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New Family of Integral Operators of Meromorphic Functions

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Abstract. We define here an integral operator $I_n(f_i, g_i)(z)$ for meromorphic functions in the punctured open unit disk. Some properties for this operator are derived.

Keywords: analytic function, meromorphic function, starlike function, convex function, integral operator.

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1 Introduction

Let $\Sigma$ denote the class of functions of the form

\[ f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \]  

which are analytic in the punctured open unit disk

\[ U^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} = U \setminus \{0\}, \]  

where $U$ is the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$.

We say that a function $f \in \Sigma$ is meromorphic starlike of order $\delta (0 \leq \delta < 1)$, and belongs to the class $\Sigma^*(\delta)$, if it satisfies the inequality

\[ -\Re \left( \frac{zf'(z)}{f(z)} \right) > \delta. \]  

A function $f \in \Sigma$ is a meromorphic convex function of order $\delta (0 \leq \delta < 1)$, if $f$ satisfies the following inequality

\[ -\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \delta, \]  

and we denote this class by $\Sigma_k(\delta)$. 
For $f \in \Sigma$, Wang et al. [13] (see also [14]) introduced and studied the subclass $\Sigma_N(\lambda)$ of $\Sigma$ consisting of functions $f(z)$ satisfying

$$-\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) < \lambda \quad (\lambda > 1, \ z \in \mathbb{U}).$$

In the literature, several integral operators of meromorphic functions in the punctured open unit disk have been investigated and studied by many authors (cf., e.g., [1-11]).

For $i = 1, 2, \cdots, n$, $c > 0$, and $\alpha_i, \gamma_i \geq 0$, we now, introduce a generalized integral operator $I_n(f_i, g_i)(z) : \Sigma^n \rightarrow \Sigma$ as follows

$$I_n(f_i, g_i)(z) = \frac{c}{z^{c+1}} \int_0^z u^{c-1} \prod_{i=1}^n (uf_i(u))^\alpha_i (-u^2 g_i'(u))^\gamma_i du, \quad (1.5)$$

where $f_i, g_i \in \Sigma$. Indeed, by varying the parameters $c, \alpha_i$ and $\gamma_i$, the operator $I_n(f_i, g_i)$ reduces to the following well-known integral operators.

(i) for $\gamma_i = 0$, we obtain the integral operator

$$H(z) = I_n(f_i)(z) = \frac{c}{z^{c+1}} \int_0^z u^{c-1} \prod_{i=1}^n (uf_i(u))^\alpha_i du, \quad (1.6)$$

introduced by Frasin [8].

(ii) For $c = 1$ and $\gamma_i = 0$, we obtain the integral operator

$$\mathcal{H}_n(z) = I_n(f_i)(z) = \frac{1}{z^2} \int_0^z \prod_{i=1}^n (uf_i(u))^\alpha_i du, \quad (1.7)$$

introduced by Mohammed and Darus [9].

(iii) For $c = 1$ and $\alpha_i = 0$, we obtain the integral operator

$$\mathcal{H}_{\gamma_1, \cdots, \gamma_n}(z) = I_n(g_i)(z) = \frac{1}{z^2} \int_0^z \prod_{i=1}^n (-u^2 g_i'(u))^\gamma_i du, \quad (1.8)$$

introduced by Mohammed and Darus [10].

(iv) If $n = 1$, $\alpha_1 = 1$, $f_1 = f$ and $\gamma_1 = 0$ we have the integral operator

$$I_c(f)(z) = \frac{c}{z^{c+1}} \int_0^z u^{c-1} f(u) du,$$
which was studied by many authors (cf., e.g., [1, 2, 6]).

For the starlikeness of the integral operator $I_n(f_i, g_i)$, we have to recall here the following Lemma.

**Lemma 1.1 ([12]).** Suppose that the function $\Psi : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfies the following condition:

$$\Re\{\Psi(is, t)\} \leq 0, \quad \left(s, t \in \mathbb{R}; \ t \leq -\frac{(1 + s^2)}{2}\right).$$

If the function $p(z) = 1 + p_1 z + \ldots$, is analytic in $\mathbb{U}$ and

$$\Re\{\Psi(p(z), zp'(z))\} > 0, \ (z \in \mathbb{U}),$$

then

$$\Re\{p(z)\} > 0 \ (z \in \mathbb{U}).$$

## 2 Main Results

In the next theorem, we place conditions for the meromorphically starlikeness of the integral operator $I_n(f_i, g_i) (z)$ which is defined in (1.5).

**Theorem 2.1.** For $i = 1, 2, \ldots, n$, let $f_i, g_i \in \Sigma$, $\alpha_i, \gamma_i \geq 0$ and let $c > 0$. If $f_i \in \Sigma^*$, $g_i \in \Sigma_k$, and $\sum_{i=1}^{n} (\alpha_i + \gamma_i) = 1$, then the general integral operator $I_n(f_i, g_i) (z)$ belongs to the meromorphic starlike function class.

**Proof.** From (1.5) it follows that

$$z^2I_n''(f_i, g_i) (z) + (c + 1)zI_n'(f_i, g_i) (z) = c \prod_{i=1}^{n} (zf_i(z))^\alpha (-z^2g_i'(z))^\gamma$$

$$= z\left(\sum_{i=1}^{n} \alpha_i \frac{zf_i'(z)}{f_i(z)} + \sum_{i=1}^{n} \gamma_i \left(\frac{zg_i''(z)}{g_i'(z)} + 1\right) + \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \gamma_i \right).$$

Which is equivalent to

$$-\frac{z^2I_n''(f_i, g_i) (z) + (c + 2)zI_n'(f_i, g_i) (z)}{zI_n'(f_i, g_i) (z) + (c + 1)I_n(f_i, g_i) (z)}$$
\[= \sum_{i=1}^{n} \alpha_i \left( -\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left( -\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i). \quad (2.3)\]

We can write (2.3) as the following

\[= \sum_{i=1}^{n} \alpha_i \left( -\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left( -\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i). \quad (2.4)\]

We define the regular function \( p \) in \( \mathbb{U} \) by

\[p(z) = -\frac{zI_n'(f_i, g_i)(z)}{I_n(f_i, g_i)(z)}, \quad (2.5)\]

and \( p(0) = 1 \). Differentiating \( p(z) \) logarithmically, we obtain

\[ -p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zI_n''(f_i, g_i)(z)}{I_n'(f_i, g_i)(z)}. \quad (2.6)\]

From (2.4), (2.5) and (2.6) we obtain

\[p(z) + \frac{zp'(z)}{-p(z) + c + 1} = \sum_{i=1}^{n} \alpha_i \left( -\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left( -\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i). \quad (2.7)\]

Let us put

\[\Psi(u, v) = u + \frac{v}{-u + c + 1}. \quad (2.8)\]

From the hypotheses of Theorem 2.1, (2.7) and (2.8) we obtain

\[\Re\{\Psi(p(z), zp'(z))\} \geq 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i) = 0. \quad (2.9)\]

Now we proceed to show that

\[\Re\{\Psi(is, t)\} \leq 0, \quad (s, t \in \mathbb{R}; \ t \leq \frac{-(1+s^2)}{2}). \]

Indeed, from (2.8), we have

\[\Re\{\Psi(is, t)\} = \Re\left\{ is + \frac{t}{-is + c + 1} \right\} = \frac{t(c+1)}{s^2 + (c+1)^2} \leq -\frac{(1+s^2)(c+1)}{2[s^2 + (c+1)^2]} < 0. \quad (2.10)\]

Thus, from (2.9), (2.10) and by using Lemma 1.1, we conclude that \( \Re\{p(z)\} > 0 \), and so

\[-\Re\left\{ \frac{zI_n'(f_i, g_i)(z)}{I_n(f_i, g_i)(z)} \right\} > 0.\]
that is $I_n(f_i, g_i)(z)$ is starlike.

Next, we place conditions for the integral operator $I_n(f_i, g_i)$ to be in the class $\Sigma_N(\lambda)$.

**Theorem 2.2.** For $i = 1, 2, \ldots, n$, let $f_i, g_i \in \Sigma$, $\alpha_i, \gamma_i \geq 0$ and let $c > 0$. If $f_i \in \Sigma^*(\delta)$, $g_i \in \Sigma_k(\delta)$, and

$$\sum_{i=1}^{n} (\alpha_i + \gamma_i) > \frac{c+1}{1-\delta}, \quad (2.11)$$

then $I_n(f_i, g_i)(z) \in \Sigma_N(\lambda), \lambda > 1$.

**Proof.** Equivalently, (2.3) can be written as

$$-\left(\frac{z f_i''(z) + f_i'(z)}{I_n(f_i, g_i)(z)} + 1\right) = \sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{g_i''(z)}{g_i(z)} - 1\right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i).$$

Therefore

$$-\left(\frac{z f_i''(z) + f_i'(z)}{I_n(f_i, g_i)(z)} + 1\right) = \sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{g_i''(z)}{g_i(z)} - 1\right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i).$$

Taking real part of both sides of (2.13), we obtain

$$-\Re\left(\frac{z f_i''(z) + f_i'(z)}{I_n(f_i, g_i)(z)} + 1\right) = \Re\left\{\sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{g_i''(z)}{g_i(z)} - 1\right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i)\right\} + \sum_{i=1}^{n} \alpha_i \left(-\Re\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \Re\left(-\frac{g_i''(z)}{g_i(z)} - 1\right) + c + 2 - \sum_{i=1}^{n} (\alpha_i + \gamma_i) \leq \left|\frac{z f_i''(z) + f_i'(z)}{I_n(f_i, g_i)(z)} + 1\right| + \sum_{i=1}^{n} \alpha_i \left(-\Re\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \Re\left(-\frac{g_i''(z)}{g_i(z)} - 1\right) + c + 2 - \sum_{i=1}^{n} (\alpha_i + \gamma_i).$$

(2.14)
Let
\[
\lambda = \left| \frac{(c+1)I_{n}(f_{i},g_{i})(z)}{zI_{n}(f_{i},g_{i})(z)} \left[ \sum_{i=1}^{n} \alpha_{i} \left( -\frac{zf_{i}'(z)}{f_{i}(z)} \right) + \sum_{i=1}^{n} \gamma_{i} \left( -\frac{zg_{i}''(z)}{g_{i}'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} (\alpha_{i} + \gamma_{i}) \right] \right|
\]
\[
+ \sum_{i=1}^{n} \alpha_{i} \left( -\frac{zg_{i}'(z)}{g_{i}(z)} \right) + \sum_{i=1}^{n} \gamma_{i} \Re \left( -\frac{zg_{i}''(z)}{g_{i}'(z)} - 1 \right) + c + 2 - \sum_{i=1}^{n} (\alpha_{i} + \gamma_{i}).
\]
Since
\[
\left| \frac{(c+1)I_{n}(f_{i},g_{i})(z)}{zI_{n}(f_{i},g_{i})(z)} \left[ \sum_{i=1}^{n} \alpha_{i} \left( -\frac{zf_{i}'(z)}{f_{i}(z)} \right) + \sum_{i=1}^{n} \gamma_{i} \left( -\frac{zg_{i}''(z)}{g_{i}'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} (\alpha_{i} + \gamma_{i}) \right] \right| > 0,
\]
f_{i} \in \Sigma^{*}(\delta), g_{i} \in \Sigma_{k}(\delta), then we have
\[
\lambda > c + 2 - (1 - \delta) \sum_{i=1}^{n} (\alpha_{i} + \gamma_{i}).
\]
Then, by the hypothesis (2.11), we have \(\lambda > 1\). Therefore, \(I_{n}(f_{i},g_{i})(z) \in \Sigma_{N}(\lambda), \lambda > 1\).

If we set \(\gamma_{i} = 0\) in Theorem 2.2, then we have [8, Theorem 2.6].

Further, Putting \(c = 1, \gamma_{i} = 0\) in Theorem 2.2, we get

**Corollary 2.3.** For \(i = 1, 2, \ldots, n\), let \(f_{i} \in \Sigma, \alpha_{i} \geq 0\). If \(f_{i} \in \Sigma^{*}(\delta)\), and
\[
\sum_{i=1}^{n} \alpha_{i} > \frac{2}{1 - \delta},
\]
then \(H_{n}(z) \in \Sigma_{N}(\lambda), \lambda > 1\).

In addition, taking \(c = 1, \alpha_{i} = 0\) in Theorem 2.2, we receive

**Corollary 2.4.** For \(i = 1, 2, \ldots, n\), let \(g_{i} \in \Sigma, \gamma_{i} \geq 0\). If \(g_{i} \in \Sigma_{k}(\delta)\), and
\[
\sum_{i=1}^{n} \gamma_{i} > \frac{2}{1 - \delta},
\]
then \(H_{\gamma_{1},\ldots,\gamma_{n}}(z) \in \Sigma_{N}(\lambda), \lambda > 1\).

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**References**


