Title: Coefficients for certain analytic functions related to arguments of $f'(z)$ (On Schwarzian Derivatives and Its Applications)

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Citation: 数理解析研究所講究録 (2013), 1824: 1-7

Issue Date: 2013-02

URL: http://hdl.handle.net/2433/194732

Type: Departmental Bulletin Paper

Textversion: publisher

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Coefficients for certain analytic functions related to arguments of \( f'(z) \)

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Abstract

For some real \( \delta_1 \) and \( \delta_2 \) \((-\pi < \delta_2 < 0 < \delta_1 < \pi)\), the properties of the coefficients of functions \( f(z) \), normalized by \( f(0) = f'(0) - 1 = 0 \) and satisfying the conditions \( \sup \{ \arg f'(z) \} = \delta_1 \) and \( \inf \{ \arg f'(z) \} = \delta_2 \), are discussed.

1 Introduction

Let \( \mathcal{A} \) be the class of functions \( f(z) \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \), and let \( \mathcal{P} \) be the class of functions \( p(z) \) of the form

\[
p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k
\]

which are analytic in \( \mathbb{U} \) and satisfy the condition

\[
\text{Re}(p(z)) > 0 \quad (z \in \mathbb{U}).
\]

A function \( p(z) \in \mathcal{P} \) is said to be the Carathéodory function. The following lemma is well-known and it can be found in excellent books by Duren [1] or by Pommerenke [4].

Lemma 1.1 If \( p(z) \in \mathcal{P} \), then the coefficient estimates

\[
|c_k| \leq 2
\]

for each \( k \) \((k = 1, 2, 3, \cdots)\) are obtained. Equality holds true for the function \( p(z) \) given by

\[
p(z) = \frac{1 + z}{1 - z} = 1 + \sum_{k=1}^{\infty} 2z^k.
\]

We say that \( f(z) \in \mathcal{R}(\delta_1, \delta_2) \) if \( f(z) \in \mathcal{A} \) satisfies the following conditions

\[
\sup \{ \arg f'(z) \} = \delta_1 \quad (z \in \mathbb{U}) \quad \text{and} \quad \inf \{ \arg f'(z) \} = \delta_2 \quad (z \in \mathbb{U})
\]

for some real \( \delta_1 \) and \( \delta_2 \) \((-\pi < \delta_2 < 0 < \delta_1 < \pi)\) and \( f'(z) \neq 0 \) in \( \mathbb{U} \).

2010 Mathematics Subject Classification: Primary 30C45.

Keywords and Phrases: Carathéodory function, close-to-convex function, coefficient bound.
In particular, for some real $\delta$ ($0 < \delta < \pi$), we write $\mathcal{R}(\delta, \delta - \pi) \equiv \mathcal{R}_\delta$ which means that if $f(z) \in \mathcal{R}_\delta$, then $f(z)$ satisfies
\[
\text{Re} \left( e^{i(\delta - \delta)} f'(z) \right) > 0 \quad (z \in \mathbb{U}).
\]

By Noshiro-Warschawski Theorem (for detail, see [3], [6]), it is well-known that all functions $f(z) \in \mathcal{R}_\delta$ are univalent in $\mathbb{U}$ and belong to the classical family of univalent functions $\mathcal{S}$. In fact, all functions $f(z) \in \mathcal{R}_\delta$ are close-to-convex univalent in $\mathbb{U}$. The class $\mathcal{R} \equiv \mathcal{R}_\delta$ was studied and many results were established (cf. [2]). For a function $f(z) \in \mathcal{R}(\delta_1, \delta_2)$, supposing that
\[
q(z) = \frac{e^{-i\varphi} f'(z)^b + i \sin \varphi}{\cos \varphi}
\]
where $X = \frac{\delta_1 - \delta_2}{\pi}$ and $\varphi = \frac{(\delta_1 + \delta_2)\pi}{2(\delta_1 - \delta_2)}$, we see that $q(z)$ is a member of the class $\mathcal{P}$. Furthermore, setting
\[
f'(z)^b = 1 + \sum_{k=1}^{\infty} b_kw^k,
\]
for a function $f(z) \in \mathcal{A}$, we have the following theorem by the help of Lemma 1.1 We can find this result, for example, in [5, Theorem 4]. However, a proof is included for the benefit of the readers.

**Theorem 1.2** If $f(z) \in \mathcal{R}(\delta_1, \delta_2)$, then
\[
|b_k| \leq 2 \cos \varphi \quad (k = 1, 2, 3, \cdots),
\]
where $\varphi = \frac{(\delta_1 + \delta_2)\pi}{2(\delta_1 - \delta_2)}$. Equality holds true for $f(z)$ given by
\[
f'(z)^b = \frac{1 + e^{i2\varphi}z}{1 - z}.
\]

**Proof.** Noting that
\[
f'(z)^b = \{ (\cos \varphi)q(z) - i \sin \varphi \} e^{i\varphi} = 1 + \sum_{k=1}^{\infty} (e^{i\varphi} \cos \varphi) c_k z^k
\]
for some $q(z) \in \mathcal{P}$, we know that $b_k = (e^{i\varphi} \cos \varphi) c_k$. Therefore, we obtain that
\[
|b_k| = |e^{i\varphi}| \cdot |\cos \varphi| \cdot |c_k| \leq 2 \cos \varphi.
\]
If we consider $f(z)$ given by
\[
f'(z)^b = \frac{1 + e^{i2\varphi}z}{1 - z} = 1 + (1 + e^{i2\varphi}) \sum_{k=1}^{\infty} z^k,
\]
then we see that
\[
|b_k| = \sqrt{2(1 + \cos 2\varphi)} = 2 \cos \varphi \quad (k = 1, 2, 3, \cdots).
\]
2 Main results

Our first result is contained in the following theorem.

**Theorem 2.1** If \( f(z) \in \mathcal{R}(\delta_1, \delta_2) \), then the coefficients of \( f(z) \) are represented as follows:

\[
a_n = \frac{1}{n} \sum_{m=1}^{n-1} \binom{X}{m} \left( \sum_{l_1+l_2+\cdots+l_m=n-1} b_{l_1} b_{l_2} \cdots b_{l_m} \right) \quad (n = 2, 3, 4, \cdots),
\]

where \( l_1, l_2, \cdots, l_m \in \mathbb{N} = \{1, 2, 3, \cdots\} \) and \( X = \frac{\delta_1 - \delta_2}{\pi} \).

**Proof.** We first remark that

\[
f'(z) = 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} = \left( 1 + \sum_{k=1}^{\infty} b_k z^k \right)^X = 1 + \sum_{m=1}^{\infty} \left\{ \binom{X}{m} \left( \sum_{k=1}^{\infty} b_k z^k \right)^m \right\}.
\]

Then, considering the coefficient of \( z^{n-1} \) with

\[
\left( \sum_{k=1}^{\infty} b_k z^k \right)^m = (b_1 z + b_2 z^2 + b_3 z^3 + \cdots)^m,
\]

we have that

\[
\left( 1 + \sum_{k=1}^{\infty} b_k z^k \right)^X = 1 + \sum_{m=1}^{\infty} \left\{ \binom{X}{m} \left( \sum_{l_1+l_2+\cdots+l_m=n-1} b_{l_1} b_{l_2} \cdots b_{l_m} \right) \right\} z^{n-1}.
\]

Thus, we know that

\[
a_n = \sum_{m=1}^{n-1} \binom{X}{m} \left( \sum_{l_1+l_2+\cdots+l_m=n-1} b_{l_1} b_{l_2} \cdots b_{l_m} \right)
\]

which completes the proof of the theorem. \(\square\)

By virtue of Theorem 1.2 and Theorem 2.1, we derive

**Theorem 2.2** If \( f(z) \in \mathcal{R}(\delta_1, \delta_2) \), then it follows that

\[
|a_n| \leq \frac{1}{n} \sum_{m=1}^{n-1} \left\{ \binom{n-2}{m-1} \frac{2^m}{m!} \left( \prod_{j=0}^{m-1} |j - X| \cos^m \varphi \right) \right\} \quad (n = 2, 3, 4, \cdots).
\]

**Proof.** By Theorem 1.2, Theorem 2.1 and the triangle inequality, we obtain that
\[ |a_n| \leq \frac{1}{n} \sum_{m=1}^{n-1} \left( \begin{array}{l} X_m \\ m \end{array} \right) \left( \sum_{l_1+l_2+\cdots+l_m=n-1} |b_{l_1}||b_{l_2}| \cdots |b_{l_m}| \right) \]

\[ \leq \frac{1}{n} \sum_{m=1}^{n-1} \frac{|X||X-1|\cdots|X-m+1|}{m!} 2^m \cos^m \varphi \left( \sum_{l_1+l_2+\cdots+l_m=n-1} 1 \right) \]

\[ = \frac{1}{n} \sum_{m=1}^{n-i} \left\{ \left( \begin{array}{l} n -2m -1 \\ m-1 \end{array} \right) \frac{2^m}{m!} \left( \prod_{j=0}^{m-1} |j-X| \right) \cos^m \varphi \right\}. \]

Taking \( \delta_1 = \delta \) and \( \delta_2 = \delta - \pi \) for some \( \delta \) \((0 < \delta < \pi)\) in Theorem 2.2, we can immediately see that \( X = 1 \) and \( \varphi = \delta - \frac{\pi}{2} \). Therefore, we have the following corollary.

**Corollary 2.3** If \( f(z) \in \mathcal{R}_\delta \), then it follows that

\[ |a_n| \leq \frac{2}{n} \sin \delta \quad (n = 2, 3, 4, \cdots). \]

The result is sharp for

\[ f(z) = e^{i\delta} z - (1 - e^{i\delta}) \log(1 - z) = z - \sum_{n=2}^{\infty} \frac{2ie^{i\delta} \sin \delta}{n} z^n. \]

**Proof.** The coefficient estimates in the corollary are readily obtained by Theorem 2.2. To prove the sharpness, we define the function \( P(z) \) given by

\[ P(z) = \frac{e^{-i\delta} - e^{i\delta} z}{1 - z} \quad (z \in U). \]

Then,

\[ |z| = \left| \frac{P(z) - e^{-i\delta}}{P(z) - e^{i\delta}} \right| < 1 \]

which implies that

\[ P(z)\overline{P(z)} - e^{i\delta} P(z) - e^{-i\delta} \overline{P(z)} + 1 < P(z)\overline{P(z)} - e^{-i\delta} P(z) - e^{i\delta} \overline{P(z)} + 1. \]

Thus, we have that

\[ (e^{i\delta} - e^{-i\delta}) \left( \frac{P(z) - P(z)}{P(z) - P(z)} \right) > 0, \]

that is, that

\[ -4 \sin \delta \cdot \text{Im} (P(z)) > 0. \]
Therefore, $P(z)$ satisfies

$$-\text{Im} \left( P(z) \right) > 0 \quad (z \in \mathbb{U}).$$

This leads us that

$$\text{Re} \left( e^{i(\frac{\pi}{2} - \delta)} f'(z) \right) = \text{Re} (iP(z)) = -\text{Im} \left( P(z) \right) > 0 \quad (z \in \mathbb{U}).$$

Therefore, we know that $f(z) = e^{i2\delta}z - (1 - e^{i2\delta}) \log(1 - z) \in \mathcal{R}_\delta$ and

$$|a_n| = \left| -\frac{2ie^{i\delta} \sin \delta}{n} \right| = \frac{2}{n} \sin \delta.$$

\[\square\]

\textbf{Remark 2.4} Putting $\delta = \frac{\pi}{4}$ in Corollary 2.3, we have that

$$f(z) = iz - (1 - i) \log(1 - z) = z + \sum_{n=2}^{\infty} \frac{1-i}{n} z^n.$$

This function $f(z)$ maps the open unit disk $\mathbb{U}$ onto the following domain.

3 \hspace{1cm} \textbf{Appendix}

In this section, for some real $\delta_1$ and $\delta_2$ \((-\pi < \delta_2 < 0 < \delta_1 < \pi\), we define the subclass $Q(\delta_1, \delta_2)$ of $\mathcal{A}$ as follows:

$$Q(\delta_1, \delta_2) = \left\{ f(z) \in \mathcal{A} : \sup \left( \arg \frac{f(z)}{z} \right) = \delta_1, \inf \left( \arg \frac{f(z)}{z} \right) = \delta_2 \text{ and } \frac{f(z)}{z} \neq 0 \ (z \in \mathbb{U}) \right\}.$$
When $\delta_1 = \delta$ and $\delta_2 = \delta - \pi$ for some $\delta$ ($0 < \delta < \pi$), we write $Q(\delta, \delta - \pi) \equiv Q_{\delta}$ and we know the next relation between $R(\delta_1, \delta_2)$ and $Q(\delta_1, \delta_2)$.

**Remark 3.1**

$$f(z) \in Q(\delta_1, \delta_2) \text{ if and only if } \int_0^z \frac{f(\xi)}{\xi} d\xi = z + \sum_{n=2}^\infty \frac{a_n}{n} z^n \in R(\delta_1, \delta_2).$$

Applying the above remark and Theorem 2.2, we deduce the following theorem.

**Theorem 3.2** If $f(z) \in Q(\delta_1, \delta_2)$, then

$$|a_n| \leq \sum_{m=1}^{n-1} \left\{ \binom{n-2m-1}{m-1} \frac{2^m}{m!} \prod_{j=0}^{m-1} |j-X| \cos^m \varphi \right\} (n=2,3,4,\cdots).$$

Setting $\delta_1 = \delta$ and $\delta_2 = \delta - \pi$ for some $\delta$ ($0 < \delta < \pi$) in Theorem 3.2, we have

**Corollary 3.3** If $f(z) \in Q_{\delta}$, then

$$|a_n| \leq 2 \sin \delta \quad (n=2,3,4,\cdots).$$

*The result is sharp for $f(z)$ given by*

$$f(z) = \frac{z - e^{i\delta} z^2}{1 - z} = z - \sum_{n=2}^\infty (2ie^{i\delta} \sin \delta) z^n.$$

**Remark 3.4** If we take $\delta = \frac{\pi}{4}$ in Corollary 3.3, we obtain that

$$f(z) = \frac{z - iz^2}{1 - z} = z + \sum_{n=2}^\infty (1 - i) z^n.$$

This function $f(z)$ maps the open unit disk $U$ onto the following domain.
References


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