<table>
<thead>
<tr>
<th>Title</th>
<th>Remarks on weak covering properties (General and Geometric Topology today and their problems)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Sakai, Masami</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2013), 1833: 70-72</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/194867">http://hdl.handle.net/2433/194867</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Remarks on weak covering properties

酒井 政美 (Masami Sakai)
神奈川大学 (Kanagawa University)

All spaces considered here are regular. We recall some definitions. A space $X$ is said to be Menger [8] (resp., weakly Menger [3]) if for every sequence $\{U_n : n \in \omega\}$ of open covers of $X$, there are finite subfamilies $V_n \subset U_n (n \in \omega)$ such that $\bigcup\{ \bigcup V_n : n \in \omega\} = X$ (resp., $\bigcup\{ \bigcup V_n : n \in \omega\}$ is dense in $X$). A space $X$ is said to be weakly Lindelöf if every open cover $U$ has a countable subfamily $V$ such that $\bigcup V$ is dense in $X$.

The implications of these notions are as follows.

weakly Menger $\rightarrow$ weakly Lindelöf

Menger $\rightarrow$ Lindelöf

Babinkostova, Pansera and Scheepers posed the following question.

**Question** ([2, Question 32]) Is there a Lindelöf space which is not weakly Menger?

In this note, we show that this question is affirmative.

A space is said to be $K_{\sigma\delta}$ if it is the intersection of countably many $\sigma$-compact spaces. A $K_{\sigma\delta}$-space is Lindelöf. A space $X$ is said to satisfy the countable chain condition (shortly, CCC) if each pairwise disjoint family of nonempty open subsets of $X$ is countable. The weight (resp., $\pi$-weight) of a space $X$ is denoted by $w(X)$ (resp., $\pi w(X)$). The continuum hypothesis is denoted by $\text{CH}$, and $\mathfrak{c}$ is the continuum.

**Lemma 0.1** ([1, Theorem 5']) Under CH, if a space $X$ is a CCC Baire space with $\pi w(X) \leq \mathfrak{c}$, then $X$ contains a dense hereditarily Lindelöf subspace.

**Theorem 0.2** (1) There is a $K_{\sigma\delta}$ CCC Čech-complete space which is not weakly Menger,

(2) under CH, there is a hereditarily Lindelöf space which is not weakly Menger.

Proof. (1). Let $X$ be a Tychonoff CCC space with $w(X) = \mathfrak{c}$ which is not weakly Menger. For example, let $X = F[\mathbb{P}]$ be the Pixley-Roy hyperspace over the space $\mathbb{P}$ of irrationals. Indeed, Tychonoff CCC, $w(X) = \mathfrak{c}$ and not being weakly Menger follow from [4, Theorem 3.3.(b)], [6, Theorem 2.5.(b)] and [3, Theorem 2A] respectively. Fix a sequence $\{U_n : n \in \mathbb{N}\}$ of open covers of $X$ such that for any finite subfamilies $V_n \subset U_n (n \in \mathbb{N})$, $\bigcup\{ \bigcup V_n : n \in \mathbb{N}\}$ is not dense in $X$. Let $cX$ be a compactification of $X$ with $w(cX) = \mathfrak{c}$. For
each $n \in \mathbb{N}$, we take an open family $\mathcal{G}_n = \{U' : U \in \mathcal{U}_n\}$ in $cX$ satisfying $U' \cap X = U$ for all $U \in \mathcal{U}_n$. Let $G_n = \bigcup \mathcal{G}_n$, and $G = \bigcap\{G_n : n \in \omega\}$. For simplicity, we may assume $G_{n+1} \subset G_n$. Obviously $G$ satisfies CCC and $w(G) = c$. Moreover, considering the open covers $G_n|G = \{U' \cap G : U \in \mathcal{U}_n\}$ in $G$, we can easily see that $G$ is not weakly Menger.

For each finite sequence $s \in \mathbb{N}^\omega$, we can inductively define a nonempty open set $W_s$ in $cX$ satisfying the following conditions: 

(i) for each $n \in \mathbb{N}$, $\{\overline{W}_s : s \in \mathbb{N}^n\}$ is pairwise disjoint, and $\bigcup\{\overline{W}_s : s \in \mathbb{N}^n\}$ is a dense subspace of $G_n$,

(ii) for each $s \in \mathbb{N}^\omega$ and $n \in \mathbb{N}$, $\overline{W}_{s^n} \subset W_s$.

Finally let $Y = \bigcap\{\bigcup_{s \in \mathbb{N}^n} \overline{W}_s : n \in \mathbb{N}\}$. Obviously $Y$ is a $K_{\sigma\delta}$-space with $w(Y) \leq c$. Since $Y$ is dense and $G_\delta$ in $cX$, it is a CCC Čech-complete space. Moreover, since $G$ is not weakly Menger, $Y$ is not weakly Menger.

(2). Apply Lemma 0.1 to the space $Y$ in (1). Then we have a dense hereditarily Lindelöf space $Z$ in $Y$. Since $Y$ is not weakly Menger, $Z$ is not weakly Menger. □

A paracompact Čech-complete space is metrizable if it has a $G_\delta$-diagonal [5, 5.1.1]. A Čech-complete CCC space with a point-countable base is second countable [7, Theorem 1.5']. Therefore the space $Y$ in Theorem 0.2 (1) has neither a $G_\delta$-diagonal nor a point-countable base.

For a space $X$ and a subspace $A \subset X$, we denote by $X_A$ the space obtained by isolating all points of $X \setminus A$. If $X$ is regular, so is $X_A$.

**Theorem 0.3** There is a Lindelöf space with both a $G_\delta$-diagonal and a point-countable base which is not weakly Menger.

Proof. Let $\mathbb{C}$ be the Cantor set. Let $\{B_0, B_1\}$ be a Bernstein partition of $\mathbb{C}$ [5, 5.5.4.(a)], in other words $\mathbb{C} = B_0 \cup B_1, B_0 \cap B_1 = \emptyset$ and for every uncountable compact set $K \subset \mathbb{C}$, $K \cap B_i \neq \emptyset$ ($i = 0, 1$). Note that both $B_0$ and $B_1$ are uncountable and dense in $\mathbb{C}$, and for every open set $U \subset \mathbb{C}$ containing $B_i$ ($i = 0, 1$), $\mathbb{C} \setminus U$ is countable. Let $D$ be a countable dense subset in $\mathbb{C}$ which is contained in $B_0$. Since the set $\mathbb{C} \setminus D$ is a dense and co-dense $G_\delta$-subset of $\mathbb{C}$, it is homeomorphic to the irrationals $\mathbb{P}$ [5, 6.2.A.(a)]. It is well known that $\mathbb{P}$ is not Menger, so there is a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of $\mathbb{C} \setminus D$ such that for any finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ ($n \in \omega$), $\bigcup\{\mathcal{V}_n : n \in \omega\}$ is not a cover of $\mathbb{C} \setminus D$.

For this sequence $\{\mathcal{U}_n : n \in \omega\}$, we observe that for any finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ ($n \in \omega$), $\bigcup\{\mathcal{V}_n : n \in \omega\}$ does not cover $B_0 \setminus D$. Assume that there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ ($n \in \omega$) such that $\bigcup\{\mathcal{V}_n : n \in \omega\}$ covers $B_0 \setminus D$. For each $V \in \bigcup\{\mathcal{V}_n : n \in \omega\}$, take an open set $V'$ in $\mathbb{C}$ with $V' \cap (\mathbb{C} \setminus D) = V$. Let $K = \mathbb{C} \setminus \bigcup\{V' : V \in \bigcup_{n \in \omega} \mathcal{V}_n\}$. Then $K \subset D \cup B_1$. If $K$ is uncountable,
then $K \setminus D$ is an uncountable complete separable metric space. Therefore, by the Cantor-Bendixson theorem and [5, 4.5.5], $K \setminus D$ contains a subset which is homeomorphic to $\mathbb{C}$. This is a contradiction, because of $K \setminus D \subset B_1$. Thus $K$ is countable. Let $K \cap B_1 = \{b_n : n \in \omega\}$, and take some $U_n \in \mathcal{U}_n$ with $b_n \in U_n$. Then $\mathcal{V}_n' = \mathcal{V}_n \cup \{U_n\}$ is a finite subfamily of $\mathcal{U}_n$ and $\bigcup\{\mathcal{V}_n : n \in \omega\}$ covers $\mathbb{C} \setminus D$. This is a contradiction.

Let $X = \mathbb{C}_{B_1}$. Obviously $X$ is a Lindel"{o}f space with a $G_\delta$-diagonal and a point-countable base. We see that the space $X$ is not weakly Menger. For each $n \in \omega$, let $W_n = \{U \cup D : U \in \mathcal{U}_n\}$. Then $W_n$ is an open cover of $X$. By the observation in the preceding paragraph, for any finite subfamilies $W_n' \subset W_n \ (n \in \omega)$, there is a point $r \in (B_0 \setminus D) \setminus \bigcup\{\bigcup W_n' : n \in \omega\}$. Since the point $r$ is isolated in $X$, $\bigcup\{\bigcup W_n' : n \in \omega\}$ is not dense in $X$. □

References


