数理解析研究所講究録
第 1826 巻 2013 年 180-185

SIEGEL EISENSTEIN SERIES, HECKE OPERATORS, AND FOURIER EXPANSIONS

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ABSTRACT. We discuss the action of Hecke operators on Siegel Eisenstein series in the case of degree 2 and square-free level.

1. INTRODUCTION

Determining representation numbers of quadratic forms is a classical problem in number theory, and elliptic modular forms have been used to great advantage in studying this problem. The number of times a positive definite quadratic form Q represents an integer t is given by the tth Fourier coefficient of the theta series attached to Q, and this theta series is one of our basic examples of a modular form. It is well-known that the average theta series lies in the space spanned by Eisenstein series (this weighted average is taken over the genus of Q, which consists of all quadratic forms that locally everywhere are isometric to Q). In the case of integral weight, the Fourier expansions for the Eisenstein series are well-known (see, for instance, [15]); then, realising the average theta series as a linear combination of Eisenstein series, one obtains closed-form formulas for the average representation numbers (see, for instance, [17]).

Siegel introduced generalised theta series to study how often a given quadratic form Q represents any other quadratic form T; these generalised theta series are our prototypical examples of Siegel modular forms. Currently, the study of Siegel modular forms is a very active area of research, and there are many different approaches used, both to provide new proofs of known results, and to obtain new insights, tools, and of course theorems.

Two fundamental problems that have not been completely solved are that of finding explicit Fourier series expansions for all Siegel Eisenstein series, and that of determining the action of Hecke operators on all Siegel Eisenstein series. In the case of elliptic modular forms, for any (integral) weight, level and character, as we know the Fourier expansions of a basis for the space of Eisenstein series, we can use these to determine the action of the Hecke operators on Eisenstein series. However, in the case of Siegel modular forms, closed-form formulas for Fourier coefficients of Eisenstein series are only known in certain cases: In [9], formulas for the degree 2, level 1 Eisenstein series are developed. These are also developed in [4] (chapter II) using the Fourier expansion for the level 1, index 1 Jacobi Eisenstein series (see [4] chapter I), and the connection between Jacobi forms and degree 2

2010 *Mathematics Subject Classification*: Primary 11F41

Siegel forms revealed by the first proof of the Saito-Kurokawa correspondence (proved in the series of papers [11], [12], [13], [1], [19]). Then in [7], [8], Katsurada combines the induction formula for local densities with a functional equation to obtain formulas for the Fourier coefficients of the Eisenstein series with Siegel degree at least 3 and level 1. In [10], Kohnen gives an explicit linear version of the Ikeda lift (this lift was conjectured by Duke and Imamoglu); using this, he obtains formulas for the Fourier coefficients of the Eisenstein series of any even Siegel degree and level 1. Then in [2], Choie and Kohnen modify Kohnen's approach to yield formulas for the Fourier coefficients of the Eisenstein series of any odd Siegel degree and level 1. In [14], Mizuno modifies the approach of [4], using (among other things) a converse theorem of Imai [6] to obtain formulas for one of the Eisenstein series of Siegel degree 2, odd square-free level, and primitive character. Quite recently, in [16], Takemori uses p-adic Siegel modular forms to develop formulas for one of the Eisenstein series of Siegel degree 2, arbitrary level, and primitive character. All these Fourier coefficient formulas are rather complicated.

Recently [18], we determined the action of Hecke operators on all Eisenstein series of Siegel degree 2 and square-free level by intricate but elementary methods, without any use of known Fourier coefficients of Siegel Eisenstein series. The idea of the approach is described below; it relies merely on the definition of the Eisenstein series and the explicit set of matrices described in [5] that give the action of the Hecke operators. Via this concrete approach, we find that the natural basis for the space of degree 2 Eisenstein series of square-free level $\mathcal{N}$ and character χ consists of eigenforms for all Hecke operators $T(p)$, $T_j(p^2)$ $(1 \le j \le n)$ where p is a prime not dividing $\mathcal{N}$, and we compute the eigenvalues. For primes $q|\mathcal{N}$, we obtain Hecke relations among these Eisenstein series when $\chi^2 \ne 1$; we use these to diagonalise the basis to obtain a basis consisting of eigenforms for all $T(p)$, $T_j(p^2)$, and we compute the eigenvalues (see Proposition 2.1 below). Additionally, we note that these Hecke relations can be used with known Eisenstein series Fourier coefficients to generate the Fourier coefficients of other Eisenstein series. In particular, when $\chi = 1$, we note that one can use the Fourier expansion for the level 1 Eisenstein series to generate the Fourier expansions for the basis of Eisenstein series of square-free level $\mathcal{N}$ and trivial character; this has recently been carried out by Martin Dickson [3].

Currently we are in the process of extending this work to arbitrary Siegel degree. For trivial character, we are again finding that for primes $q|\mathcal{N}$, the action of $T(q)$, $T_j(q^2)$ yield sufficiently many Hecke relations to allow us to generatre the Fourier expansions of all basis elements with square-free level $\mathcal{N}$ from the Fourier expansion of the level 1 Eisenstein series.

2. Definitions and results

For $n \in \mathbb{Z}_+$, the symplectic group $Sp_n(\mathbb{Z})$ is the set of $2n \times 2n$ matrices (written in the form of $n \times n$ blocks)

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_{2n}(\mathbb{Z}) : \ A\,{}^tB,\ C\,{}^tD \text{ are symmetric, } A\,{}^tD - B\,{}^tC = I \right\}.$$

For $\mathcal{N} \in \mathbb{Z}_+$, $\Gamma_0(\mathcal{N})$ is the subgroup of $Sp_n(\mathbb{Z})$ consisting of those matrices where the block C is congruent to 0 modulo $\mathcal{N}$; Γ_∞ is the subgroup of $Sp_n(\mathbb{Z})$ consisting of those matrices where the block C is equal to 0. Each 0-dimensional cusp for the Siegel upper half-space

$$\mathcal{H}_{(n)} = \{X + iY : \; X, Y \in \mathbb{R}^{n,n}_{\text{sym}}, \; Y > 0 \;\}$$

and each Eisenstein series in the natural basis for the subspace of $\Gamma_0(\mathcal{N})$-Siegel Eisenstein series corresponds to an element of

$$\Gamma_\infty \backslash Sp_n(\mathbb{Z}) / \Gamma_0(\mathcal{N}).$$

For $\gamma_0 \in Sp_n(\mathbb{Z})$, the weight k Eisenstein series corresponding to $\Gamma_\infty \gamma_0 \Gamma_0(\mathcal{N})$ is defined by

$$\mathbb{E}_{\gamma_0}(\tau) = \sum_\gamma \overline{\chi}(\det D_\gamma)\, 1(\tau)|\gamma_0\gamma$$

where $\Gamma_\infty\gamma_0\gamma$ varies over the $\Gamma_0(\mathcal{N})$-orbit of $\Gamma_\infty\gamma_0$, and for $\gamma = \begin{pmatrix} * & * \\ M & N \end{pmatrix} \in Sp_n(\mathbb{Z})$, $1(\tau)|\gamma = \det(M\tau + N)^{-k}$; here $\tau \in \mathcal{H}_{(n)}$.

From now on, suppose $\mathcal{N}$ is square-free. We can show that the elements of $\Gamma_\infty \backslash Sp_n(\mathbb{Z}) / \Gamma_0(\mathcal{N})$ correspond to factorisations of $\mathcal{N}$ as a product of $n+1$ positive integers as follows: With $\mathcal{N}_0 \cdots \mathcal{N}_n = \mathcal{N}$, we have $\Gamma_\infty \begin{pmatrix} * & * \\ M & N \end{pmatrix}$ and $\Gamma_\infty \begin{pmatrix} * & * \\ M' & N' \end{pmatrix}$ in the same $\Gamma_0(\mathcal{N})$-orbit if and only if $\operatorname{rank}_q M = \operatorname{rank}_q M'$ for each prime $q|\mathcal{N}$ (here $\operatorname{rank}_q M$ denotes the rank of M over $\mathbb{Z}/q\mathbb{Z}$). Thus we can parameterise our basis of the space of Eisenstien series by these factorisations of $\mathcal{N}$, labeling the basis elements as $\mathbb{E}_{(\mathcal{N}_0,\dots,\mathcal{N}_n)}$. (If $\chi_q^2 = 1$ for any prime $q|\mathcal{N}_i$, $0 < i < n$, the above series for $\mathbb{E}_{(\mathcal{N}_0,\dots,\mathcal{N}_n)}$ is not well-defined. If we try to build $\mathbb{E}_{(\mathcal{N}_0,\dots,\mathcal{N}_n)}$ from $\Gamma(\mathcal{N})$-Eisenstein series when $\chi_q^2 \neq 1$ for some $q|\mathcal{N}_i$, $0 < i < n$, we get 0. So we only have $\mathbb{E}_{(\mathcal{N}_0,\dots,\mathcal{N}_n)}$ in our basis when $\chi_q^2 = 1$ for all primes $q|\mathcal{N}_1 \cdots \mathcal{N}_{n-1}$.)

Now consider the case that $n = 2$, and let $\rho = (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$ where $\mathcal{N}_0\mathcal{N}_1\mathcal{N}_2 = \mathcal{N}$. Consider p a prime not dividing the level $\mathcal{N}$; the sum for $\mathbb{E}_\rho(\tau)|T(p)$ involves terms such as

$$\begin{aligned}
&p^{k-3}\chi(p)\overline{\chi}_\rho(M,N) \\
&\quad \cdot \det\left(M \begin{pmatrix} 1 & \\ & p \end{pmatrix} G^{-1} \left(\tau\, {}^tG^{-1} + \begin{pmatrix} u & \\ & 0 \end{pmatrix} \right) \begin{pmatrix} 1/p & \\ & 1 \end{pmatrix} + N \right)^{-k} \\
&= p^k p^{k-3}\chi(p)\overline{\chi}_\rho(M,N) \\
&\quad \cdot \det\left(M \begin{pmatrix} 1 & \\ & p \end{pmatrix} G^{-1}\tau + \left(M \begin{pmatrix} u & \\ & 0 \end{pmatrix} + N \right) {}^tG \begin{pmatrix} p & \\ & 1 \end{pmatrix} \right)^{-k}
\end{aligned}$$

where u varies over $(\mathbb{Z}/p\mathbb{Z})^\times$, G varies over $SL_n(\mathbb{Z})/\mathcal{K}$,

$$\mathcal{K} = \left\{ \gamma \in SL_n(\mathbb{Z}) : \; \gamma \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \; (p) \; \right\},$$

and $\chi_\rho(M,N) = \chi(\det D_\gamma)$ where $\Gamma_\infty \begin{pmatrix} * & * \\ M & N \end{pmatrix} = \Gamma_\infty \gamma_0 \gamma$ for $\gamma \in \Gamma_0(\mathcal{N})$. The other terms can be similarly massaged to be of the shape

$$p^{2k-3}\chi(p^2)\overline{\chi}_\rho(M,N)\det(pM\tau+N)^{-k},$$

and

$$p^{2k-3}\overline{\chi}_\rho(M,N)\det(M\tau+MY+pN)^{-k}$$

with $Y \in \mathbb{Z}^{2,2}_{\text{sym}}$ varying modulo p. We demonstrate how we proceed with the latter terms.

Suppose $p \nmid \mathcal{N}$; decompose $\mathbb{E}_\rho$ as $\mathbb{E}_0 + \mathbb{E}_1 + \mathbb{E}_2$ where each E_i is supported on pairs $(M\ N)$ with $\operatorname{rank}_p M = i$. To demonstrate our technique, consider the case $\operatorname{rank}_p M = 1$. Adjust the $SL_2(\mathbb{Z})$-equivalence class representative $(M\ N)$ to assume that p divides the lower row of M; set

$$(M'\ N') = \begin{pmatrix} 1 & \\ & \frac{1}{p} \end{pmatrix} (M \quad MY + pN).$$

The upper row of M is non-zero modulo p, as is the lower row of N; so $(M'\ N')$ is a coprime symmetric pair, with $\operatorname{rank}_p M' \geq 1$. Also, $p^{-k}\det(M'\tau + N')^{-k} = \det(M\tau+MY+pN)^{-k}$. We can choose $Y \equiv 0\ (\mathcal{N})$; then $(M'\ N') \equiv \begin{pmatrix} 1 & \\ & 1/p \end{pmatrix} (M\ pN)\ (\mathcal{N})$ and so

$$\begin{aligned} p^{-k}\chi_{\mathcal{N}_0\mathcal{N}_2}(p)\cdot\overline{\chi}_\rho(M',N')\det(M'\tau+N')^{-k} \\ = \overline{\chi}_\rho(M,N)\det(M\tau+MY+pN)^{-k}. \end{aligned}$$

Reversing, given a coprime symmetric pair $(M'\ N')$, we need to count the number of times an element of $SL_2(\mathbb{Z})(M'\ N')$ is generated through the preceeding process. Thus as we vary $E \in SL_2(\mathbb{Z})$, $Y \in \mathbb{Z}^{2,2}_{\text{sym}}$ modulo p, we need to count how often

$$(M\ N) = \begin{pmatrix} 1 & \\ & p \end{pmatrix} E \left(M' \quad \frac{1}{p}(N' - M'Y) \right)$$

represents distinct, integral $SL_2(\mathbb{Z})$-equivalence classes with $\operatorname{rank}_p M = 1$. For $E \in SL_2(\mathbb{Z})$, we have $\begin{pmatrix} 1 & \\ & p \end{pmatrix} E \begin{pmatrix} 1 & \\ & 1/p \end{pmatrix} \in SL_2(\mathbb{Z})$ if and only if $E \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}\ (p)$ (so using such E does not change the $SL_2(\mathbb{Z})$-equivalence class of $(M\ N)$). Thus we need to consider $E \in \mathcal{K}\backslash SL_2(\mathbb{Z})$; so we can take $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, α varying modulo p.

(a) Say $\operatorname{rank}_p M' = 1$; assume p divides row 2 of M'. For 1 choice of E, p divides row 1 of EM', and then since $M = \begin{pmatrix} 1 & \\ & p \end{pmatrix} EM'$, $\operatorname{rank}_p M \neq 1$. So take any other choice of E (p choices). To have N integral, we need to choose Y so that

$$E(N' - M'Y) \equiv \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \quad (p);$$

there are p choices for Y that satisfy this condition. Then we necessarily have that $(M\ N)$ is a coprime pair. This gives us a contribution to $\mathbb{E}_\rho|T(p)$ of $p^{-k}\chi_{\mathcal{N}_0\mathcal{N}_2}(p)\cdot p^2{\cdot}p^{2k-3}\mathbb{E}_1$.

(b) Say $\operatorname{rank}_p M' = 2$. To have N integral we need

$$E(N' - M'Y) \equiv \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \quad (p),$$

and to have $(M\ N)$ coprime we need

$$E(N' - M'Y) \not\equiv 0 \ \ (p).$$

So we have $p+1$ choices for E, and for each of these, $p-1$ choices for Y. This gives us a contribution to $\mathbb{E}_\rho|T(p)$ of $p^{-k}\chi_{\mathcal{N}_0\mathcal{N}_2}(p)\cdot(p^2-1){\cdot}p^{2k-3}\mathbb{E}_2$.

We continue in this manner to evaluate the action of $T(p)$ and $T_1(p^2)$. This gives us the following (Propositions 3.3 and 3.4 in [18]):

Proposition 2.1. *For $p\nmid\mathcal{N}$, we have*

$$\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}|T(p) = \lambda(p)\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}$$

where

$$\lambda(p) = \left(\chi_{\mathcal{N}_0\mathcal{N}_1}(p)p^{k-1} + \chi_{\mathcal{N}_2}(p)\right)\left(\chi_{\mathcal{N}_0}(p)p^{k-2} + \chi_{\mathcal{N}_1\mathcal{N}_2}(p)\right),$$

and

$$\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}|T_1(p^2) = \lambda_1(p^2)\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}$$

where

$$\lambda_1(p^2) = (p+1)\left(\chi_{\mathcal{N}_0}(p^2)p^{2k-3} + \chi(p)p^{k-3}(p-1) + \chi_{\mathcal{N}_2}(p^2)\right).$$

For a prime $q|\mathcal{N}$, we proceed as before, but now the the comparison of $\chi_\rho(M,N)$ and $\chi_\rho(M',N')$ contributes character sums on χ_q. When $q|\mathcal{N}_0$ we get the Hecke relation

$$\begin{aligned}\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}|T(q) &= \lambda(q)\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)} \\ &\quad + a\mathbb{E}_{(\mathcal{N}_0/q,q\mathcal{N}_1,\mathcal{N}_2)} + b\mathbb{E}_{(\mathcal{N}_0/q,\mathcal{N}_1,q\mathcal{N}_2)},\end{aligned}$$

where $a\neq 0$ iff $\chi_q = 1$, $b\neq 0$ iff $\chi_q^2 = 1$, and

$$\lambda(q) = \chi_{\mathcal{N}_1}(q)\chi_{\mathcal{N}_2}(q^2).$$

For $q|\mathcal{N}_1$, we have

$$\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}|T(q) = \lambda(q)\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)} + c\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1/q,q\mathcal{N}_2)}$$

where $c\neq 0$ iff $\chi_q = 1$ and

$$\lambda(q) = \chi_{\mathcal{N}_0\mathcal{N}_2}(q)q^{k-1};$$

for $q|\mathcal{N}_2$, we have

$$\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}|T(q) = \lambda(q)\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}$$

where

$$\lambda(q) = \chi_{\mathcal{N}_0}(q^2)\chi_{\mathcal{N}_1}(q)q^{2k-3}.$$

With $q|\mathcal{N}$, we have similar statements for $\mathbb{E}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)}|T_1(q^2)$, where

$$\lambda_1(q^2) = \begin{cases} \chi_{\mathcal{N}_2}(q^2)(q+1) & \text{if } q|\mathcal{N}_0, \\ \chi_{\mathcal{N}_0}(q^2)q^{2k-3} + \chi_{\mathcal{N}_2}(q^2)q & \text{if } q|\mathcal{N}_1, \\ \chi_{\mathcal{N}_0}(q^2)(q+1)q^{2k-3} & \text{if } q|\mathcal{N}_2. \end{cases}$$

Using these results, we construct a basis

$$\left\{ \widetilde{\mathbb{E}}_{(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2)} : \mathcal{N}_0\mathcal{N}_1\mathcal{N}_2 = \mathcal{N},\ \chi^2_{\mathcal{N}_1} = 1 \right\}$$

for $\mathcal{E}_k^{(2)}(\mathcal{N},\chi)$ consisting of eigenforms for the full Hecke algebra; their eigenvalues are the λ, λ_1 given above.

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