ON SPECIAL VALUES OF TENSOR PRODUCT L-FUNCTIONS OF AN INNER FORM OF GSP(4) AND GL(2)

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ABSTRACT. We consider the Rankin-Selberg integral which represents degree 8 tensor product $L$-functions for quaternion unitary groups and $GL_2$. Using this integral representation, we prove the algebraic nature of special values.

1. SET UP

Let $F$ be a number field and $E$ a quadratic extension. For each $n \in \mathbb{N}$, we define the similitude unitary group $G_n = GU(n, n)$:

$G_n(F) = \{ g \in GL(2n, E) \mid \sigma^g J_n g = \lambda_n(g) J_n, \lambda_n(g) \in F^\times \}$

where $\sigma$ is non-trivial element in $\text{Gal}(E/F)$ and

$J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}.$

Let $E \subset D$ be a quaternion algebra over $F$. For $x \in D$, we mean the canonical involution by $\overline{x}$. For a matrix $A = (a_{ij})$ with entries in $D$, we denote the matrix $(\overline{a_{ij}})$ by $\overline{A}$.

Let us define the quaternion similitude unitary group $H_D$ by

$H_D(F) = \{ g \in GL(2, D) \mid \overline{\sigma} g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \lambda(g) \in F^\times \}.$

When $D \simeq M_2(F)$, we have an isomorphism

$H_D(F) \simeq GSp(4, F) = G_2(F) \cap GL(4, F).$

We note that we can take $\epsilon \in F^\times$ such that

$D \simeq \left\{ \begin{pmatrix} a & \epsilon b \\ b^\sigma & a^\sigma \end{pmatrix} \mid a, b \in E \right\}.$

Thus we may suppose that $D \subset \text{Mat}_{2 \times 2}(E)$, so that we can consider $H_D$ as a subgroup of $GL(4, E)$. In fact, $H_D$ can be embedded into $G_2$, and we fix it. Let us define a subgroup $H$ of $G_1 \times G_2$ by

$H = \{(g_1, h_2) \in G_1 \times H_D \mid \lambda_1(g_1) = \lambda_2(h_2)\},$

and we regard $H$ as a subgroup of $G_3$ by the following embedding

$H \ni \begin{pmatrix} (a & b) \\ (c & d) \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & A & 0 & B \\ c & 0 & d & 0 \\ 0 & C & 0 & D \end{pmatrix} \in G_3.$
2. Global integral

Let $P = MN$ denote the Siegel parabolic subgroup of $G_3$ where

\[
M(F) = \left\{ \begin{pmatrix} g & 0 & \lambda \cdot (tg^\sigma)^{-1} \\ 0 & 1 & 0 \end{pmatrix} \mid g \in \text{GL}_3(E), \lambda \in F^\times \right\},
\]

\[
N(F) = \left\{ \begin{pmatrix} 1_3 & X \\ 0 & 1_3 \end{pmatrix} \mid tX^\sigma = X \in \text{Mat}_{3 \times 3}(E) \right\}.
\]

Let $v$ be a character of $\mathbb{A}_E^\times/E^\times$ and $\tau$ a character of $\mathbb{A}_F^\times/F^\times$ Then we define a character $v \otimes \tau$ of $P(\mathbb{A}_F)$ by

\[
(v \otimes \tau)[\begin{pmatrix} g & 0 & \lambda \cdot (tg^\sigma)^{-1} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1_3 & X \\ 0 & 1_3 \end{pmatrix}] = v(\det g) \cdot \tau(\lambda).
\]

Let $\delta_P$ denote the modulus character of $P(\mathbb{A}_F)$. Then let $I(s, v \otimes \tau)$ denote the normalized degenerate principal series representation $\text{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}((v \otimes \tau) \cdot \delta_P^s)$ of $G(\mathbb{A}_F)$.

Here we employ the normalized induction so that $I(s, v \otimes \tau)$ is unitarizable when $\text{Re}(s) = 0$. Then for a holomorphic section $f^{(s)}$ of $I(s, v \otimes \tau)$ we have the Siegel Eisenstein series defined by

\[
E(g, f^{(s)}) = \sum_{\gamma \in P(F) \backslash G(F)} f^{(s)}(\gamma g).
\]

This series is absolutely convergent in the right half plane $\text{Re}(s) > \frac{1}{2}$ (Langlands [5]).

Let $\sigma$ be an irreducible cuspidal representation of $GL_2(\mathbb{A}_F)$ and let $\chi$ be a character of $\mathbb{A}_E^\times/E^\times$ such that

\[
\chi|_{\mathbb{A}_F^\times} = \omega_{\sigma}
\]

where $\omega_{\sigma}$ denotes the central character of $\sigma$. Since we have the isomorphism

\[
G_1(F) \simeq (GL(2, F) \times E^\times) /\{(a, a^{-1}) \mid a \in F^\times\},
\]

we can regard $\sigma \otimes \chi$ as the irreducible cuspidal automorphic representation of $G_1(\mathbb{A}_F)$ and we denote it by $\pi$. Let $V_\pi$ be the space of automorphic forms for $\pi$.

Let $(\Pi, V_\Pi)$ be an irreducible cuspidal automorphic representation of $H_D(\mathbb{A}_F)$. Let $\omega_\Pi$ denote the central character of $\Pi$. Then we study a global integral defined by

\[
Z(f^{(s)}, \phi, \Phi) = \int_{Z(\mathbb{A}_F)H(F) \backslash H(\mathbb{A}_F)} E(f^{(s)}, h)\Psi(g_1)\Phi(h_2)dh
\]

for $f^{(s)} \in I(s, v \otimes \tau)$, $\Psi \in V_\pi$ and $\Phi \in V_\Pi$, where $Z = Z_G \cap H$, $Z_{G_3}$ denotes the center of $G_3$, and $h = (g_1, h_2) \in H$. Here in order for the integral (2.0.2) to be well-defined, we assume that

\[
\omega_\Pi \cdot \omega_{\sigma} \cdot \tau^2 \cdot (v|_{\mathbb{A}_F^\times})^3 = 1.
\]

Proposition 2.1. For $\text{Re}(s) \gg 0$, we have

\[
Z(f^{(s)}, \Psi, \Phi) = \int_{S(\mathbb{A}_F) \backslash H(\mathbb{A}_F)} f^{(s)}(\eta h)W_\Psi(g_1)B_\Phi(h_2)dh
\]
where $B_{\Phi}$ is the Bessel model of $\Phi$ with respect to a non-split torus and $W_{\Psi}$ is the Whittaker model of $\Psi$, and $S$ is defined as follows: Let us define the Bessel subgroups $R$ of $H_{D}$ by

$$R(F) = \left\{ \begin{pmatrix} a^\sigma & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^\sigma & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 & \varepsilon b & c \\ 0 & 1 & c^\sigma & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_{2}(F) \mid a \in E^\times, b \in F, c \in E \right\}.$$  

Then a subgroup $S$ of $H$ is defined by

$$S = \{(\varphi(r), r) \mid r \in R\}$$

where we denote

$$\varphi \begin{bmatrix} a^\sigma & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^\sigma & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 & \varepsilon b & c \\ 0 & 1 & c^\sigma & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}.$$  

Remark. Our integral representation is a generalization to the similitude quaternion unitary case of Saha’s interpretation [11] of Furusawa’s integral [2]. Note that we unfold the Rankin-Selberg integral involving the Siegel Eisenstein series on $G_{3}$ directly without recourse to the Klingen Eisenstein series on $G_{2}$. Thus even when $H_{D} \simeq \text{GSp}(4)$, our local integral is totally different from Saha’s.

In order for our investigation to be non-vacuous, we assume that

II has a Bessel model of non-split type.

We note that by the result of Li [6], any irreducible cuspidal automorphic representation of $H_{D}(\mathbb{A})$ has a Bessel model of this type if $D$ does not split. Moreover if $D \simeq \text{Mat}_{2 \times 2}(F)$, i.e., $H_{D} \simeq \text{GSp}(4)$, II has a Whittaker model or a Bessel model of some type. If II is associated to a holomorphic cusp form, it is non-generic, and Pitale-Schmidt [8] shows that it does not have a Bessel model of split type. Thus such automorphic representations satisfy the above assumption.

The uniqueness of Bessel model is expected for any irreducible admissible representations of $H_{D}(F_{v})$. However as far as the author knows, there is no reference which proves the uniqueness in general. For example, for unramified representations of $\text{GSp}(4, F_{v})$, Sugano [12] proves the uniqueness. Then by the uniqueness of Bessel model and Whittaker model, we obtain

$$Z(s) = \prod_{v \notin S} Z_v(W_{\Psi,v}, B_{\Phi,v}, f_v^{(s)}) \cdot Z_S(W_{\Psi,S}, B_{\Phi,S}, f_S^{(s)}).$$

Here $S$ is a finite set of places such that any place $v \notin S$ is finite and satisfies

1. $2$ does not divide $v$
2. $E_{v}/F_{v}$ is unramified quadratic extension or $E_{v} \simeq F_{v} \oplus F_{v}$
3. $\Pi_{v}, \pi_{v}, \nu_{v}, \tau_{v}$ are unramified.
4. $D(F_{v}) \simeq \text{Mat}_{2 \times 2}(F_{v})$.

Then Furusawa and Ichino computed unramified local integrals explicitly.
Proposition 2.2 (Furusawa-Ichino, Appendix in [7]). Suppose $v \notin S$. For normalized spherical vectors $W_v, B_v$ and $f_v^{(s)}$, we have

$$Z_v(s) = \prod_{i=1}^{3} L \left( 6s + i, \nu|_{F_v^x} \cdot \varepsilon_{E_v/F_v}^{i+3} \right)^{-1} \cdot L \left( 3s + \frac{1}{2}, \Pi \times \sigma \times (\nu|_{F_v^x})^2 \times \tau \right)$$

where we normalize the measure on $H(F_v)$ suitably, and $\varepsilon_{E_v/F_v}$ is the quadratic character of $F_v^x$ corresponding to $E_v$ via local class field theory.

3. Main Theorem

Assume that

$$H_D(\mathbb{R}) \simeq \text{GSp}(4, \mathbb{R}) \quad \text{and} \quad F = \mathbb{Q}.$$ 

We possibly have $D \simeq \text{Mat}_{2 \times 2}(\mathbb{Q})$. We suppose that the central characters of $\Pi$ and $\pi$ are trivial.

Suppose that the archimedean component $\Pi_{\infty}$ of $\Pi$ is the holomorphic discrete series of $\text{PGSp}(4, \mathbb{R})$ with Harish-Chandra parameter $\ell(e_1 + e_2)$ with even integer $\ell$ where we define

$$e_i \begin{pmatrix} t_1 & t_2 \\ t_1^{-1} & t_2^{-1} \end{pmatrix} = t_i \quad t_i \in \mathbb{G}_m.$$ 

Suppose that $\sigma$ is a cuspidal automorphic representation associated to a new form of weight $\ell$. Then we consider an automorphic form $\Psi \in V_\sigma$ as the automorphic form on $G_1(\mathbb{A})$ by extending it trivially, i.e.

$$\Psi(ag) = \Psi(g)$$

for $a \in \mathbb{A}_E^\times$ and $g \in \text{GL}(2, \mathbb{A}_Q)$.

Theorem 3.1. Suppose that $\ell > 6$. Let $\tilde{\Phi} \in V_{\Pi}$ and $\Psi \in V_\sigma$ be arithmetic automorphic forms in the sense of Harris [4]. Then for an integer $m$ such that $2 < m \leq \frac{\ell}{2} - 1$, we have

$$\frac{L(m, \Pi \times \sigma)}{\pi^{4m} \langle \Psi \otimes \Phi, \Psi \otimes \Phi \rangle} \in \overline{\mathbb{Q}}$$

and

$$\left( \frac{L(m, \Pi \times \sigma)}{\pi^{4m} \langle \Psi \otimes \Phi, \Psi \otimes \Phi \rangle} \right)^\tau = \frac{L(m, \Pi^\tau \times \sigma^\tau)}{\pi^{4m} \langle \Psi^\tau \otimes \Phi^\tau, \Psi^\tau \otimes \Phi^\tau \rangle}$$

for all $\tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Here we define

$$\langle \Psi \otimes \Phi, \Psi \otimes \Phi \rangle = \int_{Z(H(\mathbb{A}_Q)H(\mathbb{Q}) \backslash H(\mathbb{A}_Q))} |\Psi(g_1)\Phi(h_2)|^2 dh$$

where we denote $h = (g_1, h_2) \in H(\mathbb{A}_Q)$, and $dh$ is the Tamagawa measure on $H(\mathbb{A}_Q)$.

We can prove this by a similar way with Garrett-Harris [3]. For a detail of the proof, we refer to [7].
3.1. **Period Relation.** Let $(\Pi, V_{\Pi})$ be an irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, A_{\mathbb{Q}})$ as in Theorem 3.1. Further we assume that $\Pi$ is tempered and non-endoscopic. We suppose that there exists an irreducible cuspidal automorphic representation $(\Pi_{D}, V_{\Pi_{D}})$ of $H_{D}(A_{\mathbb{Q}})$ such that for every place $v$ such that $H_{D}(Q_{v}) \simeq \mathrm{GSp}(4, \mathbb{Q}_{v})$,

$$\Pi_{v} \simeq \Pi_{D,v}.$$ 

Then $\Pi_{D}$ satisfies the condition in Theorem 3.1. Comparing the equations in Theorem 3.1 for $\Pi$ and $\Pi_{D}$, we obtain the following relation.

**Corollary 3.1.** For any arithmetic forms $\Phi \in V_{\Pi}$ and $\Phi_{D} \in V_{\Pi_{D}}$, we have

$$\langle \Phi, \Phi \rangle / \langle \Phi_{D}, \Phi_{D} \rangle \in \overline{\mathbb{Q}}$$

and

$$((\langle \Phi, \Phi \rangle / \langle \Phi_{D}, \Phi_{D} \rangle))^{\tau} = \langle \Phi^{\tau}, \Phi^{\tau} \rangle / \langle \Phi_{D}^{\tau}, \Phi_{D}^{\tau} \rangle$$

for any $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Here we define the pairing $\langle \Phi_{D}, \Phi_{D} \rangle$ by

$$\langle \Phi, \Phi \rangle = \int_{Z_{H_{D}(A_{\mathbb{Q}})H_{D}(\mathbb{Q})\backslash H_{D}(A_{\mathbb{Q}})}} |\Phi_{D}(h)|^{2} \, dh$$

where $dh$ is the Tamagawa measure on $H_{D}(A_{\mathbb{Q}})$, and we define $\langle \Phi, \Phi \rangle$ similarly.

3.2. **Remarks on Theorem 3.1.**

3.2.1. **Critical point.** The critical points in Theorem 3.1 does not cover all critical points on the right half plane $\Re(s) > 0$. Indeed the critical points for $s = \frac{1}{2}$ and $\frac{1}{6}$ are not included due to the analytic property of Eisenstein series.

3.2.2. **Split case.** When $H_{D} \simeq \mathrm{GSp}(4)$, similar results are proved by many people. Furusawa [2] discovered an integral representation of this $L$-function and he proved the algebraicity at the rightmost critical point for Siegel cusp forms and elliptic cusp form of full level. Pitale-Schmidt [9] extended his result with respect to the level of elliptic cusp forms, and Saha [10] extended with respect to both of levels of Siegel cusp forms and elliptic cusp form. Saha [11] also proved the algebraicity for other critical points combining the pull-back formula and differential operators. On the other hand, Böcherer-Heim [1] showed the algebraicity at all critical points in the full modular balanced mixed weight case using Heim’s integral representation.

3.2.3. **Yoshida’s Conjecture.** When the irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, A_{\mathbb{Q}})$ is associated to a Siegel cusp form, our result is compatible with Yoshida’s calculation [13] on Deligne period.

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REFERENCES


