<table>
<thead>
<tr>
<th>Title</th>
<th>A Report on Studies of Relative Randomness (Proof theory and complexity)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Peng, NingNing</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2013), 1832: 154-157</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/194845">http://hdl.handle.net/2433/194845</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
A Report on Studies of Relative Randomness

NingNing Peng *
Mathematical Institute, Tohoku University
Sendai-shi, Miyagi-ken, 980-8578, Japan
sa8m42@math.tohoku.ac.jp

Abstract

We report some results of our recent studies. Let $\Gamma$ be a set of (Turing) oracles. A set $Z$ is called $\Gamma$-random if $Z$ is ML-random relative to $A$ for all $A \in \Gamma$. We use $L$ and $G$ to denote the set of low sets and the set of 1-generic sets, respectively. In [7], Yu proved that $L$-randomness is equivalent to $\emptyset'$-Schnorr randomness, where $\emptyset'$ denotes the halting problem. We show that $(L \cap G)$-randomness is still equivalent to $\emptyset'$-Schnorr randomness. We also proved that $(L \cap MLR)$-randomness is equivalent to $\emptyset'$-Schnorr randomness.

1 Introduction

For a definition of random sequences, many approaches have been made until a definition was proposed by Martin-Löf [3] in 1966, which for the first time included all standard statistical properties of random sequences. The relativized randomness was first studied by Gaifman and Snir. We say that a set is n-random if it is ML-random relative to $\emptyset^{(n-1)}$. So it is 1-random if it is ML-random. 2-random if it is ML-random relative to $\emptyset'$. 2-randomness was first studied by Kurtz [6]. He also considered weak 2-randomness, an interesting notion lying strictly between Martin-Löf randomness and 2-randomness. In this report, we will introduce other randomness notions which between Martin-Löf randomness and 2-randomness.

$\Gamma$-randomness was first studied in [9], and is strongly connected with Yu’s research [7]. The $\Gamma$-randomness notion could sometimes produce alternative proofs of existing results. For instance, some properties of $\emptyset'$-Schnorr randomness are proved more easily by the characterization due to $L$-randomness than the usual methods. In section 3, we will report some new characterizations of $L$-randomness. The detail proof of these results will be published in the future literature.

*This research was partially supported by RIMS. The author would like to thank Prof. Toshio Suzuki for many helpful remarks. The full version of this paper will appear soon.
2 Preliminaries

The collection of binary strings is denoted by $2^{<\mathbb{N}}$, i.e., the set of all functions from $\{0, \ldots, n\}$ to $\{0, 1\}$ for some $n \in \mathbb{N}$. We use $\sigma, \tau, \cdots$ to denote the elements of $2^{<\mathbb{N}}$. Let $2^\mathbb{N}$ denote the set of infinite binary sequences. Subsets of $\mathbb{N}$ can be identified with element of $2^\mathbb{N}$. These are also called reals. For sets $A, B$, Let $A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$, namely the set which is $A$ on the even bit positions and $B$ on the odd positions.

For $\sigma \in 2^{<\mathbb{N}}$, we write $|\sigma|$ for the length of $\sigma$. Equivalently, $|\sigma| = \#\text{dom}(\sigma)$. Here the cardinality of a set $A$ is denoted by $\#A$. The empty string is denoted by $\lambda$. For strings $\sigma$ and $\tau$, let $\sigma \preceq \tau$ denotes that $\sigma$ is a prefix of $\tau$, i.e., $\text{dom}(\sigma) \subseteq \text{dom}(\tau)$ and $\sigma(m) = \tau(m)$ holds for each $m \in \text{dom}(\sigma)$. The concatenation of two strings $\sigma$ and $\tau$ is denoted by $\sigma\tau$. For a set $A$, $A \mid n$ is the prefix of $A$ of length $n$. A topology of $2^\mathbb{N}$ is induced by basic open sets $[\sigma] = \{X \in 2^\mathbb{N} : X \succeq \sigma\}$ for all strings $\sigma \in 2^{<\mathbb{N}}$. So each open set of $2^\mathbb{N}$ is generated by a subset of $2^{<\mathbb{N}}$, that is $[S]^\ast = \{X \in 2^\mathbb{N} : \exists \sigma \in S \sigma \preceq X\}$.

With this topology, $2^\mathbb{N}$ is called the Cantor space.

The Lebesgue measure on $2^\mathbb{N}$ is induced by giving each basic open set $[\sigma]$ measure $\mu([\sigma]) := 2^{-|\sigma|}$. for each string $\sigma$. If a class $G \subseteq 2^\mathbb{N}$ is open then $\mu(G) = \sum_{\sigma \in B} 2^{-|\sigma|}$ where $B$ is a prefix-free set of strings such that $G = \bigcup_{\sigma \in B} [\sigma]$. A class $C \subseteq 2^\mathbb{N}$ is called null if $\mu(C) = 0$. If $2^\mathbb{N} - C$ is null we say that $C$ is conull.

3 $\Gamma$-randomness

ML-randomness is a central notion of algorithmic randomness for subsets of $\mathbb{N}$, which defined in the following way.

**Definition 1** (Martin-Löf [3]).

(i) A Martin-Löf test, or ML-test for short, is a uniformly c.e. sequence $(G_m)_{m \in \mathbb{N}}$ of open sets such that $\forall m \in \mathbb{N} \mu(G_m) \leq 2^{-m}$.

(ii) A set $Z \subseteq \mathbb{N}$ fails the test if $Z \in \bigcap_m G_m$, otherwise $Z$ passes the test.

(iii) $Z$ is ML-random if $Z$ passes each ML-test. Let MLR denote the class of ML-random sets. Let non-MLR denote its complement in $2^\mathbb{N}$.

Following Schnorr [10], we will look at other natural notion of randomness, which refine the notion of Martin-Löf randomness.

**Definition 2** (Schnorr [10]). A Schnorr test is a ML-test $(G_m)_{m \in \mathbb{N}}$ such that $\mu G_m$ is computable uniformly in $m$. A set $Z \subseteq \mathbb{N}$ fails the test if $Z \in \bigcap_m G_m$, otherwise $Z$ passes the test. $Z$ is Schnorr random if $Z$ passes each Schnorr test.

We recall some definitions in [9].

**Definition 3.** Let $\Gamma \subset \omega^\omega$. A set $Z$ is $\Gamma$-random if $Z$ is ML-random relative to $f$ for all $f \in \Gamma$. Any ML-test relative to $f \in \Gamma$ is called a $\Gamma$-test.
For \( f \in \omega^\omega \), we say \( f \)-random and \( f \)-test instead of \( \{ f \} \)-random and \( \{ f \} \)-test, respectively. Recall that a set \( A \) is low if \( A' \leq_T \emptyset' \). In particular, \( \Gamma \)-randomness is called \( \text{L-randomness} \) if \( \Gamma \) is the set of low sets.

Since a ML-test is a uniformly c.e. sequence \((G_m)_{m \in \mathbb{N}} \) of open sets such that \( \forall m \in \mathbb{N} \mu G_m \leq 2^{-m} \). Thus, we can define an \( \text{L} \)-test to be a sequence \((G_m)_{m \in \mathbb{N}} \) of open sets, which is uniformly c.e in some low set, such that \( \forall m \in \mathbb{N} \mu G_m \leq 2^{-m} \).

The randomness notions between ML-randomness and 2-randomness have been extensively investigated in the literature by many researchers. In 2012, Yu [7] show that L-randomness lying strictly between Martin-Löf randomness and 2-randomness.

**Theorem 1** (Yu [7]). **L-randomness is equivalent to \( \emptyset' \)-Schnorr randomness.**

In [8], we also give another characterization of \( \text{L} \)-randomness. Let \( \text{PA} \) denote the set of all functions of \( \text{PA} \) degrees.

**Proposition 1** (Peng, Higuchi, Yamazaki and Tanaka [8]). **L-randomness is equivalent to \( \text{L} \cap \text{PA} \)-randomness.**

Let \( \mathbb{G} \) denote the set of all 1-generic elements of \( 2^\omega \). Here, recall that an element \( Z \) of \( 2^\omega \) is 1-generic if for any c.e. subset \( W \) of \( 2^{<\omega} \), there exists \( \sigma < Z \) such that either \( \sigma \in W \) or \( [\sigma] \cap W = \emptyset \) holds. It is well-known that any 1-generic element \( Z \) of \( 2^\omega \) is generalized low, i.e., \( Z \oplus \emptyset' \) computes \( Z' \). Thus a 1-generic element of \( 2^\omega \) is computable relative to \( \emptyset' \) if and only if it is low.

Now we have the following theorem.

**Theorem 2.** **(\( \text{L} \cap \mathbb{G} \))-randomness is equivalent to \( \emptyset' \)-Schnorr randomness.**

The following answer a question in [8].

**Theorem 3.** **(\( \text{L} \cap \text{MLR} \))-randomness is equivalent to \( \emptyset' \)-Schnorr randomness.**

A natural of Turing reducibility from the point of view of ML-randomness is the LR-reducibility which was introduced in [5].

**Definition 4** (Nies [5]). For any \( A, B \subseteq \mathbb{N} \), we say that \( A \) is \( \text{LR} \)-reducible to \( B \), abbreviated \( A \leq_{LR} B \), if

\[
\forall X (X \text{ is } B-\text{random} \Rightarrow X \text{ is } A-\text{random})
\]

Intuitively this means that if oracle \( A \) can identify some patterns on some real \( x \), oracle \( B \) can also find patterns on \( x \). In other words, \( B \) is at least as good as \( A \) for this purpose.


**Theorem 4** (David, [2]). **For any low real \( X, Y \), there exists a low c.e. real \( Z \) such that \( X, Y \leq_{LR} Z \).**
We also show some similar results as follows.

**Theorem 5.** For any low real $X, Y$, there exists a low 1-generic real $Z$ such that $X, Y \leq_{LR} Z$.

The above can be shown from theorem 2.

**Theorem 6.** For any low real $X, Y$, there exists a low Martin-Löf random real $Z$ such that $X, Y \leq_{LR} Z$.

This follows from theorem 3.

**Acknowledgments**

We would like to thank Prof. Kazuyuki Tanaka and Prof. Takeshi Yamazaki, Dr. Kojiro Higuchi for their valuable comments and discussions.

**References**


