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Kyoto University
A Report on Studies of Relative Randomness

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Abstract

We report some results of our recent studies. Let \( \Gamma \) be a set of (Turing) oracles. A set \( Z \) is called \( \Gamma \)-random if \( Z \) is ML-random relative to \( A \) for all \( A \in \Gamma \). We use \( \mathbb{L} \) and \( \mathbb{G} \) to denote the set of low sets and the set of 1-generic sets, respectively. In [7], Yu proved that \( \mathbb{L} \)-randomness is equivalent to \( \emptyset' \)-Schnorr randomness, where \( \emptyset' \) denotes the halting problem. We show that \( (\mathbb{L} \cap \mathbb{G}) \)-randomness is still equivalent to \( \emptyset' \)-Schnorr randomness. We also proved that \( (\mathbb{L} \cap \mathbb{MLR}) \)-randomness is equivalent to \( \emptyset' \)-Schnorr randomness.

1 Introduction

For a definition of random sequences, many approaches have been made until a definition was proposed by Martin-Löf [3] in 1966, which for the first time included all standard statistical properties of random sequences. The relativized randomness was first studied by Gaifman and Snir. We say that a set is \( n \)-random if it is ML-random relative to \( \emptyset^{(n-1)} \). So it is 1-random if it is ML-random. 2-random if it is ML-random relative to \( \emptyset' \). 2-randomness was first studied by Kurtz [6]. He also considered weak 2-randomness, an interesting notion lying strictly between Martin-Löf randomness and 2-randomness. In this report, we will introduce other randomness notions which between Martin-Löf randomness and 2-randomness.

\( \Gamma \)-randomness was first studied in [9], and is strongly connected with Yu's research [7]. The \( \Gamma \)-randomness notion could sometimes produce alternative proofs of existing results. For instance, some properties of \( \emptyset' \)-Schnorr randomness are proved more easily by the characterization due to \( \mathbb{L} \)-randomness than the usual methods. In section 3, we will report some new characterizations of \( \mathbb{L} \)-randomness. The detail proof of these results will be published in the future literature.

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2 Preliminaries

The collection of binary strings is denoted by $2^{<\mathbb{N}}$, i.e., the set of all functions from \{0, \ldots, n\} to \{0, 1\} for some $n \in \mathbb{N}$. We use $\sigma, \tau, \cdots$ to denote the elements of $2^{<\mathbb{N}}$. Let $2^\mathbb{N}$ denote the set of infinite binary sequences. Subsets of $\mathbb{N}$ can be identified with element of $2^\mathbb{N}$. These are also called reals. For sets $A, B$, Let $A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$, namely the set which is $A$ on the even bit positions and $B$ on the odd positions.

For $\sigma \in 2^{<\mathbb{N}}$, we write $|\sigma|$ for the length of $\sigma$. Equivalently, $|\sigma| = \#\text{dom}(\sigma)$. Here the cardinality of a set $A$ is denoted by $\#A$. The empty string is denoted by $\lambda$. For strings $\sigma$ and $\tau$, let $\sigma \preceq \tau$ denotes that $\sigma$ is a prefix of $\tau$, i.e., $\text{dom}(\sigma) \subseteq \text{dom}(\tau)$ and $\sigma(m) = \tau(m)$ holds for each $m \in \text{dom}(\sigma)$. The concatenation of two strings $\sigma$ and $\tau$ is denoted by $\sigma \tau$. For a set $A$, $A \upharpoonright n$ is the prefix of $A$ of length $n$. A topology of $2^\mathbb{N}$ is induced by basic open sets $[\sigma] = \{X \in 2^\mathbb{N} : X \succeq \sigma\}$ for all strings $\sigma \in 2^{<\mathbb{N}}$. So each open set of $2^\mathbb{N}$ is generated by a subset of $2^{<\mathbb{N}}$, that is $[S]^\omega = \{X \in 2^\mathbb{N} : \exists \sigma \in S \ \sigma \preceq X\}$. With this topology, $2^\mathbb{N}$ is called the Cantor space.

The Lebesgue measure on $2^\mathbb{N}$ is induced by giving each basic open set $[\sigma]$ measure $\mu([\sigma]) := 2^{-|\sigma|}$, for each string $\sigma$. If a class $G \subseteq 2^\mathbb{N}$ is open then $\mu(G) = \sum_{\sigma \in B} 2^{-|\sigma|}$ where $B$ is a prefix-free set of strings such that $G = \bigcup_{\sigma \in B} [\sigma]$. A class $C \subseteq 2^\mathbb{N}$ is called null if $\mu(C) = 0$. If $2^\mathbb{N} - C$ is null we say that $C$ is conull.

3 \(\Gamma\)-randomness

ML-randomness is a central notion of algorithmic randomness for subsets of $\mathbb{N}$, which defined in the following way.

Definition 1 (Martin-Löf [3]). (i) A Martin-Löf test, or ML-test for short, is a uniformly c.e. sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\forall m \in \mathbb{N} \ \mu(G_m) \leq 2^{-m}$.

(ii) A set $Z \subseteq \mathbb{N}$ fails the test if $Z \in \bigcap_m G_m$, otherwise $Z$ passes the test.

(iii) $Z$ is ML-random if $Z$ passes each ML-test. Let MLR denote the class of ML-random sets. Let non-MLR denote its complement in $2^\mathbb{N}$.

Following Schnorr [10], we will look at other natural notion of randomness, which refine the notion of Martin-Löf randomness.

Definition 2 (Schnorr [10]). A Schnorr test is a ML-test $(G_m)_{m\in\mathbb{N}}$ such that $\mu G_m$ is computable uniformly in $m$. A set $Z \subseteq \mathbb{N}$ fails the test if $Z \in \bigcap_m G_m$, otherwise $Z$ passes the test. $Z$ is Schnorr random if $Z$ passes each Schnorr test.

We recall some definitions in [9].

Definition 3. Let $\Gamma \subset \omega^\omega$. A set $Z$ is $\Gamma$-random if $Z$ is ML-random relative to $f$ for all $f \in \Gamma$. Any ML-test relative to $f \in \Gamma$ is called a $\Gamma$-test.
For $f \in \omega^\omega$, we say $f$-random and $f$-test instead of \{f\}-random and \{f\}-test, respectively. Recall that a set $A$ is low if $A' \leq_T \emptyset$. In particular, $\Gamma$-randomness is called \textit{L-randomness} if $\Gamma$ is the set of low sets.

Since a ML-test is a uniformly c.e. sequence $(G_m)_{m \in \mathbb{N}}$ of open sets such that $\forall m \in \mathbb{N} \ \mu G_m \leq 2^{-m}$. Thus, we can define an $L$-test to be a sequence $(G_m)_{m \in \mathbb{N}}$ of open sets, which is uniformly c.e in some low set, such that $\forall m \in \mathbb{N} \ \mu G_m \leq 2^{-m}$.

The randomness notions between ML-randomness and 2-randomness have been extensively investigated in the literature by many researchers. In 2012, Yu [7] show that L-randomness lying strictly between Martin-Löf randomness and 2-randomness.

**Theorem 1** (Yu [7]). \textit{L-randomness is equivalent to $\emptyset'$-Schnorr randomness.}

In [8], we also give another characterization of $L$-randomness. Let PA denote the set of all functions of PA degrees.

**Proposition 1** (Peng, Higuchi, Yamazaki and Tanaka [8]). \textit{L-randomness is equivalent to $L \cap \text{PA}$-randomness.}

Let $\mathcal{G}$ denote the set of all 1-generic elements of $2^\omega$. Here, recall that an element $Z$ of $2^\omega$ is 1-generic if for any c.e. subset $W$ of $2^{<\omega}$, there exists $\sigma \prec Z$ such that either $\sigma \in W$ or $[\sigma] \cap W = \emptyset$ holds. It is well-known that any 1-generic element $Z$ of $2^\omega$ is generalized low, i.e., $Z \oplus \emptyset'$ computes $Z'$. Thus a 1-generic element of $2^\omega$ is computable relative to $\emptyset'$ if and only if it is low.

Now we have the following theorem.

**Theorem 2.** \textit{(L $\cap \mathcal{G}$)-randomness is equivalent to $\emptyset'$-Schnorr randomness.}

The following answer a question in [8].

**Theorem 3.** \textit{(L $\cap \text{MLR}$)-randomness is equivalent to $\emptyset'$-Schnorr randomness.}

A natural of Turing reducibility from the point of view of ML-randomness is the LR-reducibility which was introduced in [5].

**Definition 4** (Nies [5]). For any $A, B \subseteq \mathbb{N}$, we say that $A$ is LR-reducible to $B$, abbreviated $A \leq_{LR} B$, if

$$\forall X(X \text{ is } B \text{-random } \Rightarrow X \text{ is } A \text{-random})$$

Intuitively this means that if oracle $A$ can identify some patterns on some real $x$, oracle $B$ can also find patterns on $x$. In other words, $B$ is at least as good as $A$ for this purpose.


**Theorem 4** (David, [2]). \textit{For any low real $X, Y$, there exists a low c.e. real $Z$ such that $X, Y \leq_{LR} Z$.}
We also show some similar results as follows.

**Theorem 5.** For any low real $X, Y$, there exists a low 1-generic real $Z$ such that $X, Y \leq_{LR} Z$.

The above can be shown from theorem 2.

**Theorem 6.** For any low real $X, Y$, there exists a low Martin-Löf random real $Z$ such that $X, Y \leq_{LR} Z$.

This follows from theorem 3.

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**References**


