

Choose ν such that

$$\nu = -a/2 \tag{23}$$

then we obtain

$$\phi_{2-\frac{a}{2}} \cdot z + \phi_{1-\frac{a}{2}} \cdot (2cz + \tfrac{a}{2}) = 0 \quad . \tag{24}$$

from (22), using (23).

Set

$$\phi_{1-\frac{a}{2}} = u = u(z) \qquad (\phi = u_{\frac{a}{2}-1}), \tag{25}$$

we have then

$$u_1 + u\cdot\left(2c + \tfrac{a}{2}z^{-1}\right) = 0 \tag{26}$$

from (24). The solution to this (variable separable form) equation is given by

$$u = K(e^{(2c+\frac{a}{2}z^{-1})_{-1}})^{-1} \tag{27}$$

$$= K(e^{2cz+\frac{a}{2}\log z})^{-1} \tag{28}$$

$$= Ke^{-2cz}\cdot z^{-\frac{a}{2}} . \tag{29}$$

Therefore, we obtain

$$\phi = u_{\frac{a}{2}-1} = K(e^{-2cz}\cdot z^{-\frac{a}{2}})_{\frac{a}{2}-1} \tag{30}$$

from (25).

Inversely (30) satisfies equation (24) clearly.

Hence we obtain

$$\varphi = Ke^{cz}(e^{-2cz}\cdot z^{-\frac{a}{2}})_{\frac{a}{2}-1} \tag{31}$$

$$= Ke^{i\sqrt{b}\,z}(e^{-2i\sqrt{b}\,z}\cdot z^{-\frac{a}{2}})_{\frac{a}{2}-1} \equiv \varphi_{[1](a,b)} \tag{2}$$

from (11) with (14).

Next, changing the order

$e^{-2i\sqrt{b}z}$ and $z^{-\frac{a}{2}}$ in parenthesis $(\ \cdot\ )_{\frac{a}{2}-1}$ in (2)

we obtain other solution $\varphi_{[2](a,b)}$ which is different from (2) for $(\frac{a}{2}-1)\notin Z_0^+$,

that is,

$$\varphi = Ke^{i\sqrt{b}\,z}(z^{-\frac{a}{2}}\cdot e^{-2i\sqrt{b}\,z})_{\frac{a}{2}-1} \equiv \varphi_{[2](a,b)} \quad . \tag{3}$$

(Refer to Theorem D.)

II) Case $\lambda = -c$;

Set $-\sqrt{b}$ instead of $\sqrt{b}$ in (2) and (3), we obtain (4) and (5) respectinely.

Proof of Group II.

Set

$$\varphi = z^{\beta}\phi \qquad (\phi = \phi(z)), \tag{33}$$

we have then

$$\phi_2 \cdot z + \phi_1 \cdot (2\beta + a) + \phi \cdot \{(\beta^2 - \beta + a\beta)z^{-1} + bz\} = 0 \qquad (34)$$

from (10), ausing (33).

Here we choose β such that

$$\beta^2 - \beta + a\beta = \beta(\beta - 1 + a) = 0 \ , \qquad (35)$$

that is,

$$\beta = \begin{cases} 0 & (36) \\ 1 - a \, . & (37) \end{cases}$$

When $\beta = 0$, (34) is reduced to (10), therefore, we have the same solutions as Group I.

When $\beta = 1 - a$ we obtain

$$\phi_2 \cdot z + \phi_1 \cdot (2 - a) + \phi \cdot bz = 0 \qquad (38)$$

from (34) .

The particular solutions to this equation are given by (refer to the proof of Group I)

$$\phi_{[1]} = Ke^{i\sqrt{b}z}(e^{-2i\sqrt{b}z} \cdot z^{\frac{a}{2}-1})_{-\frac{a}{2}} \qquad (39)$$

$$\phi_{[2]} = Ke^{i\sqrt{b}z}(z^{\frac{a}{2}-1} \cdot e^{-2i\sqrt{b}z})_{-\frac{a}{2}} \qquad (40)$$

$$\phi_{[3]} = Ke^{-i\sqrt{b}z}(e^{2i\sqrt{b}z} \cdot z^{\frac{a}{2}-1})_{-\frac{a}{2}} \qquad (41)$$

$$\phi_{[4]} = Ke^{-i\sqrt{b}z}(z^{\frac{a}{2}-1} \cdot e^{2i\sqrt{b}z})_{-\frac{a}{2}} \qquad (42)$$

setting $2 - a$ instead of a in the right hand side of(2), (3), (4),and (5), respectively.

Now we have

$$\varphi = z^{1-a}\phi \ . \qquad (43)$$

from (33), hence we obtain

$$\varphi = z^{1-a}\phi_{[1]} = Kz^{1-a}e^{i\sqrt{b}z}(e^{-2i\sqrt{b}z} \cdot z^{\frac{a}{2}-1})_{-\frac{a}{2}} \equiv \overset{\circ}{\varphi}_{[1](a,b)} \qquad (6)$$

$$\varphi = z^{1-a}\phi_{[2]} \equiv \overset{\circ}{\varphi}_{[2](a,b)} \qquad (7)$$

$$\varphi = z^{1-a}\phi_{[3]} \equiv \overset{\circ}{\varphi}_{[3](a,b)} \qquad (8)$$

$$\varphi = z^{1-a}\phi_{[4]} \equiv \overset{\circ}{\varphi}_{[4](a,b)} \qquad (9)$$

using (39) ~ (42).

§3. Familiar Forms of The Solutions

Theorem 2. *We have the more familiar form (expanded form) presentations for the solutions to the differential equation (1) in §2. as follows.*

Group I.

(i) $$\varphi_{[1](a,b)} = K(-2i\sqrt{b})^{\frac{a}{2}-1} z^{-\frac{a}{2}} e^{-i\sqrt{b}z}\ {}_2F_0(1-\tfrac{a}{2}, \tfrac{a}{2}; \tfrac{i}{2\sqrt{b}z}) \quad (|i/2\sqrt{b}\,z|<1) \qquad (1)$$

(ii) $$\varphi_{[2](a,b)} = K(-e^{-i\pi\frac{a}{2}} \frac{\Gamma(a-1)}{\Gamma(\frac{a}{2})}) z^{1-a} e^{-i\sqrt{b}z}\ {}_1F_1(1-\tfrac{a}{2}; 2-a; 2i\sqrt{b}\,z) \qquad (2)$$

$$(|\Gamma(a-1-k)/\Gamma(\tfrac{a}{2})|<\infty,\ |2i\sqrt{b}z|<1)$$

(iii) $$\varphi_{[3](a,b)} = K(2i\sqrt{b})^{\frac{a}{2}-1} z^{-\frac{a}{2}} e^{i\sqrt{b}z}\ {}_2F_0(1-\tfrac{a}{2}, \tfrac{a}{2}; \tfrac{-i}{2\sqrt{b}z}) \quad (|-i/2\sqrt{b}\,z|<1) \qquad (3)$$

(iv) $$\varphi_{[4](a,b)} = K(-e^{-i\pi\frac{a}{2}} \frac{\Gamma(a-1)}{\Gamma(\frac{a}{2})}) z^{1-a} e^{i\sqrt{b}z}\ {}_1F_1(1-\tfrac{a}{2}; 2-a; -2i\sqrt{b}\,z) \qquad (4)$$

$$(|\Gamma(a-1-k)/\Gamma(\tfrac{a}{2})|<\infty,\ |-2i\sqrt{b}\,z|<1)$$

Group II.

(i) $$\mathring{\varphi}_{[1](a,b)} = K(-2i\sqrt{b})^{-\frac{a}{2}} z^{-\frac{a}{2}} e^{-i\sqrt{b}z}\ {}_2F_0(\tfrac{a}{2}, 1-\tfrac{a}{2}; \tfrac{i}{2\sqrt{b}z}) \quad (|i/2\sqrt{b}\,z|<1) \qquad (5)$$

(ii) $$\mathring{\varphi}_{[2](a,b)} = K(e^{i\pi\frac{a}{2}} \frac{\Gamma(1-a)}{\Gamma(1-\frac{a}{2})}) e^{-i\sqrt{b}z}\ {}_1F_1(\tfrac{a}{2}; a; 2i\sqrt{b}\,z) \qquad (6)$$

$$(|\Gamma(1-a-k)/\Gamma(1-\tfrac{a}{2})|<\infty,\ |2i\sqrt{b}z|<1)$$

(iii) $$\mathring{\varphi}_{[3](a,b)} = K(2i\sqrt{b})^{-\frac{a}{2}} z^{-\frac{a}{2}} e^{i\sqrt{b}z}\ {}_2F_0(\tfrac{a}{2}, 1-\tfrac{a}{2}; \tfrac{-i}{2\sqrt{b}z}) \quad (|-i/2\sqrt{b}\,z|<1) \qquad (7)$$

(iv) $$\mathring{\varphi}_{[4](a,b)} = K(e^{i\pi\frac{a}{2}} \frac{\Gamma(1-a)}{\Gamma(1-\frac{a}{2})}) e^{i\sqrt{b}z}\ {}_1F_1(\tfrac{a}{2}; a; -2i\sqrt{b}\,z) \qquad (8)$$

$$(|\Gamma(1-a-k)/\Gamma(1-\tfrac{a}{2})|<\infty,\ |-2i\sqrt{b}\,z|<1)$$

where ${}_pF_q(\cdots\cdots)$ *is the generalized Gausss hypergeometric function,*

Proof of Group I. We have

(i) $$\varphi_{[1](a,b)} = Ke^{i\sqrt{b}z}(e^{-2i\sqrt{b}z} z^{-\frac{a}{2}})_{\frac{a}{2}-1} \qquad (\S 2.(2))$$

$$= Ke^{i\sqrt{b}z} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{a}{2})}{k!\,\Gamma(\frac{a}{2}-k)} (e^{-2i\sqrt{b}z})_{\frac{a}{2}-1-k} (z^{-\frac{a}{2}})_k \qquad (9)$$

$$= Ke^{i\sqrt{b}z}\sum_{k=0}^{\infty}\frac{(-1)^k[1-\frac{a}{2}]_k}{k!}\{(-2i\sqrt{b})^{\frac{a}{2}-1-k}e^{-2i\sqrt{b}z}\}\{e^{-i\pi k}\frac{\Gamma(\frac{a}{2}+k)}{\Gamma(\frac{a}{2})}z^{-\frac{a}{2}-k}\} \quad (10)$$

$$(|\Gamma(\tfrac{a}{2}+k)/\Gamma(\tfrac{a}{2})|<\infty)\ .$$

$$= K(-2i\sqrt{b})^{\frac{a}{2}-1}z^{-\frac{a}{2}}e^{-i\sqrt{b}z}\sum_{k=0}^{\infty}\frac{[1-\frac{a}{2}]_k[\frac{a}{2}]_k}{k!}\left(\frac{i}{2\sqrt{b}z}\right)^k \quad (|\tfrac{i}{2\sqrt{b}z}|<1) \quad (11)$$

$$= K(-2i\sqrt{b})^{\frac{a}{2}-1}z^{-\frac{a}{2}}e^{-i\sqrt{b}z}\ {}_2F_0(1-\tfrac{a}{2},\tfrac{a}{2};\tfrac{i}{2\sqrt{b}z}) \quad (1)$$

using Lemma (iv), since

$$\Gamma(\lambda-k)=(-1)^{-k}\frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(k+1-\lambda)}=(-1)^{-k}\frac{\Gamma(\lambda)}{[1-\lambda]_k} \quad (k\in Z_0^+) \quad (12)$$

and

$$(e^{\lambda z})_\nu=\lambda^\nu e^{\lambda z} \quad \text{(Refer to [1] Vol. 1 and [2])} \quad (13)$$

(i i) $\varphi_{[2](a,b)} = Ke^{i\sqrt{b}z}(z^{-\frac{a}{2}}\cdot e^{-2i\sqrt{b}z})_{\frac{a}{2}-1}$ (§2.(3))

$$= Ke^{i\sqrt{b}z}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{a}{2})}{k!\,\Gamma(\frac{a}{2}-k)}(z^{-\frac{a}{2}})_{\frac{a}{2}-1-k}(e^{-2i\sqrt{b}z})_k \quad (14)$$

$$= Ke^{i\sqrt{b}z}\sum_{k=0}^{\infty}\frac{(-1)^k[1-\frac{a}{2}]_k}{k!}\{e^{-i\pi(\frac{a}{2}-1-k)}\frac{\Gamma(a-1-k)}{\Gamma(\frac{a}{2})}z^{-a+1+k}\}\{(-2i\sqrt{b})^k e^{-2i\sqrt{b}z}\} \quad (15)$$

$$(|\Gamma(a-1-k)/\Gamma(\tfrac{a}{2})|<\infty)\ .$$

$$= K(-e^{-i\pi\frac{a}{2}}\frac{\Gamma(a-1)}{\Gamma(\frac{a}{2})})z^{1-a}e^{-i\sqrt{b}z}\sum_{k=0}^{\infty}\frac{[1-\frac{a}{2}]_k}{k!\,[2-a]_k}(2i\sqrt{b}z)^k \quad (16)$$

$$(|2i\sqrt{b}z|<1)$$

(i i) $\varphi_{[2](a,b)} = K(-e^{-i\pi\frac{a}{2}}\frac{\Gamma(a-1)}{\Gamma(\frac{a}{2})})\,z^{1-a}\,e^{-i\sqrt{b}z}\ {}_1F_1(1-\tfrac{a}{2};2-a;2i\sqrt{b}z)$ (2)

using Lemma (iv).

(iii) $\varphi_{[3](a,b)} = Set\ -\sqrt{b}\ instead\ of\ \sqrt{b}\ in\ (1).$ (3)

(iv) $\varphi_{[4](a,b)} = Set\ -\sqrt{b}\ instead\ of\ \sqrt{b}\ in\ (2).$ (4)

Proof of Group II. We have

(i) $$\varphi^{\circ}_{[1](a,b)} = K z^{1-a} e^{i\sqrt{b}z} (e^{-2i\sqrt{b}z} \cdot z^{\frac{a}{2}-1})_{-\frac{a}{2}} \qquad (\S 2.(6))$$

$$= K z^{1-a} e^{i\sqrt{b}z} \sum_{k=0}^{\infty} \frac{\Gamma(1-\frac{a}{2})}{k!\,\Gamma(1-\frac{a}{2}-k)} (e^{-2i\sqrt{b}z})_{-\frac{a}{2}-k} (z^{\frac{a}{2}-1})_k \qquad (17)$$

$$= K z^{1-a} e^{i\sqrt{b}z} \sum_{k=0}^{\infty} \frac{(-1)^k [\frac{a}{2}]_k}{k!} \{(-2i\sqrt{b})^{-\frac{a}{2}-k} e^{-2i\sqrt{b}z}\}\{e^{-i\pi k} \frac{\Gamma(1-\frac{a}{2}+k)}{\Gamma(1-\frac{a}{2})} z^{\frac{a}{2}-1-k}\} \qquad (18)$$

$$(|\Gamma(1-\tfrac{a}{2}+k)/\Gamma(1-\tfrac{a}{2})| < \infty).$$

$$= K(-2i\sqrt{b})^{-\frac{a}{2}} z^{-\frac{a}{2}} e^{-i\sqrt{b}z} \sum_{k=0}^{\infty} \frac{[\frac{a}{2}]_k [1-\frac{a}{2}]_k}{k!} \left(\frac{i}{2\sqrt{b}z}\right)^k \quad (|\tfrac{i}{2\sqrt{b}z}| < 1) \qquad (19)$$

$$= K(-2i\sqrt{b})^{-\frac{a}{2}} z^{-\frac{a}{2}} e^{-i\sqrt{b}z}\, {}_2F_0(\tfrac{a}{2}, 1-\tfrac{a}{2}; \tfrac{i}{2\sqrt{b}z}) \qquad (5)$$

using Lemma (iv).

(ii) $$\varphi^{\circ}_{[2](a,b)} = K z^{1-a} e^{i\sqrt{b}z} (z^{\frac{a}{2}-1} \cdot e^{-2i\sqrt{b}z})_{-\frac{a}{2}} \qquad (\S 2.(7))$$

$$= K z^{1-a} e^{i\sqrt{b}z} \sum_{k=0}^{\infty} \frac{\Gamma(1-\frac{a}{2})}{k!\,\Gamma(1-\frac{a}{2}-k)} (z^{\frac{a}{2}-1})_{-\frac{a}{2}-k} (e^{-2i\sqrt{b}z})_k \qquad (20)$$

$$= K z^{1-a} e^{i\sqrt{b}z} \sum_{k=0}^{\infty} \frac{(-1)^k [\frac{a}{2}]_k}{k!} \{e^{i\pi(\frac{a}{2}+k)} \frac{\Gamma(1-a-k)}{\Gamma(1-\frac{a}{2})} z^{a-1+k}\}\{(-2i\sqrt{b})^k e^{-2i\sqrt{b}z}\} \qquad (21)$$

$$(|\Gamma(1-a-k)/\Gamma(1-\tfrac{a}{2})| < \infty).$$

$$= K(e^{i\pi\frac{a}{2}} \frac{\Gamma(1-a)}{\Gamma(1-\frac{a}{2})}) e^{-i\sqrt{b}z} \sum_{k=0}^{\infty} \frac{[\frac{a}{2}]_k}{k!\,[a]_k} (2i\sqrt{b}z)^k \qquad (22)$$

$$(|2i\sqrt{b}z| < 1)$$

$$= K(e^{i\pi\frac{a}{2}} \frac{\Gamma(1-a)}{\Gamma(1-\frac{a}{2})})\, e^{-i\sqrt{b}z}\, {}_1F_1(\tfrac{a}{2}; a; 2i\sqrt{b}z) \qquad (6)$$

using Lemma (iv).

(iii) $\varphi^{\circ}_{[3](a,b)}$ = *Set* $-\sqrt{b}$ *instead of* $\sqrt{b}$ *in* (5). (7)

(iv) $\varphi^{\circ}_{[4](a,b)}$ = *Set* $-\sqrt{b}$ *instead of* $\sqrt{b}$ *in* (6). (8)

Notice that we have

$$\varphi_{[1](a,b)} = \varphi^{\circ}_{[1](a,b)} \quad \text{except the constant coefficient,} \qquad (23)$$

and

$$\varphi_{[3](a,b)} = \varphi^{\circ}_{[3](a,b)} \quad \text{except the constant coefficient,} \qquad (24)$$

§4. Some special cases

[I] Case of $a = 0$;

Corollary 1. *Let be $\varphi = \varphi(z) \in F$,then the homogeneous ordinary differential equation*

$$\varphi_2 + \varphi \cdot b = 0 \qquad (b \neq 0) \tag{1}$$

has particular solutions of the forms (fractional differintegrated form)

Group I.

(i)
$$\varphi = Ke^{i\sqrt{b}z}(e^{-2i\sqrt{b}z} \cdot 1)_{-1} \equiv \varphi_{[1](0,b)} \tag{2}$$

(ii)
$$\varphi = Ke^{i\sqrt{b}z}(1 \cdot e^{-2i\sqrt{b}z})_{-1} \equiv \varphi_{[2](0,b)} \tag{3}$$

(iii)
$$\varphi = Ke^{-i\sqrt{b}z}(e^{2i\sqrt{b}z} \cdot 1)_{-1} \equiv \varphi_{[3](0,b)} \tag{4}$$

(iv)
$$\varphi = Ke^{-i\sqrt{b}z}(1 \cdot e^{2i\sqrt{b}z})_{-1} \equiv \varphi_{[4](0,b)} \tag{5}$$

Group II.

(i)
$$\varphi = Kz e^{i\sqrt{b}z}(e^{-2i\sqrt{b}z} \cdot z^{-1})_0 \equiv \overset{\circ}{\varphi}_{[1](0,b)} \tag{6}$$

(ii)
$$\varphi = Kz e^{i\sqrt{b}z}(z^{-1} \cdot e^{-2i\sqrt{b}z})_0 \equiv \overset{\circ}{\varphi}_{[2](0,b)} \tag{7}$$

(iii)
$$\varphi = Kz e^{-i\sqrt{b}z}(e^{2i\sqrt{b}z} \cdot z^{-1})_0 \equiv \overset{\circ}{\varphi}_{[3](0,b)} \tag{8}$$

(iv)
$$\varphi = Kz e^{-i\sqrt{b}z}(z^{-1} \cdot e^{2i\sqrt{b}z})_0 \equiv \overset{\circ}{\varphi}_{[4](0,b)} \tag{9}$$

where $K(\neq 0)$ is an arbitrary constant.

Note. Notice that we have ;

$$\overset{\circ}{\varphi}_{[1](0,b)} = \overset{\circ}{\varphi}_{[2](0,b)} \quad \text{and} \quad \overset{\circ}{\varphi}_{[3](0,b)} = \overset{\circ}{\varphi}_{[4](0,b)} \;. \tag{10}$$

Proof.

Set $a = 0$ in Theorem 1.

Because we have directly (from (6) and (7))

$$\overset{\circ}{\varphi}_{[1](0,b)} = Ke^{-i\sqrt{b}z} = \overset{\circ}{\varphi}_{[2](0,b)} \tag{11}$$

since

$$(e^{-2i\sqrt{b}z} \cdot z^{-1})_0 = e^{-2i\sqrt{b}z} z^{-1} = (z^{-1} \cdot e^{-2i\sqrt{b}z})_0 \;. \tag{12}$$

Or calculating them as follows (for its expanded form) ;

$$\overset{\circ}{\varphi}_{[1](0,b)} = Kz\, e^{i\sqrt{b}z}(e^{-2i\sqrt{b}z} \cdot z^{-1})_0 \tag{6}$$

$$= Kze^{i\sqrt{b}z} \sum_{k=0}^{\infty} \frac{\Gamma(1)}{k!\,\Gamma(1-k)} (e^{-2i\sqrt{b}z})_{-k}(z^{-1})_k \tag{13}$$

$$= Kz\, e^{i\sqrt{b}\,z}\{e^{-2i\sqrt{b}\,z}z^{-1} + \frac{\Gamma(1)}{\Gamma(0)}(-2i\sqrt{b})^{-1}e^{-2i\sqrt{b}\,z}(-z^{-2})$$

$$+\frac{\Gamma(1)}{2!\,\Gamma(-1)}(-2i\sqrt{b})^{-2}e^{-2i\sqrt{b}\,z}(2z^{-3}) + \cdots\cdots\} \qquad (14)$$

$$= Ke^{-i\sqrt{b}\,z}\,. \qquad (15)$$

using Lemma (iv). And

$$\overset{\circ}{\varphi}_{[2](0,b)} = Kz e^{i\sqrt{b}\,z}(z^{-1}\cdot e^{-2i\sqrt{b}\,z})_0 \qquad (7)$$

$$= Kz\, e^{i\sqrt{b}\,z}\sum_{k=0}^{\infty}\frac{\Gamma(1)}{k!\,\Gamma(1-k)}(z^{-1})_{-k}(e^{-2i\sqrt{b}\,z})_k \qquad (16)$$

$$= Kz\, e^{i\sqrt{b}\,z}\{z^{-1}e^{-2i\sqrt{b}\,z} + \frac{\Gamma(1)}{\Gamma(0)}(z^{-1})_{-1}(e^{-2i\sqrt{b}\,z})_1 + \frac{\Gamma(1)}{2!\,\Gamma(-1)}(z^{-1})_{-2}(e^{-2i\sqrt{b}\,z})_2 + \cdots\cdots\} \qquad (17)$$

$$= Kz\, e^{i\sqrt{b}\,z}\{z^{-1}e^{-2i\sqrt{b}\,z} + \frac{1}{\Gamma(0)}(\log z)(-2i\sqrt{b}\,)e^{-2i\sqrt{b}\,z}$$

$$+\frac{1}{2!\,\Gamma(-1)}(\log z)_{-1}(-2i\sqrt{b})^2 e^{-2i\sqrt{b}\,z} + \cdots\cdots\} \qquad (18)$$

$$= Ke^{-i\sqrt{b}\,z}\,, \qquad (19)$$

using

$$(z^{-1})_{-2} = ((z^{-1})_{-1})_{-1} = (\log z)_{-1} = z\log z - z\ . \qquad (20)$$

Note. Equation (1) is the simple harmonic vibration one.

Corollary 1'. Familiar forms of the solutions in Corollary I are shown as follows.

Group I.

(i) $$\varphi_{[1](0,b)} = K\frac{i}{2\sqrt{b}}(\cos\sqrt{b}\,z - i\sin\sqrt{b}\,z) \qquad (21)$$

(ii) $$\varphi_{[2](0,b)} = K\frac{1}{\sqrt{b}}\sin\sqrt{b}\,z \qquad (22)$$

(iii) $$\varphi_{[3](0,b)} = K\frac{-i}{2\sqrt{b}}(\cos\sqrt{b}\,z + i\sin\sqrt{b}\,z) \qquad (23)$$

(iv) $$\varphi_{[4](0,b)} = K\frac{1}{\sqrt{b}}\sin\sqrt{b}\,z \qquad (24)$$

Group II.

(i) $$\overset{\circ}{\varphi}_{[1](0,b)} = K(\cos\sqrt{b}\,z - i\sin\sqrt{b}\,z) \qquad (25)$$

(ii) $$\overset{\circ}{\varphi}_{[2](0,b)} = K(\cos\sqrt{b}\,z - i\sin\sqrt{b}\,z) \qquad (26)$$

(iii) $$\overset{\circ}{\varphi}_{[3](0,b)} = K(\cos\sqrt{b}\,z + i\sin\sqrt{b}\,z) \qquad (27)$$

(iv) $$\overset{\circ}{\varphi}_{[4](0,b)} = K(\cos\sqrt{b}\,z + i\sin\sqrt{b}\,z)\ . \qquad (28)$$

[II] Case of $a = 2$;

Corollary 2. *Let be $\varphi = \varphi(z) \in F$,then the homogeneous ordinary differential equation*

$$\varphi_2 + \varphi_1 \cdot \tfrac{2}{z} + \varphi \cdot b = 0 \qquad (z \neq 0) \qquad (29)$$

has particular solutions of the forms (fractional differintegrated form)

Group I.

(i) $$\varphi = Ke^{i\sqrt{b}z}(e^{-2i\sqrt{b}z} \cdot z^{-1})_0 \equiv \varphi_{[1](2,b)} \qquad (30)$$

(ii) $$\varphi = Ke^{i\sqrt{b}z}(z^{-1} \cdot e^{-2i\sqrt{b}z})_0 \equiv \varphi_{[2](2,b)} \qquad (31)$$

(iii) $$\varphi = Ke^{-i\sqrt{b}z}(e^{2i\sqrt{b}z} \cdot z^{-1})_0 \equiv \varphi_{[3](2,b)} \qquad (32)$$

(iv) $$\varphi = Ke^{-i\sqrt{b}z}(z^{-1} \cdot e^{2i\sqrt{b}z})_0 \equiv \varphi_{[4](2,b)} \qquad (33)$$

Group II.

(i) $$\varphi = Kz^{-1}e^{i\sqrt{b}z}(e^{-2i\sqrt{b}z} \cdot 1)_{-1} \equiv \overset{\circ}{\varphi}_{[1](2,b)} \qquad (34)$$

(ii) $$\varphi = Kz^{-1}e^{i\sqrt{b}z}(1 \cdot e^{-2i\sqrt{b}z})_{-1} \equiv \overset{\circ}{\varphi}_{[2](2,b)} \qquad (35)$$

(iii) $$\varphi = Kz^{-1}e^{-i\sqrt{b}z}(e^{2i\sqrt{b}z} \cdot 1)_{-1} \equiv \overset{\circ}{\varphi}_{[3](2,b)} \qquad (36)$$

(iv) $$\varphi = Kz^{-1}e^{-i\sqrt{b}z}(1 \cdot e^{2i\sqrt{b}z})_{-1} \equiv \overset{\circ}{\varphi}_{[4](2,b)} \qquad (37)$$

where $K(\neq 0)$ is an arbitrary constant.

Note. Notice that we have ;

$$\varphi_{[1](2,b)} = \varphi_{[2](2,b)} \quad and \quad \varphi_{[3](2,b)} = \varphi_{[4](2,b)}\ . \qquad (38)$$

Proof. Set $a = 2.$ in Theorem 1.

Corollary 2'. Familiar forms of the solutions in Corollary 2 are shown as follows.

Group I.

(i) $$\varphi_{[1](2,b)} = Kz^{-1}(\cos\sqrt{b}\,z - i\sin\sqrt{b}\,z) \qquad (39)$$

(ii) $$\varphi_{[2](2,b)} = Kz^{-1}(\cos\sqrt{b}\,z - \sin\sqrt{b}\,z) \qquad (40)$$

(iii) $$\varphi_{[3](2,b)} = Kz^{-1}(\cos\sqrt{b}\,z + i\sin\sqrt{b}\,z) \qquad (41)$$

(iv) $$\varphi_{[4](2,b)} = Kz^{-1}(\cos\sqrt{b}\,z + i\sin\sqrt{b}\,z) \qquad (42)$$

Group II.

(i) $$\overset{\circ}{\varphi}_{[1](2,b)} = K(\tfrac{i}{2\sqrt{b}})z^{-1}(\cos\sqrt{b}\,z - i\sin\sqrt{b}\,z) \qquad (43)$$

(ii) $$\overset{\circ}{\varphi}_{[2](2,b)} = K(\tfrac{1}{\sqrt{b}})z^{-1}\sin\sqrt{b}\,z) \qquad (44)$$

(iii) $$\overset{\circ}{\varphi}_{[3](2,b)} = K(\tfrac{-i}{2\sqrt{b}})z^{-1}(\cos\sqrt{b}\,z + i\sin\sqrt{b}\,z) \qquad (45)$$

(iv) $$\overset{\circ}{\varphi}_{[4](2,b)} = K(\tfrac{1}{\sqrt{b}})z^{-1}\sin\sqrt{b}\,z) \qquad (46)$$

where $b \neq 0$.

Proof.

We have directly (38) from (30) , (31) , (32) and (33) clearly.

Indeed we have

$$\varphi_{[1](2,b)} = Ke^{i\sqrt{b}z}(e^{-2i\sqrt{b}z} \cdot z^{-1})_0 \qquad (30)$$

$$= K\, e^{i\sqrt{b}z}\sum_{k=0}^{\infty}\frac{\Gamma(1)}{k!\,\Gamma(1-k)}(e^{-2i\sqrt{b}z})_{-k}(z^{-1})_k \qquad (47)$$

$$= Ke^{i\sqrt{b}z}\{e^{-2i\sqrt{b}z}z^{-1} + \frac{\Gamma(1)}{\Gamma(0)}(-2i\sqrt{b})^{-1}e^{-2i\sqrt{b}z}(-z^{-2})$$

$$+\frac{\Gamma(1)}{2!\,\Gamma(-1)}(-2i\sqrt{b})^{-2}e^{-2i\sqrt{b}z}(2z^{-3}) + \cdots\cdots\} \qquad (48)$$

$$= Kz^{-1}e^{-i\sqrt{b}z} = Kz^{-1}(\cos\sqrt{b}z - i\sin\sqrt{b}\,z). \qquad (39)$$

Next we have

$$\varphi_{[2](2,b)} = Ke^{i\sqrt{b}z}(z^{-1} \cdot e^{-2i\sqrt{b}z})_0 \qquad (31)$$

$$= Ke^{i\sqrt{b}z}\sum_{k=0}^{\infty}\frac{\Gamma(1)}{k!\,\Gamma(1-k)}(z^{-1})_{-k}(e^{-2i\sqrt{b}z})_k \qquad (49)$$

$$= Ke^{i\sqrt{b}z}\{z^{-1}e^{-2i\sqrt{b}z} + \frac{\Gamma(1)}{\Gamma(0)}(z^{-1})_{-1}(e^{-2i\sqrt{b}z})_1 + \frac{\Gamma(1)}{2!\,\Gamma(-1)}(z^{-1})_{-2}(e^{-2i\sqrt{b}z})_2 + \cdots\cdots\}$$

$$(50)$$

$$= Ke^{i\sqrt{b}z}\{z^{-1}e^{-2i\sqrt{b}z} + \frac{1}{\Gamma(0)}(\log z)(-2i\sqrt{b})e^{-2i\sqrt{b}z}$$

$$+\frac{1}{2!\,\Gamma(-1)}(\log z)_{-1}(-2i\sqrt{b})^2e^{-2i\sqrt{b}z} + \cdots\cdots\} \qquad (52)$$

$$= Kz^{-1}e^{-i\sqrt{b}z} = Kz^{-1}(\cos\sqrt{b}z - i\sin\sqrt{b}\,z)\,. \qquad (40)$$

And set $-\sqrt{b}$ instead of $\sqrt{b}$ in (39) and (40), we have then (41) and (42), respectively.

Next we have

$$\overset{\circ}{\varphi}_{[1](2,b)} = Kz^{-1}e^{i\sqrt{b}z}(e^{-2i\sqrt{b}z} \cdot 1)_{-1} \qquad (34)$$

$$= Kz^{-1}e^{i\sqrt{b}z}\sum_{k=0}^{\infty}\frac{\Gamma(0)}{k!\,\Gamma(0-k)}(e^{-2i\sqrt{b}z})_{-1-k}(1)_k \qquad (53)$$

$$= Kz^{-1}e^{i\sqrt{b}z}(-2i\sqrt{b})^{-1}e^{-2i\sqrt{b}z} \qquad (54)$$

$$= K(\frac{i}{2\sqrt{b}})z^{-1}e^{-i\sqrt{b}\,z} \tag{55}$$

$$= K(\frac{i}{2\sqrt{b}})z^{-1}(\cos\sqrt{b}\,z - i\sin\sqrt{b}\,z)\ . \tag{43}$$

and

$$\varphi_{[2](2,b)} = Kz^{-1}\,e^{i\sqrt{b}\,z}(1\cdot e^{-2i\sqrt{b}\,z})_{-1} \tag{35}$$

$$= Kz^{-1}\,e^{i\sqrt{b}\,z}\sum_{k=0}^{\infty}\frac{\Gamma(0)}{k!\,\Gamma(0-k)}(1)_{-1-k}(e^{-2i\sqrt{b}\,z})_k \tag{56}$$

$$= Kz^{-1}\,e^{i\sqrt{b}\,z}\{(1)_{-1}e^{-2i\sqrt{b}\,z} + (-1)(1)_{-2}(e^{-2i\sqrt{b}\,z})_1$$

$$+(1)_{-3}(e^{-2i\sqrt{b}\,z})_2 + (-1)(1)_{-4}(e^{-2i\sqrt{b}\,z})_3 + \cdots\cdots\} \tag{57}$$

$$= Kz^{-1}\,e^{i\sqrt{b}\,z}\{z\,e^{-2i\sqrt{b}\,z} + (-1)\frac{1}{2!}z^2(-2i\sqrt{b})e^{-2i\sqrt{b}\,z}$$

$$+\frac{1}{3!}z^3(-2i\sqrt{b})^2 e^{-2i\sqrt{b}\,z} + (-1)\frac{1}{4!}z^4(-2i\sqrt{b})^3 e^{-2i\sqrt{b}\,z} + \cdots\cdots\} \tag{58}$$

$$= Kz^{-1}\,e^{-i\sqrt{b}\,z}(2i\sqrt{b})^{-1}[\{1 + (2i\sqrt{b}\,z) + \frac{1}{2!}(2i\sqrt{b}\,z)^2$$

$$+\frac{1}{3!}(2i\sqrt{b}\,z)^3 + \cdots\cdots\} - 1] \tag{59}$$

$$= Kz^{-1}\,e^{-i\sqrt{b}\,z}(2i\sqrt{b})^{-1}[e^{2i\sqrt{b}\,z} - 1] \tag{60}$$

$$= K(\frac{1}{2i\sqrt{b}})z^{-1}\{e^{i\sqrt{b}\,z} - e^{-i\sqrt{b}\,z}\} \tag{61}$$

$$= K(\frac{1}{\sqrt{b}})\,z^{-1}\sin\sqrt{b}\,z \tag{44}$$

And set $-\sqrt{b}$ instead of $\sqrt{b}$ in (43) and (44), we have then (45) and (46), respectively.

§ 5. Commentary

(I) All solutions shown by (2) ~ (9) in § 2 have a fractional differintegrated form $(\cdots\cdots)_\gamma$, where the index γ is the order of differintegration.

(II) The special differential equation which is discussed in this article is much interesting one.

(III) From the homogeneous Fukuhara's equation

$$\varphi_2 + \varphi_1\cdot(a + \frac{b}{z}) + \varphi\cdot(p + \frac{q}{z} + \frac{r}{z^2}) = 0 \qquad (\varphi = \varphi(z),\ z \neq 0)\ , \tag{1}$$

we obtain

$$\varphi_2 + \varphi_1 \cdot \frac{b}{z} + \varphi \cdot p = 0 \ , \qquad (2)$$

setting $a = q = r = 0$.

Equation (2) is the same form one as § 2 (1). **That is, the equation discussed in this article is a special form of the Fukuhara's one.**

(IV) Moreover, when $b = p = 1$ we have

$$\varphi_2 + \varphi_1 \cdot \frac{1}{z} + \varphi = 0 \qquad (3)$$

from (2).

On the other hand , we obtain (3) from the homogeneous Bessel equation

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot z + \varphi \cdot (z^2 - \nu^2) = 0 , \qquad (z \neq 0) \qquad (4)$$

setting $\nu = 0$.

That is, equation (3) is the Bessel's one of order $\nu = 0$.

A particular solution to equation (3) is given by

$$\varphi = \varphi_{[1](1,1)} = Ke^{iz}(e^{-2iz} \cdot z^{-\frac{1}{2}})_{-\frac{1}{2}} \qquad (5)$$

$$= K(-2i)^{-\frac{1}{2}} z^{-\frac{1}{2}} e^{-iz} {}_2F_0(\tfrac{1}{2}, \tfrac{1}{2} ; \tfrac{i}{2z}), \qquad (|\tfrac{i}{2z}| < 1) \qquad (6)$$

$$= K(-2i)^{-\frac{1}{2}} H_0^{(2)}(z) \ , \qquad (7)$$

where $H_0^{(2)}(z)$ is the Hankel function of order 0, for example.

(refer to § 3. (1)). (cf. J. Frac. Calc. Vol.42, Nov. (2012),pp.1-19. ; Solutions to The Homogeneous Bessel Equation by means of N-Fractional Calculus Operator, by K. Nishimoto)

Note. We have

$$\varphi_2 \cdot z^2 + \varphi_1 \cdot (az^2 + bz) + \varphi \cdot (pz^2 + qz + r) = 0 \qquad (\varphi = \varphi(z), \ z \neq 0) \qquad (8)$$

from (1).

Therefore, we obtain (4) from (8), setting

$$a = 0 , \quad b = 1 , \quad p = 1 , \quad q = 0 , \quad \text{and } r = -\nu^2 . \qquad (9)$$

That is, Bessel's equation is a special one of Fukuhara's equation.

(V) Usually the special ordinary differential equations like as (1) and (2) are solved by means of Frobenius. Compare our N-fractional operator method with that of Frobenius.

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