<table>
<thead>
<tr>
<th>Title</th>
<th>Remarks on weak covering properties (General and Geometric Topology today and their problems)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Sakai, Masami</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2013), 1833: 70-72</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/194867">http://hdl.handle.net/2433/194867</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Remarks on weak covering properties

酒井 政美 (Masami Sakai)
神奈川大学 (Kanagawa University)

All spaces considered here are regular. We recall some definitions. A space $X$ is said to be Menger [8] (resp., weakly Menger [3]) if for every sequence $\{U_n : n \in \omega\}$ of open covers of $X$, there are finite subfamilies $V_n \subset U_n$ ($n \in \omega$) such that $\bigcup \{ \bigcup V_n : n \in \omega\} = X$ (resp., $\bigcup \{ \bigcup V_n : n \in \omega\}$ is dense in $X$). A space $X$ is said to be weakly Lindelöf if every open cover $U$ is dense in $X$.

The implications of these notions are as follows.

\[
\text{weakly Menger} \quad \rightarrow \quad \text{weakly Lindelöf} \\
\uparrow \\
\text{Menger} \quad \rightarrow \quad \text{Lindelöf}
\]

Babinkostova, Pansera and Scheepers posed the following question.

**Question** ([2, Question 32]) Is there a Lindelöf space which is not weakly Menger?

In this note, we show that this question is affirmative.

A space is said to be $K_{\sigma\delta}$ if it is the intersection of countably many $\sigma$-compact spaces. A $K_{\sigma\delta}$-space is Lindelöf. A space $X$ is said to satisfy the *countable chain condition* (shortly, CCC) if each pairwise disjoint family of nonempty open subsets of $X$ is countable. The weight (resp., $\pi$-weight) of a space $X$ is denoted by $w(X)$ (resp., $\pi w(X)$). The continuum hypothesis is denoted by CH, and $c$ is the continuum.

**Lemma 0.1** ([1, Theorem 5']) Under CH, if a space $X$ is a CCC Baire space with $\pi w(X) \leq c$, then $X$ contains a dense hereditarily Lindelöf subspace.

**Theorem 0.2**

(1) There is a $K_{\sigma\delta}$ CCC Čech-complete space which is not weakly Menger,

(2) under CH, there is a hereditarily Lindelöf space which is not weakly Menger.

Proof. (1). Let $X$ be a Tychonoff CCC space with $w(X) = c$ which is not weakly Menger. For example, let $X = \mathcal{F}[\mathbb{P}]$ be the Pixley-Roy hyperspace over the space $\mathbb{P}$ of irrationals. Indeed, Tychonoff CCC, $w(X) = c$ and not being weakly Menger follow from [4, Theorem 3.3.(b)], [6, Theorem 2.5.(b)] and [3, Theorem 2A] respectively. Fix a sequence $\{U_n : n \in \mathbb{N}\}$ of open covers of $X$ such that for any finite subfamilies $V_n \subset U_n$ ($n \in \mathbb{N}$), $\bigcup \{ \bigcup V_n : n \in \mathbb{N}\}$ is not dense in $X$. Let $cX$ be a compactification of $X$ with $w(cX) = c$. For
each \( n \in \mathbb{N} \), we take an open family \( \mathcal{G}_n = \{ U' : U \in \mathcal{U}_n \} \) in \( cX \) satisfying \( U' \cap X = U \) for all \( U \in \mathcal{U}_n \). Let \( \mathcal{G} = \bigcup \mathcal{G}_n \), and \( G = \bigcap \{ \mathcal{G}_n : n \in \mathbb{N} \} \).

For simplicity, we may assume \( G_{n+1} \subset G_n \). Obviously \( G \) satisfies CCC and \( w(G) = c \). Moreover, considering the open covers \( \mathcal{G}_n|G = \{ U' \cap G : U \in \mathcal{U}_n \} \) in \( G \), we can easily see that \( G \) is not weakly Menger.

For each finite sequence \( s \in \mathbb{N}^{<\omega} \), we can inductively define a nonempty open set \( W_s \) in \( cX \) satisfying the following conditions:

1. for each \( n \in \mathbb{N} \), \( \{ W_{s^n} : s \in \mathbb{N}^n \} \) is pairwise disjoint, and \( \bigcup \{ W_s : s \in \mathbb{N}^n \} \) is a dense subspace of \( G_n \),

2. for each \( s \in \mathbb{N}^{<\omega} \) and \( n \in \mathbb{N} \), \( W_{s^n} \subset W_s \).

Finally let \( Y = \bigcap \{ U_{s^n} : n \in \mathbb{N} \} \). Obviously \( Y \) is a \( K_{\sigma\delta} \)-space with \( w(Y) \leq c \). Since \( Y \) is dense and \( G_{\delta} \) in \( cX \), it is a CCC \( \check{C} \)ech-complete space. Moreover, since \( G \) is not weakly Menger, \( Y \) is not weakly Menger.

(2). Apply Lemma 0.1 to the space \( Y \) in (1). Then we have a dense hereditarily Lindelöf space \( Z \) in \( Y \). Since \( Y \) is not weakly Menger, \( Z \) is not weakly Menger. \( \square \)

A paracompact \( \check{C} \)ech-complete space is metrizable if it has a \( G_{\delta} \)-diagonal [5, 5.1.1]. A \( \check{C} \)ech-complete CCC space with a point-countable base is second countable [7, Theorem 1.5']. Therefore the space \( Y \) in Theorem 0.2 (1) has neither a \( G_{\sigma\delta} \)-diagonal nor a point-countable base.

For a space \( X \) and a subspace \( A \subset X \), we denote by \( X_A \) the space obtained by isolating all points of \( X \setminus A \). If \( X \) is regular, so is \( X_A \).

**Theorem 0.3** There is a Lindelöf space with both a \( G_{\delta} \)-diagonal and a point-countable base which is not weakly Menger.

Proof. Let \( \mathbb{C} \) be the Cantor set. Let \( \{ B_0, B_1 \} \) be a Bernstein partition of \( \mathbb{C} [5, 5.5.4.(a)] \), in other words \( \mathbb{C} = B_0 \cup B_1, B_0 \cap B_1 = \emptyset \) and for every uncountable compact set \( K \subset \mathbb{C} \), \( K \cap B_i \neq \emptyset \) \((i = 0, 1) \). Note that both \( B_0 \) and \( B_1 \) are uncountable and dense in \( \mathbb{C} \), and for every open set \( U \subset \mathbb{C} \) containing \( B_i \) \((i = 0, 1) \), \( \mathbb{C} \setminus U \) is countable. Let \( D \) be a countable dense subset in \( \mathbb{C} \) which is contained in \( B_0 \). Since the set \( \mathbb{C} \setminus D \) is a dense and co-dense \( G_{\delta} \)-subset of \( \mathbb{C} \), it is homeomorphic to the irrationals \( \mathbb{P} [5, 6.2.A.(a)] \). It is well known that \( \mathbb{P} \) is not Menger, so there is a sequence \( \{ \mathcal{U}_n : n \in \omega \} \) of open covers of \( \mathbb{C} \setminus D \) such that for any finite subfamilies \( \mathcal{V}_n \subset \mathcal{U}_n \) \((n \in \omega) \), \( \bigcup \{ \mathcal{V}_n : n \in \omega \} \) is not a cover of \( \mathbb{C} \setminus D \).

For this sequence \( \{ \mathcal{U}_n : n \in \omega \} \), we observe that for any finite subfamilies \( \mathcal{V}_n \subset \mathcal{U}_n \) \((n \in \omega) \), \( \bigcup \{ \mathcal{V}_n : n \in \omega \} \) does not cover \( B_0 \setminus D \). Assume that there are finite subfamilies \( \mathcal{V}_n \subset \mathcal{U}_n \) \((n \in \omega) \) such that \( \bigcup \{ \mathcal{V}_n : n \in \omega \} \) covers \( B_0 \setminus D \). For each \( V \in \bigcup \{ \mathcal{V}_n : n \in \omega \} \), take an open set \( V' \subset \mathbb{C} \) with \( V' \cap (\mathbb{C} \setminus D) = V \). Let \( K = \mathbb{C} \setminus \bigcup \{ V' : V \in \bigcup_{n \in \omega} \mathcal{V}_n \} \). Then \( K \subset D \cup B_1 \). If \( K \) is uncountable,
then $K \setminus D$ is an uncountable complete separable metric space. Therefore, by the Cantor-Bendixson theorem and [5, 4.5.5], $K \setminus D$ contains a subset which is homeomorphic to $\mathbb{C}$. This is a contradiction, because of $K \setminus D \subset B_1$. Thus $K$ is countable. Let $K \cap B_1 = \{b_n : n \in \omega\}$, and take some $U_n \in \mathcal{U}_n$ with $b_n \in U_n$. Then $\mathcal{V}'_n = \mathcal{V}_n \cup \{U_n\}$ is a finite subfamily of $\mathcal{U}_n$ and $\bigcup \{\mathcal{V}'_n : n \in \omega\}$ covers $\mathbb{C} \setminus D$. This is a contradiction.

Let $X = \mathbb{C}_{B_1}$. Obviously $X$ is a Lindel"of space with a $G_\delta$-diagonal and a point-countable base. We see that the space $X$ is not weakly Menger. For each $n \in \omega$, let $\mathcal{W}_n = \{U \cup D : U \in \mathcal{U}_n\}$. Then $\mathcal{W}_n$ is an open cover of $X$. By the observation in the preceding paragraph, for any finite subfamilies $\mathcal{W}'_n \subset \mathcal{W}_n (n \in \omega)$, there is a point $r \in (B_0 \setminus D) \setminus \bigcup \{\bigcup \mathcal{W}'_n : n \in \omega\}$. Since the point $r$ is isolated in $X$, $\bigcup \{\bigcup \mathcal{W}'_n : n \in \omega\}$ is not dense in $X$. □

References


