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Kyoto University
New Family of Integral Operators of Meromorphic Functions

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Abstract. We define here an integral operator $I_{n}(f_{i},g_{i})(z)$ for meromorphic functions in the punctured open unit disk. Some properties for this operator are derived.

Keywords: analytic function, meromorphic function, starlike function, convex function, integral operator.

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1 Introduction

Let $\Sigma$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{n}z^{n},$$  \hspace{1cm} (1.1)

which are analytic in the punctured open unit disk

$$U^{*} = \{z \in \mathbb{C}: 0 < |z| < 1\} = \cup \{0\},$$  \hspace{1cm} (1.2)

where $\cup$ is the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

We say that a function $f \in \Sigma$ is meromorphic starlike of order $\delta (0 \leq \delta < 1)$, and belongs to the class $\Sigma^{*}(\delta)$, if it satisfies the inequality

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta.$$  \hspace{1cm} (1.3)

A function $f \in \Sigma$ is a meromorphic convex function of order $\delta (0 \leq \delta < 1)$, if $f$ satisfies the following inequality

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta,$$  \hspace{1cm} (1.4)

and we denote this class by $\Sigma_{k}(\delta)$. 
For $f \in \Sigma$, Wang et al. [13] (see also [14]) introduced and studied the subclass $\Sigma_N(\lambda)$ of $\Sigma$ consisting of functions $f(z)$ satisfying

$$-\Re \left( \frac{zf''(z)}{f'(z)} + 1 \right) < \lambda \quad (\lambda > 1, \ z \in \mathbb{U}).$$

In the literature, several integral operators of meromorphic functions in the punctured open unit disk have been investigated and studied by many authors (cf., e.g., [1-11]).

For $i = 1, 2, \cdots, n$, $c > 0$, and $\alpha_i, \gamma_i \geq 0$, we now, introduce a generalized integral operator $I_n(f_i, g_i)(z): \Sigma^n \to \Sigma$ as follows

$$I_n(f_i, g_i)(z) = \frac{c}{z^{c+1}} \int_{0}^{z} u^{c-1} \prod_{i=1}^{n} (uf_i(u))^\alpha_i (-u^2 g_i'(u))^\gamma_i \, du,$$  \hspace{1cm} (1.5)

where $f_i, g_i \in \Sigma$. Indeed, by varying the parameters $c, \alpha_i$ and $\gamma_i$, the operator $I_n(f_i, g_i)$ reduces to the following well-known integral operators.

(i) for $\gamma_i = 0$, we obtain the integral operator

$$H(z) = I_n(f_i)(z) = \frac{c}{z^{c+1}} \int_{0}^{z} u^{c-1} \prod_{i=1}^{n} (uf_i(u))^\alpha_i \, du,$$ \hspace{1cm} (1.6)

introduced by Frasin [8].

(ii) For $c = 1$ and $\gamma_i = 0$, we obtain the integral operator

$$\mathcal{H}_n(z) = I_n(f_i)(z) = \frac{1}{z^2} \int_{0}^{z} \prod_{i=1}^{n} (uf_i(u))^\alpha_i \, du,$$ \hspace{1cm} (1.7)

introduced by Mohammed and Darus [9].

(iii) For $c = 1$ and $\alpha_i = 0$, we obtain the integral operator

$$\mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z) = I_n(g_i)(z) = \frac{1}{z^2} \int_{0}^{z} \prod_{i=1}^{n} (-u^2 g_i'(u))^\gamma_i \, du,$$ \hspace{1cm} (1.8)

introduced by Mohammed and Darus [10].

(iv) If $n = 1$, $\alpha_1 = 1$, $f_1 = f$ and $\gamma_1 = 0$ we have the integral operator

$$I_c(f)(z) = \frac{c}{z^{c+1}} \int_{0}^{z} u^{c-1} f(u) \, du,$$
which was studied by many authors (cf., e.g., [1, 2, 6]).

For the starlikeness of the integral operator $I_n(f_i, g_i)$, we have to recall here the following Lemma.

**Lemma 1.1 ([12]).** Suppose that the function $\Psi : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfies the following condition:

$$\Re\{\Psi(is, t)\} \leq 0, \quad \left(s, t \in \mathbb{R}; \ t \leq \frac{-1 + s^2}{2}\right).$$

If the function $p(z) = 1 + p_1z + ...$, is analytic in $\mathbb{U}$ and

$$\Re\{\Psi(p(z), zp'(x))\} > 0, \ (z \in \mathbb{U}),$$

then

$$\Re\{p(z)\} > 0 \ (z \in \mathbb{U}).$$

### 2 Main Results

In the next theorem, we place conditions for the meromorphically starlikeness of the integral operator $I_n(f_i, g_i)(z)$ which is defined in (1.5).

**Theorem 2.1.** For $i = 1, 2, \ldots, n$, let $f_i, g_i \in \Sigma$, $\alpha_i, \gamma_i \geq 0$ and let $c > 0$. If $f_i \in \Sigma^*$, $g_i \in \Sigma_k$, and $\sum_{i=1}^{n} (\alpha_i + \gamma_i) = 1$, then the general integral operator $I_n(f_i, g_i)(z)$ belongs to the meromorphic starlike function class.

**Proof.** From (1.5) it follows that

$$z^2I_n'(f_i, g_i)(z) + (c + 1)zI_n(f_i, g_i)(z) = c \prod_{i=1}^{n} \left(zf_i(z)\right)^{\alpha}(\!-\!z^2g_i'(z)\!)^{\gamma_i}$$

(2.1)

Differentiating both sides of (2.1) logarithmically and multiplying by $z$, we obtain

$$\frac{z^2I_n''(f_i, g_i)(z) + (c + 3)zI_n'(f_i, g_i)(z) + (c + 1)I_n(f_i, g_i)(z)}{zI_n'(f_i, g_i)(z) + (c + 1)I_n(f_i, g_i)(z)}$$

$$= \sum_{i=1}^{n} \alpha_i \frac{zf_i'(z)}{f_i(z)} + \sum_{i=1}^{n} \gamma_i \left(\frac{zg_i''(z)}{g_i'(z)} + 1\right) + \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \gamma_i.$$

(2.2)

Which is equivalent to

$$\frac{z^2I_n''(f_i, g_i)(z) + (c + 2)zI_n'(f_i, g_i)(z)}{zI_n'(f_i, g_i)(z) + (c + 1)I_n(f_i, g_i)(z)}$$
\[= \sum_{i=1}^{n} \alpha_i \left( -\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left( -\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i). \quad (2.3)\]

We can write (2.3) as the following

\[= \sum_{i=1}^{n} \alpha_i \left( -\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left( -\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i). \quad (2.4)\]

We define the regular function \( p \) in \( \mathbb{U} \) by

\[p(z) = -\frac{zI_n'(f_i, g_i)(z)}{I_n(f_i, g_i)(z)}\]

and \( p(0) = 1 \). Differentiating \( p(z) \) logarithmically, we obtain

\[-p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zI_n''(f_i, g_i)(z)}{I_n'(f_i, g_i)(z)}. \quad (2.6)\]

From (2.4), (2.5) and (2.6) we obtain

\[p(z) + \frac{zp'(z)}{-p(z) + c + 1} = \sum_{i=1}^{n} \alpha_i \left( -\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left( -\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i). \quad (2.7)\]

Let us put

\[\Psi(u, v) = u + \frac{v}{-u + c + 1}. \quad (2.8)\]

From the hypothesis of Theorem 2.1, (2.7) and (2.8) we obtain

\[\Re\{\Psi(p(z), zp'(z))\} = \sum_{i=1}^{n} \alpha_i \left( -\Re\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left\{ \Re \left( -\frac{zg_i''(z)}{g_i'(z)} - 1 \right) \right\} + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i)\]

\[> 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i) = 0. \quad (2.9)\]

Now we proceed to show that

\[\Re\{\Psi(is, t)\} \leq 0, \quad \left( s, t \in \mathbb{R}; \; t \leq \frac{-(1 + s^2)}{2} \right).\]

Indeed, from (2.8), we have

\[\Re\{\Psi(is, t)\} = \Re \left\{ is + \frac{t}{-is + c + 1} \right\} = \frac{t(c + 1) \leq -(1 + s^2)(c + 1)}{s^2 + (c + 1)^2} \leq -\frac{(1 + s^2)(c + 1)}{2 [s^2 + (c + 1)^2]} < 0. \quad (2.10)\]

Thus, from (2.9), (2.10) and by using Lemma 1.1, we conclude that \( \Re\{p(z)\} > 0 \), and so

\[-\Re \left\{ \frac{zI_n'(f_i, g_i)(z)}{I_n(f_i, g_i)(z)} \right\} > 0.]
that is $I_n(f_i, g_i)(z)$ is starlike.

Next, we place conditions for the integral operator $I_n(f_i, g_i)$ to be in the class $\Sigma_N(\lambda)$.

**Theorem 2.2.** For $i = 1, 2, \ldots, n$, let $f_i, g_i \in \Sigma$, $\alpha_i, \gamma_i \geq 0$ and let $c > 0$. If $f_i \in \Sigma^*(\delta)$, $g_i \in \Sigma_k(\delta)$, and

$$\sum_{i=1}^{n} (\alpha_i + \gamma_i) > \frac{c+1}{1-\delta}, \quad (2.11)$$

then $I_n(f_i, g_i)(z) \in \Sigma_N(\lambda)$, $\lambda > 1$.

**Proof.** Equivalently, (2.3) can be written as

$$- \frac{(z f_i'(s) + s f_i''(s))}{z f_i'(s) + s f_i''(s)} - c = \sum_{i=1}^{n} \alpha_i \left( - \frac{zf_i'(s)}{f_i(s)} \right) + \sum_{i=1}^{n} \gamma_i \left( - \frac{sg_i''(s)}{g_i(s)} - 1 \right) + 1 + \sum_{i=1}^{n} (\alpha_i + \gamma_i). \quad (2.12)$$

Therefore

$$- \left( \frac{z f_i'(s) + s f_i''(s)}{z f_i'(s) + s f_i''(s)} + 1 \right) = \sum_{i=1}^{n} \alpha_i \left( - \frac{zf_i'(s)}{f_i(s)} \right) + \sum_{i=1}^{n} \gamma_i \left( - \frac{sg_i''(s)}{g_i(s)} - 1 \right) + 1 + \sum_{i=1}^{n} (\alpha_i + \gamma_i) \quad (2.13)$$

Taking real part of both sides of (2.13), we obtain

$$- \Re \left( \frac{z f_i'(s) + s f_i''(s)}{z f_i'(s) + s f_i''(s)} + 1 \right) = \Re \left\{ \sum_{i=1}^{n} \alpha_i \left( - \frac{zf_i'(s)}{f_i(s)} \right) + \sum_{i=1}^{n} \gamma_i \left( - \frac{sg_i''(s)}{g_i(s)} - 1 \right) + 1 \right. \quad (2.14)$$

\[\left. - \sum_{i=1}^{n} (\alpha_i + \gamma_i) \right\| + \sum_{i=1}^{n} \alpha_i \left( - \Re \left( \frac{zf_i'(s)}{f_i(s)} \right) \right) + \sum_{i=1}^{n} \gamma_i \Re \left( - \frac{sg_i''(s)}{g_i(s)} - 1 \right) + c + 2 \]

\[\left. - \sum_{i=1}^{n} (\alpha_i + \gamma_i) \right\| + \sum_{i=1}^{n} \alpha_i \left( - \Re \left( \frac{zf_i'(s)}{f_i(s)} \right) \right) + \sum_{i=1}^{n} \gamma_i \Re \left( - \frac{sg_i''(s)}{g_i(s)} - 1 \right) + c + 2 \]

\[\left. - \sum_{i=1}^{n} (\alpha_i + \gamma_i) \right\| + \sum_{i=1}^{n} \alpha_i \left( - \Re \left( \frac{zf_i'(s)}{f_i(s)} \right) \right) + \sum_{i=1}^{n} \gamma_i \Re \left( - \frac{sg_i''(s)}{g_i(s)} - 1 \right) + c + 2 \]

\[\left. - \sum_{i=1}^{n} (\alpha_i + \gamma_i) \right\| + \sum_{i=1}^{n} \alpha_i \left( - \Re \left( \frac{zf_i'(s)}{f_i(s)} \right) \right) + \sum_{i=1}^{n} \gamma_i \Re \left( - \frac{sg_i''(s)}{g_i(s)} - 1 \right) + c + 2 \]
Let
\[
\lambda = \left| \frac{(c+1)I_n(f_{i}, g_{i})(z)}{zI_n(f_{i}, g_{i})(z)} \left[ \sum_{i=1}^{n} \alpha_{i} \left( -\frac{zf_{i}'(z)}{f_{i}(z)} \right) + \sum_{i=1}^{n} \gamma_{i} \left( -\frac{zg_{i}''(z)}{g_{i}'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} \left( \alpha_{i} + \gamma_{i} \right) \right] \right|
\]
\[+ \sum_{i=1}^{n} \alpha_{i} \left( -\Re \frac{zf_{i}'(z)}{f_{i}(z)} \right) + \sum_{i=1}^{n} \gamma_{i} \Re \left( -\frac{zg_{i}''(z)}{g_{i}'(z)} - 1 \right) + c + 2 - \sum_{i=1}^{n} \left( \alpha_{i} + \gamma_{i} \right).\]

Since \[\left| \frac{(c+1)I_n(f_{i}, g_{i})(z)}{zI_n(f_{i}, g_{i})(z)} \left[ \sum_{i=1}^{n} \alpha_{i} \left( -\frac{zf_{i}'(z)}{f_{i}(z)} \right) + \sum_{i=1}^{n} \gamma_{i} \left( -\frac{zg_{i}''(z)}{g_{i}'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} \left( \alpha_{i} + \gamma_{i} \right) \right] \right| > 0, f_{i}, g_{i} \in \Sigma^{*}(\delta),\]
then we have
\[
\lambda > c + 2 - (1 - \delta) \sum_{i=1}^{n} \left( \alpha_{i} + \gamma_{i} \right).
\]

Then, by the hypothesis (2.11), we have \(\lambda > 1\). Therefore, \(I_{n}(f_{i}, g_{i})(z) \in \Sigma_{N}(\lambda), \lambda > 1\).

If we set \(\gamma_{i} = 0\) in Theorem 2.2, then we have [8, Theorem 2.6].

Further, putting \(c = 1, \gamma_{i} = 0\) in Theorem 2.2, we get

Corollary 2.3. For \(i = 1, 2, \ldots, n\), let \(f_{i} \in \Sigma, \alpha_{i} \geq 0\). If \(f_{i} \in \Sigma^{*}(\delta)\), and
\[
\sum_{i=1}^{n} \alpha_{i} > \frac{2}{1 - \delta},
\]
then \(\mathcal{H}_{n}(z) \in \Sigma_{N}(\lambda), \lambda > 1\).

In addition, taking \(c = 1, \alpha_{i} = 0\) in Theorem 2.2, we receive

Corollary 2.4. For \(i = 1, 2, \ldots, n\), let \(g_{i} \in \Sigma, \gamma_{i} \geq 0\). If \(g_{i} \in \Sigma_{k}(\delta)\), and
\[
\sum_{i=1}^{n} \gamma_{i} > \frac{2}{1 - \delta},
\]
then \(\mathcal{H}_{\gamma_{1}, \ldots, \gamma_{n}}(z) \in \Sigma_{N}(\lambda), \lambda > 1\).

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References


