A Simplified Characterisation of Provably Computable Functions of the System ID₁ of Inductive Definitions (Extended Abstract)

Naohi Eguchi*
Mathematical Institute, Tohoku University, Japan

Andreas Weiermann†
Department of Mathematics, Ghent University, Belgium

Abstract
We present a simplified and streamlined characterisation of provably total computable functions of the system ID₁ of non-iterated inductive definitions. The idea of the simplification is to employ the method of operator-controlled derivations that was originally introduced by Wilfried Buchholz and afterwards applied by the second author to a streamlined characterisation of provably total computable functions of Peano arithmetic PA.

1 Introduction
As stated by Gödel’s first incompleteness theorem, any reasonable consistent formal system has an unprovable $\Pi^0_2$-sentence that is true in the standard model of arithmetic. This means that the total (computable) functions whose totality is provable in a consistent system, which are known as provably (total) computable functions, form a proper subclass of total computable functions. Hence it is natural to ask how we can describe the provably computable functions of a given system. Not surprisingly provably computable functions are closely related to provable well-ordering, i.e., ordinal analysis. Several successful applications of techniques from ordinal analysis to provably computable functions have been provided by B. Blankertz and A. Weiermann.

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Modern ordinal analysis is based on the method of *local predicativity*, that was first introduced by W. Pohlers, cf. [10, 11]. Successful applications of local predicativity to provably computable functions contain works by Blankertz and Weiermann [12] and by Weiermann [2]. However, to the authors’ knowledge, the most successful way in ordinal analysis is based on the method of *operator-controlled derivations*, an essential simplification of local predicativity, that was introduced by Buchholz [3]. In [13] the second author successfully applied the method of operator-controlled derivations to a streamlined characterisation of provably computable functions of PA. (See also [11, Section 2.1.5].) Technically this work aims to lift up the characterisation obtained in [13] to an impredicative system $\text{ID}_1$ of non-iterated inductive definitions. We introduce an ordinal notation system $\mathcal{O}(\Omega)$ and define a computable function $f^\alpha$ for a starting numerical function $f : \mathbb{N} \rightarrow \mathbb{N}$ by transfinite recursion on $\alpha \in \mathcal{O}(\Omega)$. The transfinite definition of $f^\alpha$ stems from [13]. We show that a function is provably computable in $\text{ID}_1$ if and only if it is a Kalmar elementary function in $\{s^\alpha \mid \alpha \in \mathcal{O}(\Omega) \text{ and } \alpha < \Omega\}$, where $s$ denotes the numerical successor function $m \mapsto m + 1$ and $\Omega$ denotes the least non-computable ordinal (Corollary 6.4).

This paper consists of two materials, a technical report [8] by the authors and a draft [14] by the second author. Section 3–6 consist of [8] and Section 7 consists of [14]. We mention in particular that the ordinal notation system $\mathcal{O}(\mathcal{F})$ stems from [14]. Most of proofs are omitted due to the page limitation. We note however that there is a non-trivial error in the technical report [8, p. 8, Lemma 15.5]. We restate Lemma 4.4.5, provide its proof and discuss in detail about embedding (Section 5) affected by this correction. The full details of missing proofs will appear in [7].

## 2 Preliminaries

In order to make our contribution precise, in this preliminary section we collect the central notions. We write $\mathcal{L}_{\text{PA}}$ to denote the standard language of first order theories of arithmetic. In particular we suppose that the constant 0 and the successor function symbol $S$ are included in $\mathcal{L}_{\text{PA}}$. For each natural $m$ we use the notation $\underline{m}$ to denote the corresponding numeral built from 0 and $S$. Let a set variable $X$ denote a subset of $\mathbb{N}$. We write $X(t)$ instead of $t \in X$ and $\mathcal{L}_{\text{PA}}(X)$ for $\mathcal{L}_{\text{PA}} \cup \{X\}$. Let $\text{FV}_1(A)$ denote the set of free number variables appearing in a formula $A$ and $\text{FV}_2(A)$ the set of free set variables in $A$. And then let $\text{FV}(A) := \text{FV}_1(A) \cup \text{FV}_2(A)$. For a fresh set variable $X$ we call an $\mathcal{L}_{\text{PA}}(X)$-formula $A(x)$ a *positive operator form* if $\text{FV}_1(A(x)) \subseteq \{x\}$, $\text{FV}_2(A(x)) = \{X\}$, and $X$ occurs only positively in $A$.

Let $\text{FV}_1(A(x)) = \{x\}$. For a formula $F(x)$ such that $x \in \text{FV}_1(F(x))$ we write $A(F,t)$ to denote the result of replacing in $A(t)$ every subformula $X(s)$ by $F(s)$. The language $\mathcal{L}_{\text{ID}_1}$ of the *system $\text{ID}_1$ of non-iterated inductive definitions* is defined by $\mathcal{L}_{\text{ID}_1} := \mathcal{L}_{\text{PA}} \cup \{P_A \mid A \text{ is a positive operator form}\}$ where for each positive operator...
form $A$, $P_A$ denotes a new unary predicate symbol. We write $T(L_{ID_1}, V)$ to denote the set of $L_{ID_1}$-terms and $T(L_{ID_1})$ to denote the set of closed $L_{ID_1}$-terms. The axioms of ID$_1$ consist of the axioms of Peano arithmetic PA in the language $L_{ID_1}$ and the following new axiom schemata (ID$_1$) and (ID$_2$):

(ID1) $\forall x (A(P_A, x) \rightarrow P_A(x))$.

(ID2) (The universal closure of) $\forall x (A(F, x) \rightarrow F(x)) \rightarrow \forall x (P_A(x) \rightarrow F(x))$, where $F$ is an $L_{ID_1}$-formula.

For each $n \in \mathbb{N}$ we write $I\Sigma_n$ to denote the fragment of Peano arithmetic PA with induction restricted to $\Sigma_n^0$-formulas. Let $k$ be a natural number and $f : \mathbb{N}^k \rightarrow \mathbb{N}$ a numerical function and $T$ be a system of arithmetic containing $I\Sigma_1$. Then we say that $f$ is provably total computable in $T$ or provably computable in $T$ for short if there exists a $\Sigma^0_n$-formula $A_f(x_1, \ldots, x_k, y)$ such that (i) $FV(A_f) = FV_1(A_f) = \{x_1, \ldots, x_k, y\}$, (ii) for all $\vec{m}, n \in \mathbb{N}$, $f(\vec{m}) = n$ holds if and only if $A_f(\vec{m}, n)$ is true in the standard model $\mathbb{N}$ of PA, and (iii) $\forall \vec{x} \exists! y A_f(\vec{x}, y)$ is a theorem in $T$.

### 3 A non-computable ordinal notation system $OT(\mathcal{F})$

In this section we introduce a non-computable ordinal notation system $OT(\mathcal{F}) = \langle OT(\mathcal{F}), \prec \rangle$. This new ordinal notation system is employed in the next section. For an element $\alpha \in OT(\mathcal{F})$ let $OT(\mathcal{F}) \upharpoonright \alpha$ denote the set $\{\beta \in OT(\mathcal{F}) \mid \beta < \alpha\}$.

**Definition 3.1** We define three sets $SC \subseteq H \subseteq OT(\mathcal{F})$ of ordinal terms and a set $\mathcal{F}$ of unary function symbols simultaneously. Let $0, \varphi, \Omega, S, E$ and $+$ be distinct symbols.

1. $0 \in OT(\mathcal{F})$ and $\Omega \in SC$.
2. $\{S, E\} \subseteq \mathcal{F}$.
3. If $\alpha \in OT(\mathcal{F}) \upharpoonright \Omega$, then $S(\alpha) \in OT(\mathcal{F})$ and $E(\alpha) \in H$.
4. If $\{\alpha_1, \ldots, \alpha_l\} \subseteq H$ and $\alpha_1 \geq \cdots \geq \alpha_l$, then $\alpha_1 + \cdots + \alpha_l \in OT(\mathcal{F})$.
5. If $\{\alpha, \beta\} \subseteq OT(\mathcal{F}) \upharpoonright \Omega$, then $\varphi \alpha \beta \in H$.
6. If $\alpha \in OT(\mathcal{F})$ and $\xi \in OT(\mathcal{F}) \upharpoonright \Omega$, then $\Omega^\alpha \cdot \xi \in H$.
7. If $F \in \mathcal{F}$, $\alpha \in OT(\mathcal{F})$ and $\xi \in OT(\mathcal{F}) \upharpoonright \Omega$, then $F^\alpha(\xi) \in SC$.
8. If $F \in \mathcal{F}$ and $\alpha \in OT(\mathcal{F})$, then $F^\alpha \in \mathcal{F}$. 
We write \( \omega^\alpha \) to denote \( \varphi_0 \alpha \) and \( m \) to denote \( \omega^0 \cdot m = \frac{\omega^0 \cdot \ldots \cdot \omega^0}{m \text{ many}} \).

Let \( \text{Ord} \) denote the class of ordinals and \( \text{Lim} \) the class of limit ones. We define a semantic \([\cdot]\) for \( \mathcal{O}(F) \), i.e., \(\exists\mathcal{O}(F) \to \text{Ord} \). The well ordering \( < \) on \( \mathcal{O}(F) \) is defined by \( \alpha < \beta \iff [\alpha] < [\beta] \). Let \( \Omega_1 \) denote the least non-computable ordinal \( \omega_1^{\text{CK}} \). For an ordinal \( \alpha \) we write \( \alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \ldots + \Omega_1^{\alpha_l} \cdot \beta_l \) if \( \alpha > \alpha_1 > \ldots > \alpha_l \), \( \{\beta_1, \ldots, \beta_l\} \subseteq \Omega_1 \), and \( \alpha = \Omega_1^{\alpha_1} \cdot \beta_1 + \ldots + \Omega_1^{\alpha_l} \cdot \beta_l \). Let \( \varepsilon_\alpha \) denote the \( \alpha \)th epsilon number. One can observe that for each ordinal \( \alpha < \varepsilon_{\Omega_1+1} \) there uniquely exists a set \( \{\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l\} \) of ordinals such that \( \alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \ldots + \Omega_1^{\alpha_l} \cdot \beta_l \). For a set \( K \subseteq \text{Ord} \) and for an ordinal \( \alpha \) we will write \( K < \alpha \) to abbreviate \( (\forall \xi \in K) \xi < \alpha \), and dually \( \alpha \leq K \) to abbreviate \( (\exists \xi \in K) \alpha \leq \xi \).

**Definition 3.2 (Collapsing operators)**
1. Let \( \alpha \) be an ordinal such that \( \alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \ldots + \Omega_1^{\alpha_l} \cdot \beta_l < \varepsilon_{\Omega_1+1} \). The set \( K_\Omega \alpha \) of coefficients of \( \alpha \) is defined by

\[
K_\Omega \alpha = \{\beta_1, \ldots, \beta_l\} \cup K_\Omega \alpha_1 \cup \cdots \cup K_\Omega \alpha_l.
\]

2. Let \( F : \text{Ord} \to \text{Ord} \) be an ordinal function. Then a function \( F^\alpha : \text{Ord} \to \text{Ord} \) is defined by transfinite recursion on \( \alpha \in \text{Ord} \) by

\[
\left\{
\begin{array}{l}
F^0(\xi) = F(\xi), \\
F^\alpha(\xi) = \min\{\gamma \in \text{Ord} \mid \omega^\gamma = \gamma, K_\Omega \alpha \cup \{\xi\} < \gamma \text{ and } (\forall \eta < \gamma)(\forall \beta < \alpha)(K_\Omega \beta < \gamma \Rightarrow F^\beta(\eta) < \gamma)\}.
\end{array}\right.
\]

**Corollary 3.3** Let \( F : \text{Ord} \to \text{Ord} \) be an ordinal function. Then \( F^\beta(\eta) < F^\alpha(\xi) \) holds if \((\beta < \alpha \land K_\Omega \beta \cup \{\eta\} < F^\alpha(\xi)) \) or \((\alpha < \beta \land F^\beta(\eta) < K_\Omega \alpha)\).

**Proposition 3.4** Suppose that \( \alpha < \varepsilon_{\Omega_1+1} \), a function \( F : \text{Ord} \to \text{Ord} \) has a \( \Sigma_1 \)-definition in the \( \Omega_1 \)th stage \( \mathcal{L}_{\Omega_1} \) of the constructible hierarchy \( \{\mathcal{L}_\alpha\}_{\alpha \in \text{Ord}} \) and that \( F(\xi) < \Omega_1 \) for all \( \xi < \Omega_1 \). Then \( F^\alpha \) also has a \( \Sigma_1 \)-definition in \( \mathcal{L}_{\Omega_1} \) and \( F^\alpha(\xi) < \Omega_1 \) holds for all \( \xi < \Omega_1 \).

**Proposition 3.5** For any \( \alpha \in \text{Ord} \), for any \( \eta, \xi < \Omega_1 \) and for any ordinal function \( F : \Omega_1 \to \Omega_1 \), if \( \eta < F^\alpha(\xi) \), then \( F^\alpha(\eta) \leq F^\alpha(\xi) \).

**Definition 3.6** We define the value \([\alpha] \in \text{Ord} \) of an ordinal term \( \alpha \in \mathcal{O}(F) \) by recursion on the length of \( \alpha \).

1. \([0] = 0 \) and \([\Omega] = \Omega_1 \).
2. \([\alpha + \beta] = [\alpha] + [\beta] \).
3. \([\varphi \alpha \beta] = [\varphi] [\alpha] [\beta] \), where \([\varphi] \) is the standard Veblen function, i.e.,

\[
\begin{align*}
[\varphi]0\beta &= \omega^\beta, \\
[\varphi](\alpha+1)0 &= \sup\{([\varphi] \alpha)^n0 \mid n \in \omega\}, \\
[\varphi] \varphi \gamma 0 &= \sup\{[\varphi] \alpha 0 \mid \alpha < \gamma\} & \text{if } \gamma \in \text{Lim}, \\
[\varphi](\alpha+1)(\beta+1) &= \sup\{([\varphi] \alpha)^n([\varphi] \alpha \beta + 1) \mid n \in \omega\}, \\
[\varphi] \gamma(\beta+1) &= \sup\{[\varphi] \alpha ([\varphi] \gamma \beta + 1) \mid \alpha < \gamma\} & \text{if } \gamma \in \text{Lim}, \\
[\varphi] \alpha \gamma &= \sup\{[\varphi] \alpha \beta \mid \beta < \gamma\} & \text{if } \gamma \in \text{Lim}.
\end{align*}
\]
4. $\Omega^\alpha \cdot \xi = \Omega_1^{\lceil \alpha \rceil} \cdot \lceil \xi \rceil$.

5. $[S(\alpha)] = [S](\lceil \alpha \rceil)$, where $[S]$ denotes the ordinal successor $\alpha \mapsto \alpha + 1$. Clearly $\{ [S](\xi) \mid \xi < \Omega_1 \} \subseteq \Omega_1$.

6. $[E(\alpha)] = [E](\lceil \alpha \rceil)$, where the function $[E] : \text{Ord} \to \text{Ord}$ is defined by $[E](\alpha) = \min \{ \xi \in \text{Ord} \mid \omega^\xi = \xi \text{ and } \alpha < \xi \}$. It is also clear that $\{ [E](\xi) \mid \xi < \Omega_1 \} \subseteq \Omega_1$ holds.

7. $[F^\alpha(\xi)] = [F]^\lceil \alpha \rceil(\lceil \xi \rceil)$.

**Definition 3.7** For all $\alpha, \beta \in \mathcal{O}\mathcal{T}(\mathcal{F})$, $\alpha < \beta$ if $[\alpha] < [\beta]$, and $\alpha = \beta$ if $[\alpha] = [\beta]$.

We will identify each element $\alpha \in \mathcal{O}\mathcal{T}(\mathcal{F})$ with its value $[\alpha] \in \text{Ord}$. Accordingly we will write $K_\Omega \alpha$ instead of $K_\Omega [\alpha]$ for $\alpha \in \mathcal{O}\mathcal{T}(\mathcal{F})$. Further for a finite set $K \subseteq \text{Ord}$ we write $K_\Omega K$ to denote the finite set $\bigcup_{\xi \in K} K_\Omega \xi$. By this identification, $\mathbb{H}$ is the set of additively indecomposable ordinals and $\text{SC}$ is the set of strongly critical ordinals, i.e, $\text{SC} \subseteq \mathbb{H} \subseteq \text{Lim} \cup \{1\} \subseteq \text{Ord}$.

**Corollary 3.8** $F^\alpha(\xi) < \Omega$ for any $F \in \mathcal{F}$ and $\xi < \Omega$.

**Proof.** Proof by induction over the build-up of $F \in \mathcal{F}$. \qed

**Corollary 3.9**

1. $K_\Omega 0 = K_\Omega \Omega = \emptyset$.

2. If $K_\Omega \alpha < \xi$ and $\xi \in \text{SC}$, then $K_\Omega S(\alpha) < \xi$.

3. $K_\Omega E(\alpha) = \{ E(\alpha) \}$ (since $\alpha < \Omega$).

4. If $K_\Omega \alpha \cup K_\Omega \beta < \xi$ and $\xi \in \text{SC}$, then $K_\Omega (\alpha + \beta) < \xi$.

5. $K_\Omega \varphi \alpha \beta = \{ \varphi \alpha \beta \}$ (since $\alpha, \beta < \Omega$). Further, if $\alpha, \beta < \xi$ and $\xi \in \text{SC}$, then $\varphi \alpha \beta < \xi$.

6. $K_\Omega F^\alpha(\xi) = \{ F^\alpha(\xi) \}$ (since $\xi < \Omega$).

By Corollary 3.8 each function symbol in $\mathcal{F}$ defines a weakly increasing function $F : \Omega \to \Omega$ such that $\xi < F(\xi)$ holds for all $\xi \in \Omega$. In the rest of this section let $F$ denote such a function. For a finite set $K \subseteq \text{Ord}$ we will use the notation $F[K](\xi)$ to abbreviate $F(\max(K \cup \{ \xi \}))$.

**Lemma 3.10** Let $K \subseteq \text{Ord}$ be a finite set such that $K < \Omega$. Then $(F[K])^\alpha(\xi) \leq F^\alpha[K](\xi)$ for all $\xi < \Omega$.

**Lemma 3.11** $(F^\alpha)^\beta(\xi) \leq F^{\alpha + \beta}(\xi)$ for all $\xi < \Omega$. 
4 An infinitary proof system $\text{ID}_{1}^{\infty}$

In this section we introduce the main definition of this paper, a new infinitary proof system $\text{ID}_{1}^{\infty}$, to which the new ordinal notation system $\mathcal{OT}(\mathcal{F})$ is connected, and into which every (finite) proof in $\text{ID}_{1}$ can be embedded in good order. For each positive operator form $A$ and for each ordinal term $\alpha \in (\mathcal{OT}(\mathcal{F}) \upharpoonright \Omega) \cup \{\Omega\}$ let $P_{A}^{<\alpha}$ be a new unary predicate symbol. Let us define an infinitary language $\mathcal{L}^{*}$ of $\text{ID}_{1}^{\infty}$ by $\mathcal{L}^{*} = \mathcal{L}_{PA} \cup \{\neq, \not\leq\} \cup \{P_{A}^{<\alpha}, \neg P_{A}^{<\alpha} \mid \alpha \in (\mathcal{OT}(\mathcal{F}) \upharpoonright \Omega) \cup \{\Omega\}\}$ and $A$ is a positive operator form.

Let us write $P_{A}^{<\infty}$ to denote $P_{A}$ to have the inclusion $\mathcal{L}_{ID_{1}} \subseteq \mathcal{L}^{*}$. We write $\mathcal{T}(\mathcal{L}^{*})$ to denote the set of closed $\mathcal{L}^{*}$-terms. Specifically, the language $\mathcal{L}^{*}$ contains complementary predicate symbol $\neg P$ for each predicate symbol $P \in \mathcal{L}^{*}$. We note that the negation $\neg$ nor the implication $\rightarrow$ is not included as a logical symbol. The negation $\neg A$ is defined via de Morgan’s law by $\neg(\neg P(t)) := P(t)$ for an atomic formula $P(t)$, $\neg(\alpha \land \beta) := \neg \alpha \lor \neg \beta$, $\neg(\alpha \lor \beta) := \neg \alpha \land \neg \beta$, $\forall x A := \exists x \neg A$ and $\neg \exists x A := \forall x \neg A$. The implication $A \rightarrow B$ is defined by $\neg A \lor B$. We start with technical definitions.

Definition 4.1 (Complexity measures $lh$, $rk$, $k^{\Pi}$, $k$ of $\mathcal{L}^{*}$-formulas)

1. The length $lh(A)$ of an $\mathcal{L}^{*}$-formula $A$ is the number of the symbols $P_{A}^{<\alpha}$, $\neg P_{A}^{<\alpha}$, $\lor$, $\land$, $\exists$ and $\forall$ occurring in $A$.

2. The rank $rk(A)$ of an $\mathcal{L}^{*}$-formula $A$.
   
   (a) $rk(P_{A}^{<\alpha}(t)) := rk(\neg P_{A}^{<\alpha}(t)) := \omega \cdot \alpha$.
   
   (b) $rk(A) := 0$ if $A$ is an $\mathcal{L}_{ID_{1}}$-literal.
   
   (c) $rk(\forall x A) := rk(\exists x A) := rk(A) + 1$.

3. The set $k^{\Pi}(A)$ of $\Pi$-coefficients of an $\mathcal{L}^{*}$-formula $A$.
   
   (a) $k^{\Pi}(P_{A}^{<\alpha}(t)) := \{0\}$, $k^{\Pi}(\neg P_{A}^{<\alpha}(t)) := \{0, \alpha\}$.
   
   (b) $k^{\Pi}(A) := \{0\}$ if $A$ is an $\mathcal{L}_{ID_{1}}$-literal.
   
   (c) $k^{\Pi}(A \land B) := k^{\Pi}(A \lor B) := k^{\Pi}(A) \cup k^{\Pi}(B)$.
   
   (d) $k^{\Pi}(\forall x A) := k^{\Pi}(\exists x A) := k^{\Pi}(A)$.

4. The set $k^{\Sigma}(A)$ of $\Sigma$-coefficients of an $\mathcal{L}^{*}$-formula $A$.
   
   $k^{\Sigma}(A) := k^{\Pi}(\neg A)$.

5. The set $k(A)$ of all the coefficients of an $\mathcal{L}^{*}$-formula $A$.
   
   $k(A) := k^{\Pi}(A) \cup k^{\Sigma}(A)$.

6. The set $k^{\Pi}_{1}(A)$ of $\Pi$-coefficients of an $\mathcal{L}^{*}$-formula $A$ less than $\Omega$.
   
   $k^{\Pi}_{1}(A) := k^{\Pi}(A) \mid \Omega$.

The set $k^{\Pi}_{1}(A)$ and $k_{\Omega}(A)$ are defined accordingly.
By definition $\text{rk}(A) = \text{rk}(-A)$, $k(A) = k(-A)$ and $k_{\Omega}(A) = k_{\Omega}(-A)$.

**Definition 4.2 (Complexity measures val, ord, N of $L^*$-terms)**

1. The value $\text{val}(t)$ of a term $t \in T(L_{ID_1}) = T(L_{PA})$ is the value of the closed term $t$ in the standard model $\mathbb{N}$ of the Peano arithmetic $PA$.

2. A complexity measure $\text{ord} : T(L^*) \to (\mathcal{O}\mathcal{T}(F) \cup \Omega) \cup \{\Omega\}$ is defined by

$$\begin{cases}
\text{ord}(t) := 0 & \text{if } t \in T(L_{ID_1}), \\
\text{ord}(\alpha) := \alpha & \text{if } \alpha \in \mathcal{O}\mathcal{T}(F).
\end{cases}$$

3. The norm $N(\alpha)$ of $\alpha \in \mathcal{O}\mathcal{T}(F)$.

   (a) $N(0) = 0$ and $N(\Omega) = 1$.
   (b) $N(S(\alpha)) = N(\alpha) + 1$.
   (c) $N(E(\alpha)) = N(\alpha) + 1$.
   (d) $N(\alpha + \beta) = N(\alpha) + N(\beta)$.
   (e) $N(\varphi(\alpha)) = N(\alpha) + N(\beta) + 1$.
   (f) $N(\Omega^\alpha \cdot \xi) = N(\alpha) + N(\xi) + 1$.
   (g) $N(F^\alpha(\xi)) = N(F(\xi)) + N(\alpha)$. (Note that $F(\xi) \in \mathcal{O}\mathcal{T}(F)$ if $F(\xi) \in \mathcal{O}\mathcal{T}(F)$.)

The norm is extended to a complexity measure $N : T(L^*) \to \mathbb{N}$ by

$$\begin{cases}
N(t) := \text{val}(t) & \text{if } t \in T(L_{ID_1}), \\
N(\alpha) := N(\alpha) & \text{if } \alpha \in \mathcal{O}\mathcal{T}(F).
\end{cases}$$

By definition $N(\omega^\alpha) = N(\varphi(0^\alpha)) = N(\alpha) + 1$ and $N(m) = N(\omega^m \cdot m) = m$ for any $m < \omega$. This seems to be a good point to explain why we contain the constant $\Omega$ in $\mathcal{O}\mathcal{T}(F)$. Having that $N(\Omega) = 1$ removes some technicalities.

**Definition 4.3** We define a relation $\simeq$ between $L^*$-sentences and (infinitary) propositional $L^*$-sentences.

1. $\neg P_{\mathcal{A}}^{<\alpha}(t) :\simeq \bigwedge_{\xi \in \mathcal{O}\mathcal{T}(F) \cup \alpha} \neg A(P_{\mathcal{A}}^{<\xi}, t)$ and $P_{\mathcal{A}}^{<\alpha}(t) :\simeq \bigvee_{\xi \in \mathcal{O}\mathcal{T}(F) \cup \alpha} A(P_{\mathcal{A}}^{<\xi}, t)$.

2. $A \land B :\simeq \bigwedge_{\iota \in \{0, 1\}} A_\iota$ and $A \lor B :\simeq \bigvee_{\iota \in \{0, 1\}} A_\iota$ where $A_0 \equiv A$ and $A_1 \equiv B$.

3. $\forall x A(x) :\simeq \bigwedge_{t \in T(L_{ID_1})} A(t)$ and $\exists x A(x) :\simeq \bigvee_{t \in T(L_{ID_1})} A(t)$.

We call an $L^*$-sentence $A$ a $\land$-type (conjunctive type) if $A \simeq \bigwedge_{\iota \in \{0, 1\}} A_\iota$ for some $A_\iota$, and a $\lor$-type (disjunctive type) if $A \simeq \bigvee_{\iota \in \{0, 1\}} A_\iota$ for some $A_\iota$. For the sake of simplicity we will write $\bigwedge_{\xi < \alpha} A_\xi$ instead of $\bigwedge_{\xi \in \mathcal{O}\mathcal{T}(F) \cup \alpha} A_\xi$ and write $\bigvee_{\xi < \alpha} A_\xi$ accordingly.
Lemma 4.4 1. If either $A \simeq \bigwedge_{i \in I} A_i$ or $A \simeq \bigvee_{i \in I} A_i$, then for all $i \in J$, $k^\Pi(A_i) \subseteq \{\text{ord}(i)\} \cup k^\Pi(A_i)$ and $k^\Sigma(A_i) \subseteq \{\text{ord}(i)\} \cup k^\Sigma(A_i)$.

2. For any $\alpha \in \mathcal{OT}(\mathcal{F})$, if $A \simeq \bigwedge_{\xi \in \alpha} A_\xi$, then $(\exists \sigma \in k^\Pi(A))(\forall \xi < \alpha)[\xi \leq \sigma]$.

3. For any $\mathcal{L}^*$-sentence $A$, $\text{rk}(A) = \omega \cdot \max k(A) + n$ for some $n \leq \text{lh}(A)$.

4. If $\text{rk}(A) = \Omega$, then either $A \equiv P_{A}^<\Omega(t)$ or $A \equiv \neg P_{A}^<\Omega(t)$.

5. If either $A \simeq \bigwedge_{i \in I} A_i$ or $A \simeq \bigvee_{i \in I} A_i$, then $\text{rk}(A_i) \leq \max(\{\text{rk}(A)\}) \cup \{2 \cdot \text{Nh}(\xi) + \text{lh}(A(\cdot, *)) | P_{A}^{<\xi} \text{ or } \neg P_{A}^{<\xi} \text{ occurs in } A\}$ for all $i \in J$.

Proof. We only show the non-trivial property, Property 5. By Property 3, $\text{rk}(A) = \omega \cdot \max k(A) + n$ for some $n \leq \text{lh}(A)$.

CASE. $n > 0$: In this case $\text{rk}(A) = \omega \cdot \max k(A) + n_0$ for some $n_0 < n \leq \text{lh}(A)$. Hence clearly $\text{Nh}(\text{rk}(A_i)) \leq \text{Nh}(\text{rk}(A))$.

CASE. $n = 0$: In this case without loss of generality let us assume $A$ is of the form $P_{A}^{<\alpha}(t) \simeq \bigvee_{\xi \in \alpha} A(P_{A}^{<\xi}, t)$ and hence $A_\xi \simeq A(P_{A}^{<\xi}, t)$. Let $i := \xi < \alpha$. Then $\text{rk}(A_i) = \omega \cdot \xi + n_i$ for some $n_i \leq \text{lh}(A(\cdot, t))$. Hence $\text{Nh}(\text{rk}(A)) \leq 2 \cdot \text{Nh}(\xi) + \text{lh}(A(\cdot, *))$.

Throughout this section we use the symbol $F$ to denote a weakly increasing ordinal function $F : \Omega \rightarrow \Omega$ and the symbol $f$ to denote a numerical function $f : \mathbb{N} \rightarrow \mathbb{N}$ that enjoys the following conditions.

(f.1) $f$ is a strictly increasing function such that $2m + 1 \leq f(m)$ for all $m$. Hence, in particular, $n + f(m) \leq f(n + m)$ for all $m$ and $n$.

(f.2) $2 \cdot f(m) \leq f(f(m))$ for all $m$.

We will use the notation $f[n](m)$ to abbreviate $f(n + m)$. It is easy to see that if the conditions (f.1) and (f.2) hold, then for a fixed $n$ the conditions $f[n]1$ and $(f[n]2)$ also hold.

Definition 4.5 Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a numerical function. Then a function $f^\alpha : \mathbb{N} \rightarrow \mathbb{N}$ is defined by transfinite recursion on $\alpha \in \mathcal{OT}(\mathcal{F})$ by

\begin{align*}
f^0(m) &= f(m), \\
f^\alpha(m) &= \max\{f^\beta(f^\beta(m)) | \beta < \alpha \text{ and } N(\beta) \leq f[N(\alpha)](m)\} \text{ if } 0 < \alpha.
\end{align*}

Corollary 4.6 1. If $f$ is strictly increasing, then so is $f^\alpha$ for any $\alpha \in \mathcal{OT}(\mathcal{F})$.

2. If $\beta < \alpha$ and $N(\beta) \leq f[N(\alpha)](m)$, then $f^\beta(m) < f^\alpha(m)$.

3. $f^\alpha(f^\alpha(m)) \leq f^{\alpha+1}(m)$.

We note that the function $f^\alpha$ is not a computable function in general even if $f$ is computable since the ordinal notation system $\langle \mathcal{OT}(\mathcal{F}), < \rangle$ is not a computable system.
Lemma 4.7 Let $\alpha \in \mathcal{O}(\mathcal{F})$ and $F \in \mathcal{F}$. Then $N(\alpha) \leq f^{\mathcal{O}(\alpha)}(0)$.

Lemma 4.8 Let $\{\alpha, \beta\} \subseteq \mathcal{O}(\mathcal{F}) \cap \Omega$ and $F \in \mathcal{F}$. Then $(f^{\alpha})^{\beta}(m) \leq f^{\mathcal{O}(\alpha)+\mathcal{O}(\beta)}(m)$ for all $m$.

Lemma 4.9
1. $f^{\alpha}[n](m) \leq (f[n])^{\alpha}(m)$.
2. If $n \leq m$, then $(f[n])^{\alpha}(m) \leq f^{\alpha}[f^{\alpha}(f(m))](f(m))$.

We write $f[n][m]$ to abbreviate $(f[n])(m)$ and $f[n]^{\alpha}$ to abbreviate $(f[n])^{\alpha}$.

Corollary 4.10 If $n \leq m$, then $(f[n])^{\alpha}(m) \leq f^{\alpha+2}(m)$.

We define a relation $f, F \vdash_{\rho}^{\alpha} \Gamma$ for a quintuple $(f, F, \alpha, \rho, \Gamma)$ where $\alpha < \epsilon_{\Omega+1}$, $\rho < \Omega + \omega$ and $\Gamma$ is a sequent of $\mathcal{L}$-sentences. In this paper a “sequent” means a finite set of formulas. We write $\Gamma, A$ or $A, \Gamma$ to denote $\Gamma \cup \{A\}$.

We will write $TRUE_0$ to denote the set $\{A | A$ is an $\mathcal{L}_{PA}$-literal true in the standard model $\mathbb{N}$ of PA}.

Definition 4.11 $f, F \vdash_{\rho}^{\alpha} \Gamma$ if

\[ \max\{N(F(0)), N(\alpha)\} \leq f(0), \quad K_{\Omega}^{\alpha} < F(0), \quad (HYP(f; F; \alpha)) \]

and one of the following holds.

(Ax1) $\exists A(x)$: an $\mathcal{L}_{ID_1}$-literal, $\exists s, t \in \mathcal{T}(\mathcal{L}_{ID_1})$ s.t. $\text{FV}(A) = \{x\}$, $\text{val}(s) = \text{val}(t)$ and $\{\neg A(s), A(t)\} \subseteq \Gamma$.

(Ax2) $\Gamma \cap TRUE_0 \neq \emptyset$.

(V) $\exists A \simeq \bigvee_{\iota \in J} A_{\mu} \in \Gamma$, $\exists \alpha_0 < \alpha$, $\exists \iota_0 \in J$ s.t. $N(\iota_0) \leq f(0)$, $\text{ord}(\iota_0) < \min\{\alpha, F(0)\}$ and $f, F \vdash_{\rho}^{\alpha_0} \Gamma, A_{\iota_0}$.

(A) $\exists A \simeq \bigwedge_{\iota \in J} A_{\iota} \in \Gamma$ s.t. $\max\{N(\sigma) | \sigma \in k^{\Omega}(A)\} \leq f(0)$, $k^{\Omega}(A) < F(0)$ and $(\forall \iota \in J) \ (\exists \alpha_{\iota} < \alpha) [f[N(\iota)], F[\text{ord}(\iota)] \vdash^{\alpha_{\iota}}_{\rho} \Gamma, A_{\iota}]$.

(Cl) $\exists t \in \mathcal{T}(\mathcal{L}_{ID_1})$, $\exists \alpha_0 < \alpha$ s.t. $P^{<\Omega}(t) \in \Gamma$, $\Omega < \alpha$ and $f, F \vdash_{\rho}^{\alpha_0} \Gamma, A(P^{<\Omega}_{\mathcal{A}}, t)$.

(Cut) $\exists C$: an $\mathcal{L}^*$-sentence of $\vee$-type, $\exists \alpha_0 < \alpha$ s.t. $\max\{N(\sigma) | \sigma \in k\Omega(C)\} \cup \{lh(C)\} \leq f(0)$, $k\Omega(C) < \rho$, $f, F \vdash_{\rho}^{\alpha_0} \Gamma, C$, and $f, F \vdash_{\rho}^{\alpha_0} \Gamma, \neg C$.

We will call the pair $(f, F)$ operators controlling the derivation that forms $f, F \vdash_{\rho}^{\alpha} \Gamma$. 47
In the sequel we always assume that the operator $F$ enjoys the following condition HYP($F$):

$$\eta < F(\xi) \Rightarrow F(\eta) \leq F(\xi) \quad \text{for any ordinals } \xi, \eta < \Omega. \quad \text{(HYP($F$))}$$

We note that the hypothesis HYP($F$) reflects the fact stated in Proposition 3.5. It is not difficult to see that if the condition HYP($F$) holds, then the condition HYP($F[K]$) also holds for any finite set $K < \Omega$.

**Lemma 4.12 (Inversion)** Assume that $A \simeq \bigwedge_{i \in J} A_i$. If $f, F \vdash_{\rho}^\alpha \Gamma, A$, then for all $i \in J$, $f[N(i)], F[\text{ord}(i)] \vdash_{\rho}^\alpha \Gamma, A_i$.

We write $f \circ g$ to denote the result of composing $f$ and $g$: $m \mapsto f(g(m))$.

**Lemma 4.13 (Cut-reduction)** Assume $C \simeq \bigvee_{i \in J} C_i$, $\text{rk}(C) = \rho \neq \Omega$, $\max(\{N(\sigma) \mid \sigma \in k_{\Omega}(C) \cup \{\text{lh}(C)\}) \leq f(g(0))$, and $k_{\Omega}(C) < F(0)$. If $f, F \vdash_{\rho}^\alpha \Gamma, -C$ and $g, F \vdash_{\rho}^\beta \Gamma, C$, then $f \circ g, F \vdash_{\rho}^{\alpha+\beta} \Gamma$.

For a sequent $\Gamma$ we write $k^\Pi_{\Omega}(\Gamma)$ to denote the set $\bigcup_{B \in \Gamma} k^\Pi_{\Omega}(B)$.

**Lemma 4.14 (First Cut-elimination)** Let $k < \omega$. If $f, F \vdash_{\Omega+k+2}^\alpha \Gamma$, then $f^{F(0)+1}, F \vdash_{\omega}^\alpha \Gamma$.

**Lemma 4.15 (Predicative Cut-elimination)** Assume that $\{\alpha, \beta, \gamma\} < \Omega$, $N(\alpha) \leq f^*(0)$ and $K_{\Omega}\alpha < F(0)$. If $f^*, F \vdash_{\rho}^\beta \Gamma, -C$ and $g, F \vdash_{\rho}^\beta \Gamma, C$, then $f \circ g, F \vdash_{\rho}^{\alpha+\beta} \Gamma$.

**Definition 4.16** For each $\mathcal{L}^*$-formula $B$ let $B^\alpha$ be the result of replacing in $B$ every occurrence of $P^\alpha_{\ast}$ by $P^\alpha_{\ast}$.

**Lemma 4.17 (Boundedness)** Assume that $f, F \vdash_{\rho}^\alpha \Gamma, A$. Then for all $\xi$ if $\alpha \leq \xi \leq F(0), N(\xi) \leq f(0)$ and $K_{\Omega}m < F(0)$, then $f, F \vdash_{\rho}^\alpha \Gamma, A^\xi$.

We will write $f, F \vdash_{\rho}^\alpha \Gamma$ instead of $f, F \vdash_{\rho}^\alpha \Gamma$.

**Lemma 4.18 (Impredicative Cut-elimination)**

If $f, F \vdash_{\Omega+1}^\alpha \Gamma$, then $f^{F(0)+1}, F^{\alpha+1} \vdash_{\Omega}^\alpha \Gamma$.

**Lemma 4.19 (Witnessing)** For each $j < l$ let $B_j(x)$ be a $\Delta_0^0$-$\mathcal{L}_{PA}$-formula such that $\text{FV}(B_j(x)) = \{x\}$. Let $\Gamma \equiv \exists x_0 B_0(x_0), \ldots, \exists x_{l-1} B_{l-1}(x_{l-1})$. If $f, F \vdash_{\rho}^\alpha \Gamma$ for some $\alpha \in \overline{\mathcal{O}}\mathcal{T}(\mathcal{F})$, then there exists a sequence $m_0, \ldots, m_{l-1}$ of naturals such that $\max(m_j \mid j < l) \leq f(0)$ and $B_0(m_0) \lor \cdots \lor B_{l-1}(m_{l-1})$ is true in the standard model $\mathbb{N}$ of PA.
5 Embedding ID$_1$ into ID$_1^\infty$

In this section we embed the system ID$_1$ into the infinitary system ID$_1^\infty$. Following conventions in the previous section we use the symbol f to denote a strict increasing function $f: \mathbb{N} \to \mathbb{N}$ that enjoys the conditions (f.1) and (f.2) (p. 8). Let us recall that the function symbol $E \in \mathcal{F}$ denotes the function $E: \Omega \to \Omega$ such that $E(\alpha) = \min\{\xi < \Omega \mid \omega^\xi = \xi \text{ and } \alpha < \xi\}$. It is easy to see that the condition HYP(E) holds since $E(\xi) = \varepsilon_0 \leq E(0)$ for all $\xi < E(0) = \varepsilon_0$.

Lemma 5.1 (Tautology lemma) Let $s, t \in \mathcal{T}(\mathcal{L}_{ID_1})$, $\Gamma$ be a sequent of $\mathcal{L}^*$-sentences, and $A(x)$ be an $\mathcal{L}^*$-formula such that $\text{FV}(A) = \{x\}$. If $\text{val}(s) = \text{val}(t)$, then

$$f[n], E[k_\Omega(A)] \vdash_{0}^{rk(A)-2} \Gamma, \neg A(s), A(t),$$

where $n := \max\{\{N(rk(A))\} \cup \{2 \cdot N(\sigma) + \text{lh}(A(\cdot, *)) \mid \sigma \in k_\Omega(A) \text{ and } P_A^{<\xi} \text{ or } \neg P_A^{<\xi} \text{ occurs in } A\}$.

**Proof.** By induction on $rk(A)$. Let $n := \max\{\{N(rk(A))\} \cup \{2 \cdot N(\sigma) + \text{lh}(A(\cdot, *)) \mid \sigma \in k_\Omega(A) \text{ and } P_A^{<\xi} \text{ or } \neg P_A^{<\xi} \text{ occurs in } A\}$. From Lemma 4.4.3 one can check that the condition HYP($f[n]; E(k_\Omega(A)); rk(A) \cdot 2$) holds. If $rk(A) = 0$, then $A$ is an $\mathcal{L}_{ID_1}$-literal, and hence (1) is an instance of (Ax1). Suppose that $rk(A) > 0$. Without loss of generality we can assume that $A \simeq \bigvee_{i \in J} A_i$. Let $i \in J$. By Lemma 4.4.5 we observe that $N(rk(A_i)) \leq f(n) = f[n][N(\iota)](0)$ since $2m + 1 \leq f(m)$ for all $m$ by the condition (f.1). Further by Lemma 4.4.1 $k_\Omega(rk(A_i) \cdot 2) \subseteq k_\Omega(A) \cup \{\text{ord}(\iota)\} \leq E[k_\Omega(A)][\text{ord}(\iota)]$. Summing up, we have the condition

$$\text{HYP}(f[n][N(\iota)]; E[k_\Omega(A)][\text{ord}(\iota)]; rk(A_i) \cdot 2).$$

Hence by IH we can obtain the sequent

$$f[n][N(\iota)], E[k_\Omega(A)][\text{ord}(\iota)] \vdash_{0}^{rk(A_i)-2} \Gamma, \neg A_i(s), A_i(t).$$

(2)

It is not difficult to see $\text{ord}(\iota) \leq rk(A_i) < rk(A_i) \cdot 2 + 1$ and $N(rk(A_i) \cdot 2 + 1) = N(rk(A_i) \cdot 2) + 1 \leq f[n][N(\iota)](0)$. This allows us to apply (V) to the sequent (2) yielding

$$f[n][N(\iota)], E[k_\Omega(A)][\text{ord}(\iota)] \vdash_{0}^{rk(A_i)-2+1} \Gamma, \neg A_i(s), A_i(t).$$

We can see that $rk(A_i) \cdot 2 + 1 < rk(A) \cdot 2$, $\max\{N(\sigma) \mid \sigma \in k_\Omega^n(A)\} \leq f[n](0)$ and $k_\Omega^n(A) < E[k_\Omega(A)]$. Hence we can apply (\text{V\wedge}) concluding (1). □

Lemma 5.2 Let $B_j$ be an $\mathcal{L}_{ID_1}$-sentence for each $j = 0, \ldots, l - 1$. Suppose that $B_0 \lor \cdots \lor B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then there exists a natural $k < \omega$ such that $f[m + k], E \vdash_{0}^{rk(A_i)-2+k} \Gamma, B_0, \ldots, B_{l-1}$, where $m = \max\{\{N(rk(B_j)) \mid 0 \leq j \leq l-1\} \cup \{\text{lh}(A(\cdot, *)) \mid P_A^{<\xi} \text{ or } \neg P_A^{<\xi} \text{ occurs in } B_j \text{ for some } j\}.$
Proof. Let $B_j$ be an $\mathcal{L}_{ID_1}$-sentence for each $j = 0, \ldots, l - 1$ and suppose that $B_0 \lor \cdots \lor B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then we can find a cut-free proof of the sequent $\Gamma, B_0, \ldots, B_{l-1}$ in an LK-style sequent calculus. More precisely we can find a cut-free proof $P$ of $\Gamma, B_0, \ldots, B_{l-1}$ in the sequent calculus that is known as $G3_m$. Let $h$ denote the tree height of the cut-free proof $P$. Then by induction on $h$ one can find a witnessing natural $k$ such that $f[m+k], \Gamma \vdash^\alpha \Gamma, B_0, \ldots, B_{l-1}$ for all $\alpha \geq \Omega + k$. In case $h = 0$ Tautology lemma (Lemma 5.1) can be applied since for any $\mathcal{L}_{ID_1}$-sentence $A$, $\text{rk}(A) \in \omega \cup \{\Omega + k | k < \omega \}$ and $k(A) \subseteq \{0, \Omega\}$, and hence $k_\Omega(A) = \{0\}$ and $\max\{N(\sigma) | \sigma \in k_\Omega(A)\} = 0$. \qed

Lemma 5.3 Let $m \in \mathbb{N}$ and $A(x)$ be an $\mathcal{L}_{ID_1}$-formula such that $\text{FV}(A(x)) = \{x\}$. Then for any $t \in T(\mathcal{L}_{ID_1})$ and for any sequent $\Gamma$ of $\mathcal{L}_{ID_1}$-sentences, if $\text{val}(t) = m$, then

$$f[n+m], \Gamma \vdash^{\text{rk}(A)+m+2} \Gamma, \neg A(0), \neg \forall x(A(x) \rightarrow A(S(x))), \Gamma, A(t),$$

(3)

where $n := \max\{\{N(\text{rk}(A))\} \cup \{\text{lh}(A(\cdot, \cdot)) | \text{rk}(A) \leq \Omega \}$ or $\text{rk}(A) \leq \Omega \}$ occurs in $A$.

Proof. By induction on $m$. The base case $\text{val}(t) = m = 0$ follows from Tautology lemma (Lemma 5.1). For the induction step suppose $\text{val}(t) = m + 1$. Fix a sequent $\Gamma$ of $\mathcal{L}_{ID_1}$-sentences. Then (3) holds by IH. On the other hand again by Tautology lemma,

$$f[n], \Gamma \vdash^{\text{rk}(A)+2} \Gamma, \neg A(0), \exists x(A(x) \land \neg A(S(x))), A(m), \neg A(m).$$

(4)

An application of $(\land)$ to the two sequents (3) and (4) yields

$$f[n+m], \Gamma \vdash^{\alpha+2+1} \Gamma, \neg A(0), \exists x(A(x) \land \neg A(S(x))), A(t), A(m) \land \neg A(m),$$

The final application of $(\lor)$ yields

$$f[n+m+1], \Gamma \vdash^{\text{rk}(A)+m+1+2} \Gamma, \neg A(0), \exists x(A(x) \land \neg A(S(x))), A(t).$$

Lemma 5.4 Let $\xi \leq \Omega$, $F(x)$ be an $\mathcal{L}_{ID_1}$-formula such that $\text{FV}(F(x)) = \{x\}$ and $B(X)$ be an $X$-positive $\mathcal{L}_{PA}(X)$-formula such that $\text{FV}(B) = \emptyset$. Then

$$f[n], E[K_{\Omega} \xi] \vdash^{\sigma+\alpha+1+1} \Gamma, \neg \forall x(A(F, x) \rightarrow F(x)), \neg B(P^<\xi, B(F),$$

where $\sigma := \text{rk}(F), \alpha := \text{rk}(B(P^<\xi))$ and $n := \max\{\{N(\sigma + \alpha + 1)\} \cup \{\text{lh}(B) | \text{rk}(B) \leq \Omega \} \}$. \qed

Proof. By main induction on $\xi$ and side induction on $\text{rk}(B(P^<\xi))$. Let $\text{Cl}_A(F)$ denote $\neg \forall x(A(F, x) \rightarrow F(x))$. Then $\neg \text{Cl}_A(F) \equiv \exists x(A(F, x) \land \neg F(x))$. The argument splits into several cases depending on the shape of the formula $B(X)$. 

CASE. $B(X)$ is an $\mathcal{L}_{PA}$-literal: In this case $B$ does not contain the set free variable $X$, and hence Tautology lemma (Lemma 5.1) can be applied. Note that the operator form $B$ does not occur in $B$.

CASE. $B \equiv X(t)$ for some $t \in T(\mathcal{L}_{ID_1})$: In this case $\neg B(P_{\mathcal{A}}^{<\xi}) \equiv \neg P_{\mathcal{A}}^{<\xi}(t) \equiv \bigwedge_{\eta<\xi} \neg \mathcal{A}(P_{\mathcal{A}}^{<\eta}, t)$. Let $\eta < \xi$. Then by MIH

$$f[n], E[K_{\Omega}\xi] \vdash_{0}^{(\sigma+\alpha_{\xi}+1)^2} \Gamma, \neg \mathcal{A}(F), \neg P_{\mathcal{A}}^{<\xi}(t), A(F, t), F(t)$$

where $\alpha_{\xi} := \text{rk}(A(P_{\mathcal{A}}^{<\xi}, t))$ and $n_{\xi} := \max\{N(\text{rk}(A(P_{\mathcal{A}}^{<\xi}, t))) \}$. We note that $\eta < \xi \leq \Omega$ and hence $K_{\Omega}\eta = \{\eta\} = \{\text{ord}(\eta)\}$. Hence this yields the sequent

$$f[n][N(\eta)], E[\text{ord}(\eta)] \vdash_{0}^{(\sigma+\alpha_{\eta}+1)^2} \Gamma, \neg \mathcal{A}(F), \neg P_{\mathcal{A}}^{<\eta}(t), A(F, t), F(t).$$

An application of ($\wedge$) allows us to conclude

$$f[n], E[K_{\Omega}\xi] \vdash_{0}^{(\sigma+\alpha_{\xi}+1)^2} \Gamma, \neg \mathcal{A}(F), \neg P_{\mathcal{A}}^{<\xi}(t), A(F, t), F(t).$$

On the other hand by Tautology lemma (Lemma 5.1),

$$f[n], E[K_{\Omega}\xi] \vdash_{0}^{(\sigma+\alpha_{\xi})^2} \Gamma, \neg \mathcal{A}(F), \neg P_{\mathcal{A}}^{<\xi}(t), A(F, t), F(t).$$

Another application of ($\wedge$) to the two sequents (5) and (5) yields the sequent

$$f[n], E[K_{\Omega}\xi] \vdash_{0}^{(\sigma+\alpha_{\xi}+1)^2} \Gamma, \neg \mathcal{A}(F), \neg P_{\mathcal{A}}^{<\xi}(t), A(F, t) \wedge \neg F(t), F(t).$$

An application of ($\lor$) allows us to conclude

$$f[n], E[K_{\Omega}\xi] \vdash_{0}^{(\sigma+\alpha_{\xi}+1)^2} \Gamma, \neg \mathcal{A}(F), \neg P_{\mathcal{A}}^{<\xi}(t), A(F, t).$$

The other cases can be treated in similar ways. \[\square\]

Lemma 5.5 1. $f[n], E \vdash_{0}^{\Omega^2+\omega} \forall x(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, x) \rightarrow P_{\mathcal{A}}^{<\Omega}(x))$, where $n := \max\{N(\text{rk}(A(P_{\mathcal{A}}^{<\Omega}, \Omega))), \text{lh}(A(P_{\mathcal{A}}^{<\Omega}, \Omega))\}$
2. \( f[3 + l], E \vdash_{0}^{\Omega + 2 + \omega} \Gamma, \forall \vec{y} \exists x \{ A(F(\cdot, \vec{y}), x) \rightarrow F(x, \vec{y}) \} \rightarrow \forall x \{ P_{A_{\Delta}}^\Omega(x) \rightarrow F(x, \vec{y}) \} \), where \( \vec{y} = y_0, \ldots, y_{l-1} \).

**Proof.** 1. Let \( \alpha = \text{rk}(A(P_{\Delta}^{< \Omega}, 0)) \) and \( t \in T(L_{\mathcal{D}_1}) \). By the definition of \( \text{rk} \) we can find a natural \( k \leq \text{lh}(A(P_{\Delta}^{< \Omega}, 0)) \) such that \( \alpha = \text{rk}(A(P_{\Delta}^{< \Omega}, t)) = \Omega + k \). This implies \( k(A(P_{\Delta}^{< \Omega}, t)) = \{ 0, \Omega \} \) and hence \( k(0)(A(P_{\Delta}^{< \Omega}, t)) = \{ 0 \} < E(0) \). By Tautology lemma (Lemma 5.1),

\[
f[n], E \vdash_{0}^{\Omega + 2 + k} \Gamma, P_{A_{\Delta}}^{< \Omega}(t), \neg A(P_{\Delta}^{< \Omega}, t), A(P_{\Delta}^{< \Omega}, t).
\]

Since \( \Omega < \Omega \cdot 2 + k + 1 \), we can apply the closure rule (C1) obtaining the sequent

\[
f[n], E \vdash_{0}^{\Omega + 2 + k + 1} \Gamma, \neg A(P_{\Delta}^{< \Omega}, t), P_{\Delta}^{< \Omega}(t).
\]

An application of (\( \land \)) followed by an application of (\( \lor \)) enables us to conclude

\[
f[n], E \vdash_{0}^{\Omega + 2 + \omega} \Gamma, \forall x \{ A(P_{\Delta}^{< \Omega}, x) \rightarrow P_{A_{\Delta}}^{< \Omega} x \}.
\]

2. By definition \( \text{rk}(P_{\Delta}^{< \Omega}) = \omega \cdot \Omega = \Omega \). On the other hand \( \text{rk}(F) < \omega \) and hence \( (\text{rk}(F) + \text{rk}(P_{\Delta}^{< \Omega}) + 1) \cdot 2 = \Omega \cdot 2 + 2 \). Let \( s, \vec{t} = s, t_0, \ldots, t_{l-1} \in T(L_{\mathcal{D}_1}) \). Then by the previous lemma (Lemma 5.4)

\[
f[2], E \vdash_{0}^{\Omega + 2 + 1} \neg \forall x \{ A(F(\cdot, \vec{t}), x) \rightarrow F(x, \vec{t}) \}, \neg P_{A_{\Delta}}^{< \Omega}(t), F(s, \vec{t})
\]

since \( N(\Omega + 1) = 2 \). It is not difficult to see that applications of (\( \lor \)), (\( \land \)) and (\( \land \)) in this order yield the sequent

\[
f[3], E \vdash_{0}^{\Omega + 2 + \omega} \forall x \{ A(F(\cdot, \vec{t}), x) \rightarrow F(x, \vec{t}) \} \rightarrow \forall x \{ P_{A_{\Delta}}^{< \Omega}(x) \rightarrow F(x, \vec{t}) \}
\]

Finally, \( l \)-fold application of (\( \land \)) allows us to conclude.

Let us recall that \( s \) denotes the numerical successor \( m \mapsto m + 1 \).

**Theorem 5.6** Let \( A \equiv \forall \vec{x} \exists y B(\vec{x}, y) \) be a \( \Delta_0^2 \)-sentence for a \( \Delta_0^2 \)-formula \( B(\vec{x}, y) \) such that \( \text{FV}(B(\vec{x}, y)) = \{ \vec{x}, y \} \). If \( \text{ID}_1 \vdash A \), then we can find an ordinal term \( \alpha \in \mathcal{O}\mathcal{T}(\mathcal{F}) \uparrow \Omega \) built up without the Veblen function symbol \( \varphi \) such that for all \( \vec{m} = m_0, \ldots, m_{l-1} \in \mathbb{N} \) there exists \( n \leq s^\alpha(m_0 + \cdots + m_{l-1}) \) such that \( B(\vec{m}, n) \) is true in the standard model \( \mathbb{N} \) of PA.

**Proof.** Assume \( \text{ID}_1 \vdash A \). Then there exist \( \text{ID}_1 \)-axioms \( A_0, \ldots, A_{k-1} \) such that \( (\neg A_0) \lor \cdots \lor (\neg A_{k-1}) \lor A \) is a logical consequence in the first order predicate logic with equality. Hence by Lemma 5.2,

\[
f[c_0], E \vdash_{0}^{\Omega + 3} \neg A_0, \ldots, \neg A_{k-1}, A
\]

for some constant \( c_0 < \omega \) depending on \( \text{N}(\text{rk}(A_0)), \ldots, \text{N}(\text{rk}(A_{k-1})), \text{N}(\text{rk}(A)) \) and max\{\( \text{lh}(A(\cdot, *)) \) | \( P_{A_{\Delta}}^{< k} \) or \( P_{A_{\Delta}}^{< k} \) occurs in \( A_j \) or \( A \)\}, and depending also on the tree height of a cut-free \( \text{LK} \)-derivation of the sequent \( \neg A_0, \ldots, \neg A_{k-1}, A \). By Lemma 5.3 and 5.5, for each \( j \leq k - 1 \), there exists a constant \( c_j \) depending on \( \text{rk}(A_j) \) such that \( f[c_j], E \vdash_{0}^{\Omega + 2 + \omega} A_j \). Hence \( k \)-fold application of (Cut) yields \( f[c], E \vdash_{0}^{\Omega + 3} \neg A_0, \ldots, \neg A_{k-1}, A \).
A, where $c := \max(\{k\} \cup \{c_j \mid j \leq k - 1\}) \cup \{\text{lh}(A_j) \mid j \leq k - 1\}$ and $d := \max(\{\Omega, \text{rk}(A_0), \ldots, \text{rk}(A_{k-1})\})$.

For each $n \in \mathbb{N}$ and $\alpha \in \mathcal{OT}(\mathcal{F})$ let us define ordinal $\Omega_n(\alpha)$ and $\gamma_n$ by

$$\Omega_0(\alpha) = \alpha, \quad \gamma_0 = \Omega \cdot 3,$$

$$\Omega_{n+1}(\alpha) = \Omega^{\Omega_n(\alpha)}, \quad \gamma_{n+1} = \varepsilon^{\gamma_n}(0) + 1.$$

Then $d$-fold iteration of Cut-reduction lemma (Lemma 4.13) yields the sequent $f[c]^{\gamma_d}, E \vdash_{\Omega+1}^{\Omega_n(\Omega \cdot 3)} A$. Hence Impredicative cut-elimination lemma (Lemma 4.18) yields $(f[c]^{\gamma_d})^{E^{\Omega_n(\Omega \cdot 3)}(0)}, E^{\Omega_n(\Omega \cdot 3)+1} \vdash^{E^{\Omega_n(\Omega \cdot 3)}(0)} A$.

Let $F := E^{\Omega_n(\Omega \cdot 3)+1}$ and $\beta := E^{\Omega_n(\Omega \cdot 3)}(0)$. Then $(f[c]^{\gamma_d})^\beta, F \vdash^{\phi \beta \beta} A$ holds. It is not difficult to check that $\beta < \Omega$, $N(\beta) \leq (f[c]^{\gamma_d})^\beta$, and $\text{K}_{\text{OE}}(\beta) < F(0)$. Hence Predicative cut-elimination lemma (Lemma 4.15) yields the sequent $(f[c]^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1} F \vdash_{0}^{\varphi \beta \beta} A$.

Now let $f$ denote $s^\omega$. One can check that the conditions $(s^\omega.1)$ and $(s^\omega.2)$ hold. One will also see that $s^\omega[c](m) \leq s^\omega(s^c(m)) \leq s^{\omega+c+1}(m)$ for all $m$. By these we have the inequality

$$(s[c]^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1}(0) \leq ((s^{\omega+c+1})^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1}(0).$$

Thanks to Lemma 4.8 we can find an ordinal $\alpha \in \mathcal{OT}(\mathcal{F}) \setminus \Omega$ built up without the Veblen function symbol $\varphi$ such that

$$((s^{\omega+c+1})^{\gamma_d})^{F^{\Omega \cdot \beta + \beta \cdot 2}(0)+1}(0) \leq s^\alpha(0).$$

This together with ($l$-fold application of) Inversion lemma (Lemma 4.12) yields the sequent $s^\alpha[m_0] \cdots [m_{l-1}], F \vdash^{\phi \beta \beta} \exists y B(\vec{m}, y)$,

where $\vec{m} = m_0, \ldots, m_{l-1}$. By Witnessing lemma (Lemma 4.19) we can find a natural $n \leq s^\alpha[m_0] \cdots [m_{l-1}](0) = s^\alpha(m_0 + \cdots + m_{l-1})$ such that $B(\vec{m}, n)$ is true in the standard model $\mathbb{N}$ of PA.

We say a function $f$ is elementary (in another function $g$) if $f$ is definable explicitly from the successor $s$, projection, zero $0$, addition $+$, multiplication $\cdot$, cut-off subtraction $\vdash$ (and $g$), using composition, bounded sums and bounded products.

**Corollary 5.7** Every function provably computable in $\text{ID}_1$ is elementary in $\{s^\alpha \mid \alpha \in \mathcal{OT}(\mathcal{F}) \setminus \Omega\}$. 

\[ \square \]
6 A computable ordinal notation system $\mathcal{O}(\Omega)$

In order to obtain a precise characterisation of the provably computable functions of ID$_1$, we introduce a computable ordinal notation system $(\mathcal{O}(\Omega), <)$. Essentially $\mathcal{O}(\Omega)$ is a subsystem of $\mathcal{OT}(\mathcal{F})$.

**Definition 6.1** We define three sets $\mathcal{SC} \subseteq \mathbb{H} \subseteq \mathcal{O}(\Omega)$ of ordinal terms simultaneously. Let $0$, $\Omega$, $S$, and $+$ be distinct symbols.

1. $0 \in \mathcal{O}(\Omega)$ and $\Omega \in \mathcal{SC}$.
2. If $\alpha \in \mathcal{OT}(\mathcal{F}) \uparrow \Omega$, then $S(\alpha) \in \mathcal{O}(\Omega)$.
3. If $\{\alpha_1, \ldots, \alpha_l\} \subseteq \mathbb{H}$ and $\alpha_1 \geq \cdots \geq \alpha_l$, then $\alpha_1 + \cdots + \alpha_l \in \mathcal{O}(\Omega)$.
4. If $\alpha \in \mathcal{O}(\Omega)$, then $\omega^\alpha \in \mathbb{H}$.
5. If $\alpha \in \mathcal{O}(\Omega)$ and $\xi \in \mathcal{O}(\Omega) \uparrow \Omega$, then $\Omega^\alpha \cdot \xi \in \mathbb{H}$.
6. If $\alpha \in \mathcal{O}(\Omega)$ and $\xi \in \mathcal{O}(\Omega) \uparrow \Omega$, then $S^\alpha(\xi) \in \mathcal{SC}$.

The relation $<$ on $\mathcal{O}(\Omega)$ is defined in the obvious way. One will see that $\mathcal{O}(\Omega)$ is indeed a computable ordinal notation system. Let us define the norm $N(\omega^\alpha)$ of $\omega^\alpha$ in the most natural way, i.e., $N(\omega^\alpha) = N(\alpha) + 1$.

**Lemma 6.2** Let $\alpha$ denote an ordinal term built up in $\mathcal{OT}(\mathcal{F})$ without the Veblen function symbol $\varphi$. Then there exists an ordinal term $\alpha' \in \mathcal{O}(\Omega)$ such that $\alpha \leq \alpha'$ and $N(\omega^\alpha) \leq N(\omega^{\alpha'})$.

**Proof.** By induction over the term construction of $\alpha \in \mathcal{OT}(\mathcal{F})$. In the base case let us observe that $E(\alpha) \leq S^1(\alpha)$ for all $\alpha < \Omega$ and that $N(E(\alpha)) = N(\alpha) + 1 < N(S(\alpha)) + 1 = N(S^1(\alpha))$. In the induction case we employ Lemma 3.11. \qed

**Lemma 6.3** For any ordinal term $\alpha \in \mathcal{OT}(\mathcal{F})$ built up without the Veblen function symbol $\varphi$ there exists an ordinal term $\alpha' \in \mathcal{O}(\Omega)$ such that $s^\alpha(m) \leq s^{\alpha'}(m)$ for all $m$.

**Corollary 6.4** A function is provably computable in ID$_1$ if and only if it is elementary in $\{s^\alpha \mid \alpha \in \mathcal{O}(\Omega) \uparrow \Omega\}$.

The "only if" direction follows from Corollary 5.7 and Lemma 6.3. The "if" direction can be seen as follows. One can show that for each $\alpha \in \mathcal{O}(\Omega) \uparrow \Omega$ the system ID$_1$ proves that the initial segment $(\mathcal{O}(\Omega) \uparrow \alpha, <)$ of $(\mathcal{O}(\Omega), <)$ is a well-ordering. For the full proof, we kindly refer the readers to, e.g., Pohlers [11, §29]. From this one can show that for each $\alpha \in \mathcal{O}(\Omega) \uparrow \Omega$ the function $s^\alpha$ is provably computable in ID$_1$, and hence the assertion.
7 A quick proof-theoretic analysis of ID₁

In the final section we show that the collapsing function \( F : \Omega_1 \times \varepsilon_{\Omega_1} \rightarrow \Omega_1 \); \((\xi, \alpha) \mapsto F^\alpha(\xi)\) can be used for a smooth proof-theoretic analysis of ID₁. Suppose a positive operator form \( A \). Let \( \Phi_A : \mathcal{P}(N) \rightarrow \mathcal{P}(N) \) denote the operator induced by the operator form \( A \). Namely \( \Phi_A(X) = \{ n \in N \mid N \models A(X, n) \} \) if \( X \subseteq N \). By positiveness of \( A \) the operator \( \Phi_A \) is monotone, i.e., \( X \subseteq Y \Rightarrow \Phi_A(X) \subseteq \Phi_A(Y) \), and hence \( \Phi_A \) has the least fixed point \( I_{\Phi_A} \) that corresponds to the predicate \( P_A \). Further, for an ordinal \( \alpha \), let \( I_{\Phi_A}^\alpha \) denote the \( \alpha \)-th stage of iterating \( \Phi_A \). More precisely, corresponding to the predicate \( P_A^\alpha \), \( I_{\Phi_A}^\beta \) is defined by \( I_{\Phi_A}^0 = \emptyset \) and \( I_{\Phi_A}^\alpha = \Phi_A(I_{\Phi_A}^{\alpha-1}) \). Recall that \( \Omega_1 \) denotes the least non-computable ordinal \( \omega_1^C \). From an elementary fact in generalised recursion theory, it is known that \( I_{\Phi_A}^\alpha \) is consumed at \( \alpha = \Omega_1 \), i.e., \( I_{\Phi_A}^{\Omega_1} = I_{\Phi_A} \). The norm \( |n|_{\Phi_A} \) of a natural number \( n \) is defined by \( |n|_{\Phi_A} = \min\{ \alpha \in \text{Ord} \mid n \in I_{\Phi_A}^\alpha \} \). It is natural to ask what can be said about the norm \( |n|_{\Phi_A} \) in case that \( ID_1 \vdash P_A(\bar{n}) \). An elegant proof-theoretic way to answer this question can be found in lecture notes [4] by W. Buchholz. (See [4, Theorem 9.19].) By slightly modifying the exposition in [4] we present an alternative simplified way to answer this question.

In contrast to the infinitary system \( ID_1^\infty \) we investigate the associated semiformal system \( ID_1^1 \) which is modelled following the lecture notes [4]. As until the previous section we will identify each element \( \alpha \in \mathcal{OT}(F) \) with its value \( [\alpha] \in \text{Ord} \), e.g., \( \Omega \in \mathcal{OT}(F) \) with \( \Omega_1 \in \text{Ord} \). We also follow a convention that \( F : \Omega \rightarrow \Omega \) denotes a weakly increasing function such that \( \xi < F(\xi) \) for all \( \xi < \Omega \). Further in this section we use an additional convention that \( \omega^F(\xi) = F(\xi) \), and hence \( E(\xi) \leq F(\xi) \) for all \( \xi < \Omega \). (Recall \( E(\alpha) = \min\{ \xi \in \text{Ord} \mid \omega^\xi = \xi \} \).) Let us recall that for a sequent \( \Gamma \), \( k^\Omega_\Gamma(\Gamma) \) denotes the set \( \bigcup_{B \in \Gamma} k^\Omega_\Gamma(B) \).

**Definition 7.1** \( F \vdash_\rho^\alpha \Gamma \) if \( k^\Omega_\Gamma(\Gamma) \subseteq \Omega \alpha < F(0) \) and one of the following holds.

(Ax1) \( \exists A(x) \) : an \( L_{ID_1} \)-literal, \( \exists s, t \in T(L_{ID_1}) \) s.t. \( FV(A) = \{ x \} \), \( \text{val}(s) = \text{val}(t) \) and \( \{ \neg A(s), A(t) \} \subseteq \Gamma \).

(Ax2) \( \Gamma \cap \text{TRUE}_0 \neq \emptyset \).

(V) \( \exists A \simeq \bigvee_{\iota \in J} A_\iota \in \Gamma \), \( \exists \alpha_0 < \alpha \), \( \exists \iota_0 \in J \) s.t. \( \text{ord}(\iota_0) < F(0) \), and \( F \vdash_\rho^{\alpha_0} \Gamma, A_{\iota_0} \).

(\&) \( \exists A \simeq \bigwedge_{\iota \in J} A_\iota \in \Gamma \) s.t. \( (\forall \iota \in J) (\exists \alpha_\iota < \alpha) F[\text{ord}(\iota)] \vdash_\rho^{\alpha_\iota} \Gamma, A_\iota \).

(Cl) \( \exists t \in T(L_{ID_1}) \), \( \exists \alpha_0 < \alpha \) s.t. \( P_\alpha^\Omega(t) \in \Gamma \) and \( F \vdash_\rho^{\alpha_0} \Gamma, A(\Pi_\alpha^\Omega, t) \).

(Cut) \( \exists C \) : an \( L^* \)-sentence of \( \forall \)-type, \( \exists \alpha_0 < \alpha \) s.t. \( \text{rk}(C) < \rho \), \( F \vdash_\rho^{\alpha_0} \Gamma, C \), and \( F \vdash_\rho^{\alpha_0} \Gamma, \neg C \).

**Lemma 7.2** (Inversion) Assume that \( A \simeq \bigwedge_{\iota \in J} A_\iota \). If \( F \vdash_\rho^{\alpha_0} \Gamma, A \), then \( F[\text{ord}(\iota)] \vdash_\rho^{\alpha_0} \Gamma, A_\iota \) for all \( \iota \in J \).

**Proof.** By induction on \( \alpha \). □
Lemma 7.3 (Cut-reduction) Assume that $C \simeq \bigvee_{i \in J} C_i$ and $\text{rk}(C) = \Omega + k + 1$. If $F \vdash_{\Omega+k+1} \Gamma, \neg C$ and $F \vdash_{\Omega+k+1} \Gamma, C$, then $F \vdash_{\Omega+k+1} \Gamma$.

Proof. By induction on $\beta$. $\Box$

Lemma 7.4 (Cut-elimination) Let $k < \omega$. If $F \vdash_{\Omega+k+2} \Gamma$, then $F \vdash_{\Omega+k+1} \Gamma$.

Lemma 7.5 $F[\xi]^\alpha(\xi) \leq F^\alpha(\xi)$.

Proof. By induction on $\alpha$. $\Box$

Lemma 7.6 If $\eta < \xi$ and $\alpha_\eta < \alpha$ and $K \alpha_\eta < F[\eta](0)$ then $F[\eta]^\alpha_\eta(\xi) \leq F^\alpha(\xi)$.

Lemma 7.7 If $\eta < F(0)$ and $\alpha_\eta < \alpha$ and $K \alpha_\eta < F[\eta](0)$ then $F[\eta]^\alpha_\eta(\xi) \leq F^\alpha(\xi)$.

Definition 7.8 For each $\mathcal{L}^*$-formula $B$ let $B^{\alpha,\beta}$ denote the result of replacing in $B$ every negative occurrence of $P^\alpha_A$ by $P^\alpha_A$ and every positive occurrence of $P^{\alpha\Omega}_A$ by $P^{\alpha\beta}_A$. For each sequent $\Gamma$ consisting of $\mathcal{L}^*$-formulas let $\Gamma^{\alpha,\beta} := \{B^{\alpha,\beta} \mid B \in \Gamma\}$. It is known that, viewing ID$_1$ as a subsystem of set theory in a standard way, $L_\Omega \models \text{ID}_1$ holds for the $\Omega$th stage $L_\Omega$ of the constructible hierarchy $(L_\alpha)_{\alpha \in \text{Ord}}$. We will just write $\models B$ (B is an $\mathcal{L}^*$ sentence) or $\models \Gamma$ ($\Gamma$ is an $\mathcal{L}^*$ sequent) to refer to this relation if no confusion arises.

Theorem 7.9 (Witnessing) If $F \vdash_{\Omega+1} \Gamma$, then $\models \Gamma^{F,\xi}(\xi)$ for all $\xi < \Omega$.

Proof. By induction on $\xi$. $\Box$

In embedding ID$_1$ into ID$^*_1$, we follow (very closely) the exposition in the lecture notes [4] and indicate how the operators can be adapted accordingly. As in case of embedding ID$_1$ into ID$^\alpha_1$, the condition HYP(E) on page 10 holds.

Lemma 7.10 (Tautology lemma) Let $s, t \in T(\mathcal{L}_{\text{ID}_1})$, $\Gamma$ a sequent of $\mathcal{L}^*$-sentences, and $A(x)$ be an $\mathcal{L}^*$-formula such that $\text{FV}(A) = \{x\}$. If $\text{val}(s) = \text{val}(t)$, then $F \vdash_{0}^{\text{rk}(A)^2} \Gamma, \neg A(s), A(t)$, provided $k^{\Pi}_{\Omega}(\Gamma) \cup k^{\Pi}_{\Omega}(A) < F(0)$.

Proof. By induction on $\text{rk}(A)$. $\Box$

Lemma 7.11 Let $B_j$ be an $\mathcal{L}_{\text{ID}_1}$-sentence for each $j = 0, \ldots, l - 1$. Suppose that $B_0 \vee \cdots \vee B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then there exists a natural $k < \omega$ such that $F \vdash_{0}^{\Omega^{2+k}} \Gamma, B_0, \ldots, B_{l-1}$, provided $k^{\Pi}_{\Omega}(\Gamma) < F(0)$.

This can be shown like Lemma 5.2.
Lemma 7.12 Let \( m \in \mathbb{N} \) and \( A(x) \) be an \( \mathcal{L}_{ID_{1}} \)-formula such that \( \text{FV}(A(x)) = \{x\} \).
Then for any \( t \in \mathcal{T}(\mathcal{L}_{ID_{1}}) \) and for any sequent \( \Gamma \) of \( \mathcal{L}_{ID_{1}} \)-sentences
\[
F \vdash_{0}^{(\text{rk}(A)+\text{val}(t)-2)} \Gamma, \neg A(0), \neg \forall x(A(x) \rightarrow A(S(x))), A(t),
\]
provided \( k_{\Omega}^\Pi(\Gamma) \cup k_{\Omega}^\Pi(A) < F(0) \).

Proof. By induction on \( \text{val}(t) \). \[\square\]

Lemma 7.13 Let \( \xi \leq \Omega \), \( A(x) \) be an \( \mathcal{L}_{ID_{1}} \)-formula such that \( \text{FV}(A(x)) = \{x\} \) and \( B(X) \) be an \( X \)-positive \( \mathcal{L}_{PA}(X) \)-formula such that \( \text{FV}(A) = \emptyset \).
Then
\[
F \vdash_{0}^{\text{rk}(A)+\alpha+1) \cdot 2} \Gamma, \neg \forall x(A(A(x), x) \rightarrow A(x)), \neg B(P_{\mathcal{A}}^{<\xi}), B(A),
\]
provided \( k_{\Omega}^\Pi(\Gamma) \cup k_{\Omega}^\Pi(A) \cup \{\text{ord}(\xi)\} < F(0) \) where \( \alpha := \text{rk}(B(P_{\mathcal{A}}^{<\xi})) \).

Proof. By induction on \( \text{rk}(B(P_{\mathcal{A}}^{<\xi})) \). \[\square\]

Lemma 7.14
1. \( F \vdash_{0}^{\Omega+\omega} \Gamma, \forall x(A(P_{\mathcal{A}}^{<\Omega}, x) \rightarrow P_{\mathcal{A}}^{<\Omega}(x)), \) provided \( k_{\Omega}^\Pi(\Gamma) < F(0) \).
2. \( F \vdash_{0}^{\Omega^2+\omega} \Gamma, \forall y[\forall x(A(B(\cdot, y), x) \rightarrow B(x, y)]) \rightarrow \forall x(P_{\mathcal{A}}^{<\Omega}(x) \rightarrow B(x, y))], \) provided \( k_{\Omega}^\Pi(\Gamma) \cup k_{\Omega}^\Pi(B) < F(0) \).

Let us recall that \( S \) denotes the ordinal successor.

Theorem 7.15 Let \( n \in \mathbb{N} \). If \( \text{ID}_{1} \vdash P_{\mathcal{A}}(n) \), then there exists an ordinal \( \alpha < \varepsilon_{\Omega+1} \) such that \( |n|_{\mathcal{A}} < S^{\alpha}(0) \).

Note that the latter bound is sharp in the sense that for each \( \alpha < S^{\varepsilon_{\Omega+1}}(0) := \sup\{S^{\varepsilon_{m}(\Omega+1)}(0) | m < \omega\} \) there exists an operator form \( \mathcal{A} \) and a natural number \( n \) such that \( \text{ID}_{1} \vdash P_{\mathcal{A}}(n) \) and \( \alpha \leq |n|_{\mathcal{A}} \).

8 Conclusion

In [13] the second author has started a new approach to provably total computable functions, providing a streamlined characterisation of those functions provably computable in PA. In this work we extend this approach to those functions provably computable in the system ID\(_{1}\) of non-iterated inductive definitions. The approach introduced in this work should be extended to stronger impredicative systems. The obvious next step is to extension to the system ID\(_{2}\) of an iterated inductive definitions. This extension seems to be made possible by employing an additional ordinal operator, i.e., \( f, F_{0}, F_{1} \vdash_{0}^{\rho} \Gamma \) where \( F_{0} \) is an ordinal function \( F_{0} : \Omega_{1} \rightarrow \Omega_{1} \), \( F_{1} \) is another ordinal function \( F_{1} : \Omega_{2} \rightarrow \Omega_{2} \), and \( \Omega_{2} \) denotes the least recursively regular ordinal above \( \Omega_{1} \).


References


