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Remarks on weak covering properties

酒井　政美（Masami Sakai）
神奈川大学（Kanagawa University）

All spaces considered here are regular. We recall some definitions. A space X is said to be *Menger* [8] (resp., *weakly Menger* [3]) if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$ such that $\bigcup\{\bigcup \mathcal{V}_n : n \in \omega\} = X$ (resp., $\bigcup\{\bigcup \mathcal{V}_n : n \in \omega\}$ is dense in X). A space X is said to be *weakly Lindelöf* if every open cover $\mathcal{U}$ has a countable subfamily $\mathcal{V}$ such that $\bigcup \mathcal{V}$ is dense in X.

The implications of these notions are as follows.

$$\begin{array}{ccc} \text{weakly Menger} & \rightarrow & \text{weakly Lindelöf} \\ \uparrow & & \uparrow \\ \text{Menger} & \rightarrow & \text{Lindelöf} \end{array}$$

Babinkostova, Pansera and Scheepers posed the following question.

Question ([2, Question 32]) Is there a Lindelöf space which is not weakly Menger?

In this note, we show that this question is affirmative.

A space is said to be $K_{\sigma\delta}$ if it is the intersection of countably many σ-compact spaces. A $K_{\sigma\delta}$-space is Lindelöf. A space X is said to satisfy *the countable chain condition* (shortly, *CCC*) if each pairwise disjoint family of nonempty open subsets of X is countable. The weight (resp., π-weight) of a space X is denoted by $w(X)$ (resp., $\pi w(X)$). The continuum hypothesis is denoted by CH, and $\mathfrak{c}$ is the continuum.

Lemma 0.1 ([1, Theorem 5′]) *Under CH, if a space X is a CCC Baire space with $\pi w(X) \leq \mathfrak{c}$, then X contains a dense hereditarily Lindelöf subspace.*

Theorem 0.2 *(1) There is a $K_{\sigma\delta}$ CCC Čech-complete space which is not weakly Menger,*

(2) under CH, there is a hereditarily Lindelöf space which is not weakly Menger.

Proof. (1). Let X be a Tychonoff CCC space with $w(X) = \mathfrak{c}$ which is not weakly Menger. For example, let $X = \mathcal{F}[\mathbb{P}]$ be the Pixley-Roy hyperspace over the space $\mathbb{P}$ of irrationals. Indeed, Tychonoff CCC, $w(X) = \mathfrak{c}$ and not being weakly Menger follow from [4, Theorem 3.3.(b)], [6, Theorem 2.5.(b)] and [3, Theorem 2A] respectively. Fix a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X such that for any finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \mathbb{N})$, $\bigcup\{\bigcup \mathcal{V}_n : n \in \mathbb{N}\}$ is not dense in X. Let cX be a compactification of X with $w(cX) = \mathfrak{c}$. For each $n \in \mathbb{N}$, we take an open family $\mathcal{G}_n = \{U' : U \in \mathcal{U}_n\}$ in cX satisfying $U' \cap X = U$ for all $U \in \mathcal{U}_n$. Let $G_n = \bigcup \mathcal{G}_n$, and $G = \bigcap\{G_n : n \in \mathbb{N}\}$. For simplicity, we may assume $G_{n+1} \subset G_n$. Obviously G satisfies CCC and $w(G) = \mathfrak{c}$. Moreover, considering the open covers $\mathcal{G}_n|G = \{U' \cap G : U \in \mathcal{U}_n\}$ in G, we can easily see that G is not weakly Menger.

For each finite sequence $s \in \mathbb{N}^{<\omega}$, we can inductively define a nonempty open set W_s in cX satisfying the following conditions: .

(i) for each $n \in \mathbb{N}$, $\{\overline{W}_s : s \in \mathbb{N}^n\}$ is pairwise disjoint, and $\bigcup\{\overline{W}_s : s \in \mathbb{N}^n\}$ is a dense subspace of G_n,

(ii) for each $s \in \mathbb{N}^{<\omega}$ and $n \in \mathbb{N}$, $\overline{W}_{s^\frown n} \subset W_s$.

Finally let $Y = \bigcap\{\bigcup_{s \in \mathbb{N}^n} \overline{W}_s : n \in \mathbb{N}\}$. Obviously Y is a $K_{\sigma\delta}$-space with $w(Y) \leq \mathfrak{c}$. Since Y is dense and G_δ in cX, it is a CCC Čech-complete space. Moreover, since G is not weakly Menger, Y is not weakly Menger.

(2). Apply Lemma 0.1 to the space Y in (1). Then we have a dense hereditarily Lindelöf space Z in Y. Since Y is not weakly Menger, Z is not weakly Menger. □

A paracompact Čech-complete space is metrizable if it has a G_δ-diagonal [5, 5.1.I]. A Čech-complete CCC space with a point-countable base is second countable [7, Theorem 1.5′]. Therefore the space Y in Theorem 0.2 (1) has neither a G_δ-diagonal nor a point-countable base.

For a space X and a subspace $A \subset X$, we denote by X_A the space obtained by isolating all points of $X \setminus A$. If X is regular, so is X_A.

Theorem 0.3 *There is a Lindelöf space with both a G_δ-diagonal and a point-countable base which is not weakly Menger.*

Proof. Let $\mathbb{C}$ be the Cantor set. Let $\{B_0, B_1\}$ be a Bernstein partition of $\mathbb{C}$ [5, 5.5.4.(a)], in other words $\mathbb{C} = B_0 \cup B_1, B_0 \cap B_1 = \emptyset$ and for every uncountable compact set $K \subset \mathbb{C}$, $K \cap B_i \neq \emptyset$ $(i = 0, 1)$. Note that both B_0 and B_1 are uncountable and dense in $\mathbb{C}$, and for every open set $U \subset \mathbb{C}$ containing B_i $(i = 0, 1)$, $\mathbb{C} \setminus U$ is countable. Let D be a countable dense subset in $\mathbb{C}$ which is contained in B_0. Since the set $\mathbb{C} \setminus D$ is a dense and co-dense G_δ-subset of $\mathbb{C}$, it is homeomorphic to the irrationals $\mathbb{P}$ [5, 6.2.A.(a)]. It is well known that $\mathbb{P}$ is not Menger, so there is a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of $\mathbb{C} \setminus D$ such that for any finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$, $\bigcup\{\mathcal{V}_n : n \in \omega\}$ is not a cover of $\mathbb{C} \setminus D$.

For this sequence $\{\mathcal{U}_n : n \in \omega\}$, we observe that for any finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$, $\bigcup\{\mathcal{V}_n : n \in \omega\}$ does not cover $B_0 \setminus D$. Assume that there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ $(n \in \omega)$ such that $\bigcup\{\mathcal{V}_n : n \in \omega\}$ covers $B_0 \setminus D$. For each $V \in \bigcup\{\mathcal{V}_n : n \in \omega\}$, take an open set V' in $\mathbb{C}$ with $V' \cap (\mathbb{C} \setminus D) = V$. Let $K = \mathbb{C} \setminus \bigcup\{V' : V \in \bigcup_{n \in \omega} \mathcal{V}_n\}$. Then $K \subset D \cup B_1$. If K is uncountable, then $K \setminus D$ is an uncountable complete separable metric space. Therefore, by the Cantor-Bendixson theorem and [5, 4.5.5], $K \setminus D$ contains a subset which is homeomorphic to $\mathbb{C}$. This is a contradiction, because of $K \setminus D \subset B_1$. Thus K is countable. Let $K \cap B_1 = \{b_n : n \in \omega\}$, and take some $U_n \in \mathcal{U}_n$ with $b_n \in U_n$. Then $\mathcal{V}'_n = \mathcal{V}_n \cup \{U_n\}$ is a finite subfamily of $\mathcal{U}_n$ and $\bigcup\{\mathcal{V}'_n : n \in \omega\}$ covers $\mathbb{C} \setminus D$. This is a contradiction.

Let $X = \mathbb{C}_{B_1}$. Obviously X is a Lindelöf space with a G_δ-diagonal and a point-countable base. We see that the space X is not weakly Menger. For each $n \in \omega$, let $\mathcal{W}_n = \{U \cup D : U \in \mathcal{U}_n\}$. Then $\mathcal{W}_n$ is an open cover of X. By the observation in the preceding paragraph, for any finite subfamilies $\mathcal{W}'_n \subset \mathcal{W}_n$ $(n \in \omega)$, there is a point $r \in (B_0 \setminus D) \setminus \bigcup\{\bigcup \mathcal{W}'_n : n \in \omega\}$. Since the point r is isolated in X, $\bigcup\{\bigcup \mathcal{W}'_n : n \in \omega\}$ is not dense in X. □

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