Jacquet-Langlands-Shimizu correspondence for theta lifts to $GSp(2)$ and its inner forms

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with an appendix by Ralf Schmidt

Abstract

As was first pointed out by Ibukiyama [I], the spinor $L$-functions of automorphic forms on the indefinite symplectic group $GSp(1,1)$ or the definite symplectic group $GSp^*(2)$ over $\mathbb{Q}$ right invariant by a (global) maximal compact subgroup are conjectured to be those of paramodular forms of some specified level on the symplectic group $GSp(2)$, which can be viewed as a generalization of the Jacquet-Langlands-Shimizu correspondence to the case of $GSp(2)$ and its inner forms $GSp(1,1)$ and $GSp^*(2)$.

This short note surveys our results presented at the RIMS-conference held during January 16-21 in 2012. They provide evidence of this conjecture by theta lifts from $GL(2) \times B^\times$ to the inner forms and theta lifts from $GL(2) \times GL(2)$ to $GSp(2)$ (considered by [O]), where $B$ denotes a definite quaternion algebra over $\mathbb{Q}$. Our explicit functorial correspondence given by these theta lifts are proved to be compatible with a non-archimedean local Jacquet-Langlands correspondence for $GSp(2)$ (or $GSp(4)$) and its inner forms, which is considered in the appendix by Ralf Schmidt.

1 Basic facts

1.1 Algebraic groups.

Let $B$ be a definite quaternion algebra over $\mathbb{Q}$ with the discriminant $d_B$, and let $B \ni x \mapsto \bar{x} \in B$ be the main involution of $B$. By $n$ and $\text{tr}$ we denote the reduced norm and the reduced trace of $B$ respectively.

Let $G_{nc} = GSp(1,1)$ and $G_{nc}^1 = Sp(1,1)$ be the $\mathbb{Q}$-algebraic groups defined by

$$G_{nc}(\mathbb{Q}) := \{ g \in M_2(B) \mid {}^t\bar{g}Q_{nc}g = \nu(g)Q_{nc}, \nu(g) \in \mathbb{Q}^\times \}, \quad G_{nc}^1(\mathbb{Q}) := \{ g \in G_{nc}(\mathbb{Q}) \mid \nu(g) = 1 \},$$

where $Q_{nc} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Furthermore let $G_c = GSp^*(2)$ and $G_c^1 = Sp^*(2)$ be the $\mathbb{Q}$-algebraic groups defined by

$$G_c(\mathbb{Q}) := \{ g \in M_2(B) \mid {}^t\bar{g}Q_cg = \mu(g)Q_c, \mu(g) \in \mathbb{Q}^\times \}, \quad G_c^1(\mathbb{Q}) := \{ g \in G_c(\mathbb{Q}) \mid \mu(g) = 1 \},$$

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where $Q_c := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

On the other hand, let $G' = GSp(2)$ be the $\mathbb{Q}$-algebraic group defined by
\[
G'(\mathbb{Q}) := \left\{ g \in GL_4(\mathbb{Q}) \mid t g \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}, \lambda(g) \in \mathbb{Q}^\times \right\}.
\]

We should note that $G_{nc}$ and $G_c$ are inner $\mathbb{Q}$-forms of $G'$. By $Z\mathcal{G}$ we denote the center of $\mathcal{G} = G_{nc}, G_c$ or $G_s$.

In what follows, we often put $G = G_c$ or $G_{nc}$.

### 1.2 Maximal compact subgroups.

Let $Q = Q_{nc}$ or $Q_c$. We first introduce maximal compact subgroups at the archimedean place. We put $G^1_{\infty} := \{g \in M_2(\mathbb{H}) \mid t_{\overline{g}Qg} = Q\}$, where $\mathbb{H} := B \otimes_{\mathbb{Q}} \mathbb{R}$ is the Hamilton quaternion algebra. Then $G^1_{\infty}$ is the maximal compact subgroup itself when $Q = Q_c$, and $K^0_{\infty} := \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M_2(\mathbb{H}) \mid a \pm b \in \mathbb{H}^1$ forms a maximal compact subgroup of $G^1_{\infty}$ when $Q = Q_{nc}$, and $K^0_{\infty} \simeq \mathbb{H}^1 \times \mathbb{H}^1$.

We next put $G'_{\infty} := \left\{ g \in GL_4(\mathbb{R}) \mid t g \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} g = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\}$. Then
\[
K^0_{\infty} := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A + \sqrt{1}B \in U(2) \right\}
\]
is a maximal compact subgroup of $G'_{\infty}$, where $U(2) := \{X \in M_2(\mathbb{C}) \mid t\overline{X}X = 1_2\}$ denotes the unitary group of degree two. The map $K^0_{\infty} \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in U(2)$ induces an isomorphism $K^0_{\infty} \simeq U(2)$.

Let us introduce maximal compact subgroups at non-archimedean places. We first deal with the case of $G = GSp(1,1)$ or $GSp^*(2)$. We remark that $GSp(1,1)$ and $GSp^*(2)$ are isomorphic to each other over $\mathbb{Q}_p$. We can thus identify $GSp(1,1)(\mathbb{Q}_p)$ with $GSp^*(2)(\mathbb{Q}_p)$.

We let $D$ be a divisor of $d_B$ and fix a maximal order $\mathfrak{O}$ of $B$. For $p|d_B$ let $\mathfrak{P}_p$ be the maximal ideal of the $p$-adic completion $\mathfrak{O}_p$ of $\mathfrak{O}$ and let
\[
L_p := \begin{cases} t(\mathfrak{O}_p \oplus \mathfrak{O}_p) & (p \nmid d_B \text{ or } p|D), \\ t(\mathfrak{O}_p \oplus \mathfrak{P}_p^{-1}) & (p|d_B). \end{cases}
\]

Then $K_p := \{k \in G_p \mid kL_p = L_p\}$ is a maximal compact subgroup of $G_p$ for each finite prime $p$ when $G = GSp(1,1)$ or $GSp^*(2)$. Every maximal compact subgroup of $G_p$ is conjugate to some $K_p$ by $G_p$. 

Let us next deal with the case of $GSp(2)$. When $p$ does not divide $d_B$, we put $K'_p := GSp(2)(\mathbb{Z}_p)$. When $p|d_B$ we put

$$K'_p := \left\{ \begin{array}{c}
\begin{pmatrix}
\mathbb{Z}_p & \mathbb{Z}_p & p^{-1}\mathbb{Z}_p & \mathbb{Z}_p \\
p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
p^2\mathbb{Z}_p & p^2\mathbb{Z}_p & p\mathbb{Z}_p & p^2\mathbb{Z}_p \\
p^2\mathbb{Z}_p & p^2\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
p^2\mathbb{Z}_p & p^2\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \
\end{pmatrix}
\end{array} \right\} \cap GSp(2)(\mathbb{Q}_p)
(p|D),$$

$$K'_p := \left\{ \begin{array}{c}
\begin{pmatrix}
\mathbb{Z}_p & \mathbb{Z}_p & p^{-2}\mathbb{Z}_p & \mathbb{Z}_p \\
p^2\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
p^2\mathbb{Z}_p & p^2\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
p^2\mathbb{Z}_p & p^2\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \
\end{pmatrix}
\end{array} \right\} \cap GSp(2)(\mathbb{Q}_p)
(p|D).$$

We call this open compact subgroup of $GSp(2)(\mathbb{Q}_p)$ a paramodular subgroup of $GSp(2)(\mathbb{Q}_p)$ of level $p$ or $p^2$, which is maximal when the level is $p$. We remark that $K_p \simeq K'_p$ for $p \nmid d_B$.

2 Theta lifts to $GSp(1, 1)$, $GSp^*(2)$ and $GSp(2)$.

Let $H$ and $H'$ be $\mathbb{Q}$-algebraic groups defined by

$$H(\mathbb{Q}) = GL_2(\mathbb{Q}), \ H'(\mathbb{Q}) := B^\times$$

respectively. For a positive integer $\kappa$ we let $S_{\kappa}(D)$ be the space of elliptic cusp forms of weight $\kappa$ with level $D$ (cf. [M-N-2, Section 3.1]). For a non-negative integer $\kappa'$ we let $A_{\kappa'}$ be the space of automorphic forms of weight $\sigma_{\kappa'}$ with respect to $\prod_{p<\infty} \mathcal{O}_p^\times$ (cf. [M-N-2, Section 3.2]), where $\mathcal{O}_p^\times$ denotes the unit group of $\mathcal{O}_p$.

For Hecke eigenforms $(f, f') \in S_{\kappa}(D) \times A_{\kappa'}$ let $\pi(f)$ be the automorphic representation of $GL_2(\mathfrak{h})$ generated by $f$ and JL$(\pi(f'))$ be the Jacquet-Langlands lift of the automorphic representation $\pi(f')$ generated by $f'$. The Hecke equivariant isomorphism between $A_{\kappa_2}$ and the space of new forms in $S_{\kappa_2+2}(d_B)$ (Eichler [E-1], [E-2], Shimizu [Sh]) sends a Hecke eigenform $f'$ to a primitive form JL$(f')$. The automorphic representation JL$(\pi(f'))$ is nothing but that generated by JL$(f')$.

2.1 Theta lift to $G$

For every finite prime $p < \infty$ let $\mathcal{V}_p$ be the space of locally constant compactly supported functions on $B_p^2 \times \mathcal{O}_p^\times$. Let $\mathcal{S}(\mathbb{H}^2)$ stand for the space of Schwartz functions on $\mathbb{H}^2$. When $G = G_{nc}$ (respectively $G = G_c$) we then introduce the space $\mathcal{V}_\infty$ of smooth
functions $\varphi$ on $\mathbb{H}^2 \times \mathbb{R}^\times$ such that, for each fixed $t \in \mathbb{R}^\times$, $\varphi(X, t) \in S(\mathbb{H}^2) \otimes \text{End}(V_{\frac{\kappa_1 + \kappa_2}{2}} \otimes V_{\frac{\kappa_1 - \kappa_2}{2}})$ for $(\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^2$ with $\kappa_1 \leq \kappa_2$ (respectively $S(\mathbb{H}^2) \otimes \text{End}(\mathcal{H}_{\kappa_1-4})$ for $\kappa_1 \in 2\mathbb{Z}_{\geq 0}$ with $\kappa_1 \geq 4$), where $\mathcal{H}_{\kappa_1-4}$ denotes the space of homogeneous harmonic polynomials of degree $\kappa_1 - 4$ on $\mathbb{H}^2$. We let $\varphi_{0,p} \in \mathbb{V}_p$ be the characteristic function of $L_p \times \mathbb{Z}_p^\times$.

Let $G = G_{nc}$. For $(\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^2$ with $\kappa_1 \leq \kappa_2$ we define $\varphi_{0,\infty}^{nc} = \varphi_{0,\infty}^{nc, (\kappa, \kappa)} \in \mathbb{V}_\infty$ by

$$\varphi_{0,\infty}^{nc}(X, t) = \begin{cases} t^{\frac{\kappa_2 + 3}{2}} \sigma_{\underline{\kappa}} \mapsto^{+\kappa_2} (X_1 + X_2) \otimes \sigma_{\underline{\kappa}} & (t > 0), \\
0 & (t < 0). \end{cases}$$

Let $G = G_c$. For $\kappa_1 \in 2\mathbb{Z}_{\geq 0}$ with $\kappa_1 \geq 4$, following [Lo, Definition 6.1], we define $\varphi_{0,\infty}^{c} = \varphi_{0,\infty}^{c, (\kappa, \kappa)} \in \mathbb{V}_\infty$ by

$$\varphi_{0,\infty}^{c}(X, t) = \begin{cases} t^{\frac{\kappa_1 - 1}{2}} \exp(-2\pi t \delta X_1) C(X) & (t > 0), \\
0 & (t < 0), \end{cases}$$

where $C$ is the $\text{Hom}(\mathcal{H}_{\kappa_1 - 4}, \mathcal{H}_{\kappa_1 - 4}^*) \simeq \text{End}(\mathcal{H}_{\kappa_1 - 4})$-valued function on $\mathbb{H}^2$ defined by $C(X)(h) = h(X)$ ($h \in \mathcal{H}_{\kappa_1 - 4}$), where $\mathcal{H}_{\kappa_1 - 4}^*$ denotes the dual space of $\mathcal{H}_{\kappa_1 - 4}$.

Following [M-N-1, Section 3] we introduce a metaplectic representation $r = \otimes_{v \leq \infty} r_v$ of $G(\mathbb{A}) \times H(\mathbb{A}) \times H'(\mathbb{A})$ on the restricted tensor product $\mathbb{V} = \otimes_{v \leq \infty} \mathbb{V}_v$ with respect to $\{\varphi_{0,p}\}_{p \leq \infty}$. It is associated with the standard additive character $\psi$ of $\mathbb{A}$. For $G = G_{nc}$ (respectively $G = G_c$) we define the $\text{End}(V_{\frac{\kappa_1 + \kappa_2}{2}} \otimes V_{\frac{\kappa_1 - \kappa_2}{2}})$-valued theta function (respectively $\text{End}(\mathcal{H}_{\kappa_1 - 4})$-valued theta function) $\theta_{\kappa_1, \kappa_2}(g, h, h')$ by

$$\sum_{(X, t) \in B^2 \times \mathbb{Q}^\times} r(g, h, h') \varphi_0(X, t),$$

where $\varphi_0 := \prod_{v \leq \infty} \varphi_{0,v}$ with

$$\varphi_{0,\infty} := \begin{cases} \varphi_{0,\infty}^{nc} & (G = G_{nc}), \\
\varphi_{0,\infty}^{c} & (G = G_c). \end{cases}$$

When $G = G_{nc}$ (respectively $G = G_c$), for $(\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^2$ with $\kappa_1 \leq \kappa_2$ (respectively with $\kappa_1 \geq \kappa_2$ and $\kappa_1 \geq 4$), we consider the theta lift

$$S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2} \ni (f, f') \mapsto \mathcal{L}(f, f')(g)$$

with

$$\mathcal{L}(f, f')(g) := \int_{\mathbb{R}^2_+} \overline{f(h)} \theta_{\kappa_1, \kappa_2}(g, h, h') f'(h') dh dh'.$$
By an argument similar to the proof of [M-N-1, Theorem 4.1], we verify that this is convergent on any compact subset of $G(\mathbb{A})$ when $G = G_{\text{nc}}$. On the other hand, when $G = G_{c}$, the theta function $\theta_{\kappa_{1}, \kappa_{2}}(g, h, h') f'(h')$ with a fixed $(g, h')$ can be viewed as an elliptic modular form of weight $\kappa_{1}$ and level $D$ (cf. [He, Section 6]). The convergence of the integral is thus reduced to that of the Petersson inner product of an elliptic modular form and an elliptic cusp form.

**Theorem 2.1.** Let $(\kappa_{1}, \kappa_{2}) \in (2\mathbb{Z}_{>0})^{2}$.

(1) The theta lift $\mathcal{L}(f, f')$ defines an automorphic forms, more precisely, it is left-$G(\mathbb{Q})$-invariant, right $K_{f}(D)$-invariant and right $K^{0}_{\infty}$-equivariant (respectively $G^{1}_{\infty}$-equivariant) with respect to the irreducible representation with highest weight $(\frac{\kappa_{2}+\kappa_{1}}{2}, \frac{\kappa_{1}+\kappa_{2}}{2})$ (respectively $(\frac{\kappa_{1}+\kappa_{2}}{2} - 1, \frac{\kappa_{2}-\kappa_{1}}{2} - 1)$) when $G = G_{\text{nc}}$ (respectively $G = G_{c}$). Furthermore $\mathcal{L}(f, f')$ has the trivial central character.

(2) Suppose that $(f, f')$ are Hecke eigenforms. Then $\mathcal{L}(f, f')$ is also a Hecke eigenform. Furthermore, for each $p | D$, let $\epsilon_{p}$ (respectively $\epsilon_{p}'$) be the eigenvalue for the involutive action of $\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$ (resp. a prime element $\varpi_{B, p} \in B_{p}$) on $f$ (resp. $f'$). Then $\mathcal{L}(f, f') \equiv 0$ unless $\epsilon_{p} = \epsilon_{p}'$.

(3) Assume furthermore that $1 < \kappa_{1} < \kappa_{2} + 2$ when $G = G_{\text{nc}}$ (respectively $1 < \kappa_{2} + 2 < \kappa_{1}$ when $G = G_{c}$). Then $\mathcal{L}(f, f')$ is a cusp form on $G_{\text{nc}}(\mathbb{A})$ generating, at the archimedean place, the discrete series representation with Harish-Chandra parameter $\lambda = (\frac{\kappa_{2}-\kappa_{1}}{2} + 1, \frac{\kappa_{1}+\kappa_{2}}{2})$ (respectively automorphic forms on $G_{c}(\mathbb{A})$ generating, at the archimedean place, the discrete series representation with Harish-Chandra parameter $\lambda = (\frac{\kappa_{1}+\kappa_{2}}{2}, \frac{\kappa_{2}-\kappa_{1}}{2} - 1)$) as a $(\mathfrak{g}, K^{0}_{\infty})$-module, where $\mathfrak{g}$ denotes the Lie algebra of $G^{1}_{\infty}$.

Outline of proof:

(1) The assertion is essentially due to [M-N-1, Section 4].

(2) This follows from [M-N-1, Theorem 5.1] and [M-N-1, Remark 5.2 (ii)].

(3) The fact that $\mathcal{L}(f, f')$ is cuspidal when $G = G_{\text{nc}}$ is shown in a manner similar to [M-N-2, Section 13.4]. To determine the representation type of $\mathcal{L}(f, f')$ at the Archimedean place, we use the result by Li-Paul-Tan-Zhu [L-P-T-Z, Theorem 5.1] on the archimedean theta correspondence, in which $\mathcal{L}(f, f')$ is involved. When $G = G_{c}$ this assertion then follows immediately. In view of the archimedean theta correspondence and the discrete decomposability of the cuspidal spectrum (cf. [G-G-P]), we thus see that, when $G = G_{\text{nc}}$, the archimedean component of the $G^{1}(\mathbb{A})$-module generated by $\mathcal{L}(f, f')'(gf^{*})$ with any fixed $gf \in G(\mathbb{A})$ is isomorphic to the discrete series representation in the statement as a $(\mathfrak{g}, K^{0}_{\infty})$-module.

### 2.2 Theta lift to $G'$

We next consider the theta lift from $S_{\kappa_{1}}(D_{1}) \times S_{\kappa_{2}}(D_{2})$ to automorphic forms on $G'(\mathbb{A})$. As in [O], we formulate the lift using the metaplectic representation $r'$ of $G' \times H^{1}$ considered by Harris-Kudla [Ha-K] and Roberts [R], where $H^{1}$ denotes the $\mathbb{Q}$-algebraic group defined by

$$\{(h_{1}, h_{2}) \in GL_{2} \times GL_{2} \mid \det(h_{1}) = \det(h_{2})\}.$$
Now let us introduce a quadratic space \((M_2(\mathbb{Q}), \det)\) and note that the action of \(H^1(\mathbb{Q})\) on \(M_2(\mathbb{Q})\) defined by

\[ h \cdot X = h_1^{-1}Xh_2 \quad (X \in M_2(\mathbb{Q}), \ h = (h_1, h_2) \in H^1(\mathbb{Q})) \]

induces a well-known isomorphism

\[ H_1(\mathbb{Q})/(\{(z, z) \mid z \in \mathbb{Q}^\times\}) \simeq GSO(2,2)(\mathbb{Q}). \]

We assume that \(r'\) is associated with the additive character \(\psi(\frac{1}{2}*)\) on \(\mathbb{A}\). To construct the theta lift we now recall the choice of the Schwartz function on \(M_2(\mathbb{A})^{\oplus 2}\) in [O]. At a finite place \(v = p < \infty\), we let \(\varphi_{0,p}'\) be the Schwartz function on \(M_2(\mathbb{Q}_p)^{\oplus 2}\) given by the characteristic function of

\[ \left\{ \begin{array}{c} (a_{x_1}, b_{x_1}), (a_{x_2}, b_{x_2}) \mid a_{x_1} \in D_2 \mathbb{Z}_p, \ b_{x_1} \in \mathbb{Z}_p, \ c_{x_1} \in D_1 D_2 \mathbb{Z}_p, \ d_{x_1} \in D_1 \mathbb{Z}_p, \\ a_{x_2}, b_{x_2}, c_{x_2}, d_{x_2} \in \mathbb{Z}_p, \end{array} \right\} \]

For the choice of the Schwartz function at the archimedean place we need two functions \(P_1\) and \(P_2\) on \(M_2(\mathbb{R})\) defined as follows:

\[ P_1(X) := \text{tr}(X \begin{pmatrix} -\sqrt{-1} & -1 \\ -1 & \sqrt{-1} \end{pmatrix}), \quad P_2(X) := \text{tr}(X \begin{pmatrix} -\sqrt{-1} & 1 \\ -1 & -\sqrt{-1} \end{pmatrix}) \quad (X \in M_2(\mathbb{R})) \]

Let \(\mathbb{C}[s_1, s_2]\) denote the polynomial ring of two variables \(s_1\) and \(s_2\) over \(\mathbb{C}\). As our choice of the test function at \(v = \infty\) we take the \(\mathbb{C}[s_1, s_2]\)-valued Schwartz function \(\varphi_{\infty,0}'\) on \(M_2(\mathbb{R})^{\oplus 2}\) as follows:

\[ \varphi_{\infty,0}'(X_1, X_2) := \exp(-\pi \text{tr}(tX_1X_1 + tX_2X_2))P_1(s_1X_1 + s_2X_2)^{\frac{s_1 + s_2}{2}} \]

\[ \times \left\{ \begin{array}{c} P_2(s_2X_1 - s_1X_2)^{\frac{s_2 - s_1}{2}} \quad (\kappa_1 \geq \kappa_2) \\ \bar{P}_2(s_2X_1 - s_1X_2)^{\frac{s_2 - s_1}{2}} \quad (\kappa_1 \leq \kappa_2). \end{array} \right\} \]

Put \(\varphi_0' := \otimes_{v \leq \infty} \varphi_{\infty,0}'\) and define the theta series \(\theta'_{\kappa_1, \kappa_2}(g, h)\) on \(G'(\mathbb{A}) \times H^1(\mathbb{A})\) as

\[ \sum_{(X_1, X_2) \in M_2(\mathbb{Q})^{\oplus 2}} r'(g, h) \varphi_0'(X_1, X_2). \]

We view \(f_1 \otimes f_2 := f_1f_2\) as an automorphic form on \(H^1(\mathbb{A})\) or \((H \times H)(\mathbb{A})\) for \((f_1, f_2) \in S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2)\). We embed \(\mathbb{A}^\times\) into \(H^1(\mathbb{A})\) by

\[ \mathbb{A}^\times \ni a \mapsto (a \cdot 1_2, a \cdot 1_2) \in H^1(\mathbb{A}). \]

For \((\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^{\oplus 2}\) we then define the theta lifting from \(S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2)\) to \(G'(\mathbb{A})\) by

\[ S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2) \ni (f_1, f_2) \mapsto \mathcal{L}'(f_1, f_2)(g), \]

where \(\Lambda' = \left(\frac{\kappa_1 + \kappa_2}{2}, -\frac{\kappa_1 - \kappa_2}{2}\right)\) and

\[ \mathcal{L}'(f_1, f_2)(g) := \int_{\mathbb{A}^\times \backslash H^1(\mathbb{Q}) \backslash H^1(\mathbb{A})} \theta'_{\kappa_1, \kappa_2}(g, hh')f_1 \otimes f_2(hh')dh. \]
with an invariant measure $dh$ on $\mathbb{A}^x H^1(Q) \backslash H^1(\mathbb{A})$. Here for each $g \in G'(\mathbb{A})$, we take $h' = (h'_1, h'_2) \in (H \times H)(\mathbb{A})$ so that $\nu'(g) = \det(h'_1) \det(h'_2)^{-1}$. We note that this theta lift does not depend on the choice of $h'$.

We now quote the following theorem (cf. [O]):

**Theorem 2.2.** For two non-zero primitive cusp forms $(f_1, f_2) \in S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2)$, $\mathcal{L}'(f_1, f_2)$ is a non-zero generic cusp form on $G'(\mathbb{A}) = GSp(2)(\mathbb{A})$ with the trivial central character satisfying the following properties:

1. $\mathcal{L}'(f_1, f_2)$ is a non-zero generic cusp form on $G'(\mathbb{A}) = GSp(2)(\mathbb{A})$ with the trivial central character satisfying the following properties:

   1. $\mathcal{L}'(f_1, f_2)$ is a paramodular form of level $D_1D_2$, namely, at a prime $p|N := D_1D_2$, it is right invariant by a paramodular group

$K_{p,ord_pN} := \left( \begin{array}{cccc} Z_p & Z_p & N^{-1}Z_p & Z_p \\ NZ_p & Z_p & Z_p & Z_p \\ NZ_p & NZ_p & Z_p & NZ_p \\ NZ_p & Z_p & Z_p & NZ_p \end{array} \right) \cap GSp(2)_{\mathbb{Q}_p},$

2. When $\kappa_1 \neq \kappa_2$, $\mathcal{L}'(f_1, f_2)$ lies, at the archimedean place, in the minimal $K_{\infty}^0$-type $\tau_\Lambda'$ of the large discrete series representations $\pi_{\lambda'}$ with

$\lambda' = \left( \frac{\kappa_1 + \kappa_2}{2} - 1, -\frac{|\kappa_1 - \kappa_2|}{2} \right), \quad \Lambda' = \left( \frac{\kappa_1 + \kappa_2}{2}, -\frac{|\kappa_1 - \kappa_2|}{2} \right).$

3 The Jacquet-Langlands-Shimizu correspondence for the theta lifts

3.1 Automorphic $L$-functions

We now define the spinor $L$-function for $\mathcal{L}(f, f')$, modifying the definition of [M-N-3, Section 2.6] at the archimedean place.

In [M-N-1, Section 5.1] we introduced three Hecke operators $T^i_p$ with $0 \leq i \leq 2$ for $p \nmid d_B$. Let $\Lambda^i_p$ be the Hecke eigenvalue of $T^i_p$ for $F$ with $0 \leq i \leq 2$. For $p \nmid d_B$ we put

$$Q_{F,p}(t) := 1 - p^{-\frac{3}{2}}\Lambda^1_p t + p^{-2}(\Lambda^2_p + p^2 + 1)t^2 - p^{-\frac{3}{2}}\Lambda^3_p t^3 + t^4$$

For this we note that $Q_{F,p}(p^{-s})^{-1}$ coincides with the local spinor $L$-function for an unramified principal series of the group of $GSp(2)_{\mathbb{Q}_p}$. On the other hand, in [M-N-1, Section 5.2], we introduced two Hecke operators $T^i_p$ with $0 \leq i \leq 1$ for $p|d_B$. Let $\Lambda'^i_p$ be the Hecke eigenvalue of $T^i_p$ for $F$ with $0 \leq i \leq 1$. For $p|d_B$ we put

$$Q_{F,p}(t) := \begin{cases} (1 - p^{-\frac{3}{2}}(\Lambda'^1_p - (p-1)\Lambda'^0_p)t + t^2)(1 - \Lambda'^0_p p^{-\frac{1}{2}} t) & (p|d_B), \\ (1 + \Lambda'^1_p p^{-\frac{1}{2}} t)(1 - \Lambda'^0_p p^{-\frac{1}{2}} t) & (p|D). \end{cases}$$

The first one is due to Sugano [Su, (3.4)]. The first factor of the second one comes from the numerator of the formal Hecke series.

We define the spinor $L$-function $L(F, \text{spin}, s)$ of a Hecke eigenform $F$ on $G(\mathbb{A})$ with the following properties:
• $F$ is right $K_f(D)$-invariant and right $K^0_\infty$-equivariant with respect to the irreducible representation of highest weight $(\frac{\kappa_2-\kappa_1}{2}, \frac{\kappa_1+\kappa_2}{2})$.

• $F$ generates, as a $(\mathfrak{g}, K^0_\infty)$-module, the discrete series representation with Harish Chandra parameter $(\frac{\kappa_2-\kappa_1}{2}+1, \frac{\kappa_1+\kappa_2}{2})$ with $(\kappa_1, \kappa_2) \in 2\mathbb{Z}^\oplus 2$ such that $1 < \kappa_1 < \kappa_2+2$, where recall that $\mathfrak{g}$ denotes the Lie algebra of $G^1_{\infty}$ (cf. Theorem 2.1 (3)).

The definition is as follows:

$$L(F, \text{spin}, s) := \prod_{v \leq \infty} L_v(F, \text{spin}, s),$$

where

$$L_v(F, \text{spin}, s) := \begin{cases} Q_{F,p}(p^{-s})^{-1} & (v = p < \infty), \\ \Gamma_{\mathbb{C}}(s + \frac{\kappa_1-1}{2})\Gamma_{\mathbb{C}}(s + \frac{\kappa_2+1}{2}) & (v = \infty). \end{cases}$$

By virtue of Theorem 2.1 (3) we can use this definition for $F = \mathcal{L}(f, f')$ when $(f, f')$ are Hecke eigenforms.

We generalize [M-N-3, Proposition 2.9] to have the following:

**Proposition 3.1.** The spinor $L$-function for $\mathcal{L}(f, f')$ decomposes into

$$L(\mathcal{L}(f, f'), \text{spin}, s) = L(\pi(f), s)L(\pi(f'), s),$$

where $L(\pi(f), s)$ (resp. $L(JL(\pi(f'))$, $s)$) denotes the standard $L$-function of $\pi(f)$ (resp. $JL(\pi(f'))$).

Of course, we thus see that $L(\mathcal{L}(f, f'), \text{spin}, s)$ has the meromorphic continuation and satisfies the functional equation between $s$ and $1-s$ since so do $L(\pi(f), s)$ and $L(JL(\pi(f'))$, $s)$.

We now recall that there is Novodvorsky's zeta integral of the spinor $L$-function for a generic cusp form on $G'(\mathbb{A})$ (cf. [No]). By means of the zeta integral, the theorem as follows (cf. [O]) describes the spinor $L$-function for a generic form $\mathcal{L}'(f_1, f_2)$.

**Theorem 3.2.** Let the notations be as in Theorem 2.2. Then the global spinor $L$-function of $\mathcal{L}'(f_1, f_2)$ decomposes into

$$L(\pi(f_1), s)L(\pi(f_2), s).$$

As an immediate consequence of Proposition 3.1 and this theorem we obtain the following:

**Corollary 3.3.** Let $f \in S_{\kappa_1}(D)$ be a primitive form and $f' \in A_{\kappa_2}$ be a Hecke eigenform. Then we have

$$L(\mathcal{L}(f, f'), \text{spin}, s) = L(\mathcal{L}'(f, JL(f'))), \text{spin}, s).$$
3.2 Automorphic representations generated by the theta lifts

We study locally and globally the representation $\pi(L(f, f'))$ of $G(\mathbb{A}) = GSp(1,1)(\mathbb{A})$ or $GSp^*(2)(\mathbb{A})$ generated by $L(f, f')$ (respectively the representation $\pi(L(f, JL(f'))) of G'(\mathbb{A}) = GSp(2)(\mathbb{A})$ generated by $L'(f, JL(f'))$).

(1) The case of $G$:

We first discuss the case of $G$. We note that the Lie algebra of the group $G_{\infty}/Z_{G_{\infty}}$ is isomorphic to the Lie algebra $g$ of $G_{\infty}^1$. The group $G_{\infty}/Z_{G_{\infty}}$ is isomorphic to $G_{\infty}^1$ when $G = G_{c}$ but it is neither connected or isomorphic to $G_{\infty}^1$ when $G = G_{nc}$. For $G = G_{nc}$ let $K_{\infty}$ be a maximal compact subgroup of $G_{\infty}/Z_{G_{\infty}}$. We can regard $K_{\infty}^0$ as the connected component of the identity for $K_{\infty}$. Take $\sigma := \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} \in G(\mathbb{R})$. We can then identify $K_{\infty}$ with $K_{\infty}^0 \cup K_{\infty}^0 \sigma$. For $(\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^{\oplus 2}$ with $1 < \kappa_1 + 2 < \kappa_2$ let $\pi_{\infty}^{(\kappa_1, \kappa_2)}$ be the discrete series representation of $G_{\infty}^1$ with Harish Chandra parameter $(\ell_{\kappa_2-\kappa_1} + 1, \ell_{\kappa_1+\kappa_2})$. Then we introduce another representation $\pi_{\infty}^{(\kappa_1, \kappa_2)}$ of $G_{\infty}^1$ defined by

$$\pi_{\infty}^{(\kappa_1, \kappa_2)}(g) = \pi_{\infty}^{(\kappa_1, \kappa_2)}(sg\sigma^{-1}) \quad \forall g \in G_{\infty}^1.$$

This is equivalent to the discrete series representation with Harish Chandra parameter $(\ell_{\kappa_1+\kappa_2}, \ell_{\kappa_2-\kappa_1} + 1)$, which is not isomorphic to $\pi_{\infty}^{(\kappa_1, \kappa_2)}$. There is an irreducible $(g, K_{\infty})$-module $V_{\infty}^{(\kappa_1, \kappa_2)}$ which is equivalent to $\pi_{\infty}^{(\kappa_1, \kappa_2)} \oplus \pi_{\infty, \sigma}^{(\kappa_1, \kappa_2)}$ as $(g, K_{\infty}^0)$-modules.

Proposition 3.4. Suppose that $f$ and $f'$ are Hecke eigenforms and that $1 < \kappa_1 + 2 < \kappa_2$ for $G = G_{nc}$ (respectively $1 < \kappa_2 + 2 < \kappa_1$ for $G = G_{c}$). Then the representation $\pi(L(f, f')) of G(\mathbb{A}_{\mathbb{Q}})$ is irreducible.

The point of proof is to use [N-P-S, Theorem 3.1]. We then reduce the global irreducibility to the local irreducibility at the archimedean place. When $G = G_{nc}$, we can verify that the archimedean component of $\pi(L(f, f'))$ is isomorphic to $V_{\infty}^{(\kappa_1, \kappa_2)}$.

We can therefore decompose $\pi(L(f, f'))$ into the restricted tensor product $\prod'_{v \leq \infty} \pi_{v}$ and are able to determine each local component $\pi_{v}$. To state our result on it we need several notation.

For a primitive cusp form $f \in S_{\kappa_1}(D)$ let $\pi(f)$ be an irreducible cuspidal representation of $GL_2(\mathbb{A})$, which admits a decomposition into the restricted tensor product $\pi(f) = \Pi'_{v \leq \infty} \pi(f)_{v}$. Then, for $v = p \nmid D$, $\pi(f)_{p}$ is an unramified principal series representation of $GL_2(\mathbb{Q}_{p})$. Let $\chi_{f, p}$ denote the unramified character of $\mathbb{Q}_{p}^\times$ which induces $\pi(f)_{p}$.

For a Hecke eigenform $f' \in A_{\kappa_2}$ let $\pi(f')$ be the irreducible automorphic representation of $H'(\mathbb{A})$ generated by $f'$, and let $\pi(f') = \Pi'_{v \leq \infty} \pi(f')_{v}$ be the decomposition into the restricted tensor product of local representations. When $p \nmid d_{B', \kappa_2}$, $\pi(f')_{p}$ is an unramified principal series representation of $B_{p}^{\times} \simeq GL_2(\mathbb{Q}_{p})$. We let $\chi_{f', p}$ be the unramified character of $\mathbb{Q}_{p}^{\times}$ inducing $\pi(f')_{p}$. When $p|d_{B}$, $\pi(f')_{p}$ is a character of $B_{p}$ of order at most two. Thus we have

$$\pi(f')_{p} = \delta_{p} \circ n$$
with a character $\delta_p$ of $\mathbb{Q}_p^\times$ of order at most two, where recall that the notation $n$ stands for the reduced norm of $B$ (cf. Section 1.1). In view of Theorem 2.1 (2), $\delta_p(p) = \epsilon'_p = \epsilon_p$ is necessary for $p|D$ in order that $\mathcal{L}(f,f') \neq 0$.

Following the notation of the appendix, let $\nu$ be the $p$-adic absolute value of $\mathbb{Q}_p$ and let $\xi$ be the non-trivial unramified character of $\mathbb{Q}_p^\times$ of order two for $p|d_B$. We further note that, in the appendix, the notation $\chi_1 B \rtimes \sigma$ is used for the induced representation of $GSp(1,1)(\mathbb{Q}_p)$ defined by two quasi-character $\chi$ and $\sigma$ of $\mathbb{Q}_p^\times$ when $p|d_B$. On the other hand, with three unramified quasi-characters $\chi_1, \chi_2$ and $\sigma$ of $\mathbb{Q}_p^\times$, $\chi_1 \times \chi_2 \rtimes \sigma$ denotes the unramified principal series representation of $GSp(2)(\mathbb{Q}_p)$, which is referred to as “type I” on the table of the appendix.

**Proposition 3.5.** Let the notation be as above.

1. Let $\nu = p \nmid d_B$. Then $\pi_p$ is an unramified principal series representation of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p) \simeq GSp(2)(\mathbb{Q}_p)$ given by $(\chi_{f'} - \chi_{f,p}^{-1}) \times (\chi_{f'} - \chi_{f,p}^{-1}) \rtimes \chi_{f,p}$.

2. Let $\nu = p|d_B$. When $\nu = p|d_B$, $\pi_p$ is isomorphic to the irreducible representation of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p)$ of type $I\lambda$ with $\alpha = \chi_{f,p}$ and $\chi = \chi_{f,p}' \cdot \delta_p$. When $\nu = p|D$ and $\delta_p$ is trivial (respectively non-trivial), $\pi_p$ is isomorphic to the irreducible representation of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p)$ of type $V_\alpha$ with $\sigma = \xi$ (respectively $\sigma = 1$), where, for the representations of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p)$ of type $I\lambda$ and $V_\alpha$, see the appendix.

3. When $\nu = \infty$ and $G = G_{nc}$, $\pi_{\infty}$ is isomorphic to $V^{(\kappa_1, \kappa_2)}_{\infty}$. When $\nu = \infty$ and $G = G_{c}$, $\pi_{\infty}$ is isomorphic to the irreducible representation with Harish-Chandra parameter $(\kappa_1, \kappa_2, -\frac{\kappa_1 - \kappa_2}{2} - 1)$ modulo center.

The archimedean component of $\pi(\mathcal{L}(f,f'))$ is already determined in the proof of Proposition 3.4. It thus suffices to consider the non-archimedean components. For every finite prime $p$, $\pi_p$ is a spherical representation of $G_p = GSp(1,1)(\mathbb{Q}_p)$ or $GSp(2)(\mathbb{Q}_p)$ (cf. [C]). As we see in [C], $\pi_p$ is uniquely determined by the Hecke eigenvalues. We calculate Hecke eigenvalues of $\mathcal{L}(f,f')$ explicitly in terms of eigenvalues for $(f,f')$ to obtain the assertion.

(2) The case of $G'$:

We next deal with the automorphic representation $\pi(\mathcal{L}'(f,JL(f'))) of GSp(2)(\mathbb{R})$ generated by $\mathcal{L}'(f,JL(f'))$. According to [R, Theorem 8.3], $\pi'(f,JL(f'))$ is an irreducible cuspidal representation. It thereby admits a decomposition into the restricted tensor product $\pi(\mathcal{L}'(f,JL(f'))) = \prod_{\nu<\infty} \pi'_\nu$. For each finite prime $\nu = p$, $\pi_p$ is involved in the local theta correspondence for $GSO(2,2)(\mathbb{Q}_p)$ and $GSp(2)(\mathbb{Q}_p)$, which is explicitly described in Gan-Takeda [G-T]. To describe each $\pi_p$ we use the notation of the appendix. To describe the archimedean component $\pi_{\infty}'$, we need to introduce, for two even integers $(\kappa_1, \kappa_2)$ with $1 < \kappa_1 + 2 < \kappa_2$, the irreducible admissible representation $V_{\infty}'^{(\kappa_1, \kappa_2)}$ of $GSp(2)(\mathbb{R})$ whose restriction to $Sp(2)(\mathbb{R})$ is the direct sum of the two large discrete series representation of $Sp(2)(\mathbb{R})$ with Harish Chandra parameters $(\kappa_1 + \kappa_2, 2, -\frac{\kappa_1 - \kappa_2}{2} - 1)$ and $(-\frac{\kappa_1 - \kappa_2}{2} - 1, -\frac{\kappa_1 + \kappa_2}{2})$.

**Proposition 3.6.** Let the notation be as above.

1. Let $\nu = p \nmid d_B$. Then $\pi'_p$ is isomorphic to $\pi_p$, namely an unramified principal series representation of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p) \simeq GSp(2)(\mathbb{Q}_p)$ given by $(\chi_{f'} - \chi_{f,p}^{-1}) \times (\chi_{f'} - \chi_{f,p}^{-1}) \rtimes \chi_{f,p}$.
(2) Let $v = p|d_B$. When $v = p|d_B$, $\pi'_p$ is isomorphic to the irreducible representation of $GSp(2)(\mathbb{Q}_p)$ of type $II_a$ with $\sigma = \chi_{f,p}$ and $\chi = \chi_{f,p}^{-1} \cdot \delta_p$. When $v = p|D$ and $\delta_p$ is trivial (respectively non-trivial), $\pi'_p$ is isomorphic to the irreducible representation of $GSp(2)(\mathbb{Q}_p)$ of type $V_a$ with $\sigma = \xi$ (respectively $\sigma = 1$), where, for the representations of $GSp(2)(\mathbb{Q}_p)$ of type $II_a$ and $V_a$, see the appendix.

(3) When $v = \infty$, $\pi'_\infty$ is isomorphic to $V^{(\kappa_1, \kappa_2)}$.

Using Przebinda [Prz], the representation $\pi'_\infty$ at the infinite prime $v = \infty$ is determined by the same reasoning as in the case of $GSp(1,1)(\mathbb{R})$. The representation $\pi_p$ is included in the table 2 of Section 14 or Theorem 8.2 (iv), (v), (vi) of Gan-Takeda [G-T]. Then, looking also at the table of the appendix, we have the assertion on $\pi_p$.

3.3 Conjecture and conclusion

Let $\mathcal{A}_G$ and $\mathcal{A}_{G'}$ denote the equivalence classes of irreducible automorphic representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively. We note that the $L$-group $L_G$ of $G$ is the same as the $L$-group $L_G'$ of $G'$, where $L_G = L_G'$ is the direct product of $GSp(2)(\mathbb{C})$ and the Weil group of $\mathbb{Q}$ (for the notion of $L$-group see [La] and [B] et al). As the choice of the $L$-morphism between $L_G$ and $L_G'$ we can take the identity map. The Langlands principle of functoriality predicts the following:

Conjecture 3.7 (Langlands). The $L$-morphism induced by the identity map would give rise to a natural transfer from $\mathcal{A}_G$ to $\mathcal{A}_{G'}$ which preserves $L$-functions, namely an $L$-function of an irreducible automorphic representation of $G(\mathbb{A})$ is one of some irreducible automorphic representation of $G'(\mathbb{A})$.

Let us now introduce

$$\mathcal{A}_G(K_f(D)) := \{ \pi = \prod_{v \leq \infty} \pi_v \in \mathcal{A}_G \mid \pi_p \text{ has a } K_p\text{-fixed vector for } v = p < \infty \},$$

$$\mathcal{A}_{G'}(K'_f(D)) := \{ \pi' = \prod_{v \leq \infty} \pi'_v \in \mathcal{A}_{G'} \mid \pi'_p \text{ has a } K'_p\text{-fixed vector for } v = p < \infty \},$$

where see Section 1.2 for $K_f(D)$ and $K'_f(D)$.

Based on the observation by R. Schmidt including the table of irreducible admissible representations of $G(\mathbb{Q}_p) = G_{nc}(\mathbb{Q}_p) = G(\mathbb{Q}_p)$ and $G'(\mathbb{Q}_p) = G_s(\mathbb{Q}_p)$ in the appendix (see also [RS, Section A.8]), we can formulate the conjecture as follows:

Conjecture 3.8. The above transfer would map $\mathcal{A}_G(K_f(D))$ into $\mathcal{A}_{G'}(K'_f(D))$ and an $L$-function of $\pi \in \mathcal{A}_G(K_f(D))$ is one of some $\pi' \in \mathcal{A}_{G'}(K'_f(D))$.

We remark that this was first pointed out by Ibukiyama [I] for the case of $G = G_c$ and $D = 1$. As a consequence of Corollary 3.3, Propositions 3.5 and 3.6 we have known that our theta lifts $\mathcal{L}(f, f')$ and $\mathcal{L}'(f, JL(f'))$ provide evidence of Conjecture 3.8. We state it as follows:
Theorem 3.9. Suppose that two even integers \((\kappa_1, \kappa_2)\) satisfy \(1 < \kappa_1 + 2 < \kappa_2\) when \(G = G_{nc}\) (respectively \(1 < \kappa_2 + 2 < \kappa_1\) when \(G = G_c\)). For any given primitive form \(f \in S_{\kappa_1}(D)\) and Hecke eigenform \(f' \in A_{\kappa_2}\), the map
\[
A_G(K_f(D)) \ni \pi(\mathcal{L}(f, f')) \mapsto \pi(\mathcal{L}'(f, JL(f'))) \in A_{G'}(K_{f'}(D))
\]
preserves the coincidence of the global spinor \(L\)-functions and is compatible with the non-archimedean local Jacquet-Langlands correspondence for \(G\) and \(G' = GSp(2)\) (cf. Appendix). Namely, this map satisfies the expected properties of the transfer in the conjecture.

A Appendix: The spherical representations of \(GSp(1,1)\) and local Langlands parameters for \(GSp(4)\) (by Ralf Schmidt)

Let \(F\) be a non-archimedean local field of characteristic zero. Let \(B\) be the non-split quaternion algebra over \(F\), and let \(x \mapsto \bar{x}\) be its standard involution. We consider \(GSp(1,1)\) and \(GSp(4)\) (or \(GSp(2)\)) over \(F\). Let \(\mathfrak{o}_B\) be a maximal order in \(B(F)\), and let \(\mathfrak{p}_B\) be the unique maximal ideal of \(\mathfrak{o}_B\). Let
\[
K_1 = \{g \in GSp(1,1)(F) \cap \begin{bmatrix} \mathfrak{o}_B & \mathfrak{o}_B \\ \mathfrak{o}_B & \mathfrak{o}_B \end{bmatrix} : \nu(g) \in \mathfrak{o}^\times\},
\]
\[
K_2 = \{g \in GSp(1,1)(F) \cap \begin{bmatrix} \mathfrak{o}_B & \mathfrak{p}_B \\ \mathfrak{p}_B^{-1} & \mathfrak{o}_B \end{bmatrix} : \nu(g) \in \mathfrak{o}^\times\}.
\]

We remark that these groups \(K_1\) and \(K_2\) are maximal compact subgroups of \(GSp(1,1)(F)\), and every maximal compact subgroup is conjugate to either \(K_1\) or \(K_2\).

The following table lists all irreducible, admissible representations of \(GSp(1,1)(F)\) which are constituents of representations of the form \(\chi 1_{\mathfrak{o}_B^\times} \rtimes \sigma\), where \(\chi\) and \(\sigma\) are characters of \(F^\times\). The table also lists all the irreducible, admissible representations of \(GSp(4,F)\) supported in the Borel subgroup, using the notations and classification scheme of [R-S]. Representations with the same \(L\)-parameter \(W_F' \rightarrow GSp(4,\mathbb{C})\) appear in the same row; this is nothing but the Langlands functorial transfer from \(GSp(1,1)\) to \(GSp(4)\) coming from the natural inclusion of dual groups. The actual \(L\)-parameters can be found in Table A.7 of [R-S].

The columns labeled \(K_1\) and \(K_2\) indicate, in the case when the inducing characters are unramified, the dimension of the space of \(K_1\) resp. \(K_2\) invariant vectors in a representation of \(G(F)\).
The notation $\nu$ stands for the valuation of $F$. For the IIa type representation, $\chi$ is such that $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$. For the representations in group V, the character $\xi$ is assumed to be non-trivial and quadratic.

**References**


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