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A VERY BRIEF INTRODUCTION TO VIRTUAL HAKEN CONJECTURE

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This note is a brief summary of my talk that I gave at RIMS Seminar "Representation spaces, twisted topological invariants and geometric structures of 3-manifolds" on May 30, 2012. The aim of this survey is to give an overview of works used to prove so-called virtually Haken conjecture. When I was preparing for my talk, the paper by Aschenbrenner–Friedl–Wilton [6] was very useful and I learned a lot from this.

1. THURSTON’S QUESTIONS

In 1982, Thurston asked 24 questions, saying "Here are a few questions and projects concerning 3-manifolds and Kleinian groups which I find fascinating." The first question was the famous geometrization conjecture and the questions (15)–(18) were the following [19]:

(15) Are finitely-generated Kleinian groups LERF?

A group $G$ is LERF if for every finitely generated subgroup $L < G$, and for all $g \in G \setminus L$, there exists a finite group $K$ and a homomorphism $\phi : G \to K$ such that $\phi(g) \notin \phi(L)$. See section 2 for more details.

(16) Does every hyperbolic 3-manifold have a finite-sheeted cover which is Haken?

A compact, orientable, irreducible 3-manifold $M$ is called Haken if $M$ contains an orientable, incompressible surface. $M$ is called virtually Haken if $M$ has a finite-sheeted cover that is Haken. Waldhausen asked whether every compact, orientable, irreducible 3-manifold with infinite fundamental group is virtually Haken [20]. After the proof of the geometrization conjecture, the conjecture was only open for hyperbolic 3-manifolds.

(17) Does every aspherical 3-manifold have a finite-sheeted cover with positive first Betti number?

A 3-manifold $M$ is called aspherical if all its higher homotopy groups $(\pi_i(M)$ for $i \geq 2)$ vanish. A group $G$ is said to have positive first Betti number if $\beta_1(G) = \text{rank } H_1(G; \mathbb{Q}) > 0$. A group $G$ is said to have virtually positive first Betti number if $G$ has a finite index subgroup $G' < G$ with $\beta_1(G') > 0$. A group $G$ is said to have virtually infinite first Betti number if, for any $k > 0$, there exists finite index subgroup $G' < G$ with $\beta_1(G') > k$.

A group $G$ is called large if it has a finite index subgroup $G' < G$ and an epimorphism $\phi : G \to \mathbb{Z} \ast \mathbb{Z}$. A 3-manifold $M$ is said to have corresponding properties if $\pi_1(M)$ has.

(18) Does every hyperbolic 3-manifold have a finite-sheeted cover which fibers over the circle?¹

¹After this question, Thurston wrote, “This dubious-sounding question seems to have a definite chance for a positive answer”
Let $\Sigma$ be a surface and $\phi : \Sigma \to \Sigma$ a homeomorphism. The mapping torus $T_\phi$ of $\phi$ is the manifold

$$T_\phi = \Sigma \times [0, 1]/(x, 0) \sim (\phi(x), 1).$$

A 3-manifold $M$ is said to fiber over the circle if $M$ can be obtained as a mapping torus. $M$ is called virtually fibered if $M$ has a finite-sheeted cover which fibers over the circle. There are examples of graph manifolds which are not virtually fibered [15].

Now, we have the following:

**Theorem 1.1** (Agol [3]). *All these conjectures are valid.*

2. **LOCALLY EXTENDED RESIDUALLY FINITE**

We want to know when one can lift $\pi_1$-injective immersions to embeddings in finite-sheeted covers, and LERF allows this (Scott [18]).

2.1. **Residually finite.** A group $G$ is residually finite (RF) if for every nontrivial $g \in G$, there exists a finite group $K$ and a homomorphism $\phi: G \to K$ such that $\phi(g) \neq 1$.

**Facts 2.1.** Suppose that $G$ is residually finite and finitely generated. Then following hold:

1. $G$ is Hopfian$^2$ (Mal’cev).
2. $\text{Aut}(G)$ is residually finite. (Baumslag)
3. $G$ has a solvable word problem.

**Example 2.2.** (1) Finitely-generated subgroup of $\text{GL}(n, k)$, where $k$ is a field. (Mal’cev)

(2) The fundamental group of any compact 3-manifold. (Hempel [13] and geometrization)

(3) Mapping class group of surfaces

It is known that the group $\langle a, b | b^{-1}a^2b = b^3 \rangle$ is not Hopfian, in particular, not residually finite.

**Question 2.3.** *Is every hyperbolic group residually finite?*

The expected answer seems to be NO, but....

**Theorem 2.4** (Agol–Groves–Manning [5]). *If every hyperbolic group is residually finite, then every quasi-convex subgroup of a hyperbolic group is separable.*

2.2. **LERF.** A group $G$ is LERF (locally extended residually finite) if for every finitely generated subgroup $L < G$, and for all $g \in G \setminus L$, there exists a finite group $K$ and a homomorphism $\phi : G \to K$ such that $\phi(g) \notin \phi(L)$.

**Examples 2.5.** (1) free group (Hall)

(2) surface group (Scott [18]),

(3) Bianchi groups (Agol–Long–Reid [1])

(4) Quasiconvex subgroups of word-hyperbolic Coxeter group (Haglund–Wise [12])

Not all 3-manifold groups are LERF (Burns–Karrass–Solitar [8]).

3. **CUBE COMPLEX**

Surprisingly, cube complexes play an essential rule to solve Thurston’s questions (15)–(18). Let us begin with the basic definitions.

$^2$A group $G$ is Hopfian if every homomorphic mapping of $G$ onto itself is an automorphism.
3.1. **Basic definitions.** An $n$-cube is a copy of $[-1, 1]^n$ and a 0-cube is a single point. A cube complex is a cell complex formed from cubes by identifying subcubes. The link of a 0-cube $v$ is a complex of simplices whose $n$-simplices correspond to corners of $n + 1$-cubes meeting at $v$. See Figure 1. A flag complex is a simplicial complex such that $n + 1$ vertices span an $n$-simplex if and only if they are pairwise adjacent. A cube complex $C$ is nonpositively curved if link($v$) is a flag complex for each 0-cube $v \in C^0$. A cube complex $X$ is CAT(0) if it is simply connected and nonpositively curved.

A midcube in $[-1, 1]^n$ is a subspace obtained by restricting one coordinate to 0. We then glue together midcubes in adjacent cubes whenever they meet, to get the hyperplanes of $X$. See Figure 2.

**Definition 3.1** ([11]). A cube complex is spacial if all the following hold: See Figure 3.

1. No immersed hyperplane crosses itself.
2. Each immersed hyperplane is 2-sided.
3. No immersed hyperplane self-osculates.
4. No two immersed hyperplanes inter-osculate.

**Theorem 3.2** (Haglund–Wise [11]). If $X$ is a compact special cube complex and its fundamental group $\pi_1(X)$ is word-hyperbolic, then every quasiconvex subgroup is separable.

3.2. **Salvetti complex.** Let $\Sigma$ be any graph. We build a cube complex $S_\Sigma$ as follows:

1. $S_\Sigma$ has one 0-cell;
2. $S_\Sigma$ has one (oriented) 1-cell $e_v$ for each vertex $v$ of $\Sigma$;
(3) $S_{\Sigma}$ has a square 2-cell with boundary reading $e_{u}e_{v}e_{u}e_{v}$ whenever $u$ and $v$ are joined by an edge in $\Sigma$;
(4) for $n > 2$, the $n$-skeleton is defined inductively — attach an $n$-cube to any subcomplex isomorphic to the boundary of $n$-cube which does not already bound an $n$-cube.

Let $V(\Sigma) = \{v_{1}, \ldots, v_{k}\}$ be the vertex set of the graph $\Sigma$. The right-angled Artin group (RAAG) associated to $\Sigma$ is the group given as follows:

$$A_{\Sigma} = \langle v_{1}, \ldots, v_{k} \mid [v_{i}, v_{j}] = 1 \text{ if } v_{i} \text{ and } v_{j} \text{ are connected by an edge} \rangle$$

The fundamental group of the Salvetti complex $S_{\Sigma}$ is right-angled Artin group.

The hyperplane graph of a cube complex $X$ is the graph $\Sigma(X)$ with vertex-set equal to the hyperplanes of $X$, and with two vertices joined by an edge if and only if the corresponding hyperplanes intersect.

Typing map $\phi_{X} : X \rightarrow S_{\Sigma(X)}$ is defined as follows:

(0) Each 0-cell of $X$ maps to the unique 0-cell $x_{0}$ of $S_{\Sigma(X)}$
(1) Each 1-cell $e$ of $X$ goes to the unique 1-cell in $S_{\Sigma(X)}$ which corresponds to the unique hyperplane that $e$ crosses.
(2) $\phi_{X}$ is defined inductively on higher dimensional cubes.

**Theorem 3.3** (Haglund–Wise [11]). Let $X$ be a non-positively curved cube complex. Then $X$ is special if and only if there exists a graph $\Sigma$ and there is an immersion $X \rightarrow S_{\Sigma}$ that is a local isometry at the level of the 2-skeleta.

3.3. **Compact special group.** A group is called (compact) special if it is the fundamental group of a non-positively curved (compact) special cube complex.

Let $X$ be a geodesic metric space. A subspace $Y$ is said to be quasi-convex if there exists $\kappa \geq 0$ such that any geodesic in $X$ with endpoints in $Y$ is contained within the $\kappa$-neighborhood of $Y$.

Let $\pi$ be a group with a fixed generating set $S$. A subgroup $H \subset \pi$ is said to be quasi-convex if it is a quasi-convex subspace of $\text{Cay}_{S}(\pi)$, the Cayley graph of $\pi$ with respect to the generating set $S$.

**Corollary 3.4.** (See Corollary 5.8 and Corollary 5.9 in [6].) A group is special if and only if it is a subgroup of a Right-Angled Artin Group. A group is compact special if and only if it is a quasi-convex subgroup of a Right-Angled Artin Group.

3.4. **Virtually Compact Special Theorem.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A function $f : X \rightarrow Y$ is called a quasi-isometric embedding\(^3\) if there exist constants $K \geq 1$ and $C \geq 0$ such that

$$\frac{1}{K}d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq Kd_X(x, y) + C$$

for any $x, y \in X$.

**Definition 3.5** (Quasiconvex hierarchy). The class $QH$ is defined to be the smallest class of finitely generated groups that is closed under isomorphism and satisfies the following properties.

(1) The trivial group 1 is in $QH$.

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\(^3\)Here, it is not required to preserve any algebraic structure
(2) Amalgamated product $G \cong A \ast_{C} B$ is in $\mathcal{Q}H$ if $A, B \in \mathcal{Q}H$ and $C$ is finitely generated and the inclusion map $C \hookrightarrow A \ast_{C} B$ is a quasi-isometric embedding.

(3) HNN extension $G \cong A \ast_{C} B$ is in $\mathcal{Q}H$ if $A \in \mathcal{Q}H$ and $C$ is finitely generated and the inclusion map $C \hookrightarrow A \ast_{C}$ is a quasi-isometric embedding.

**Theorem 3.6** (Virtually Compact Special Theorem for $\mathcal{Q}H$ [21]). If $G \in \mathcal{Q}H$ is word-hyperbolic, then $G$ is virtually compact special.

This theorem has an application to one-relater groups.

**Corollary 3.7** ([21]). Every one-relater group with torsion is virtually compact special.

Let $N$ be a closed, hyperbolic 3-manifold which contains an incompressible geometrically finite surface. Thurston showed that $N$ admits a hierarchy of geometrically finite surfaces. A subgroup of $\pi_{1}(N)$ is geometrically finite if and only if it is quasiconvex. Combining these results and virtually compact special theorem, we get the following:

**Theorem 3.8** (Wise). Let $N$ be a closed hyperbolic 3-manifold which contains an incompressible geometrically finite surface, then $\pi_{1}(N)$ is virtually compact special.

### 3.5. Surface subgroups

Let us recall some notions in Kleinian group theory. A Fuchsian group is a discrete subgroup of $\text{PSL}(2, \mathbb{R})$. A Kleinian group is a discrete subgroup of $\text{PSL}(2, \mathbb{C})$. A quasifuchsian group is a Kleinian group $G$ that is conjugate to a Fuchsian group by a quasiconformal automorphism of $\hat{\mathbb{C}}$.

Fix an identification of $\pi_{1}(N)$ with a discrete subgroup of $\text{PSL}(2, \mathbb{C})$. $N$ is said to contain a dense set of quasifuchsian surface groups if for each great circle $C$ of $\partial \mathbb{H}^{3} = S^{2}$ there exists a sequence of $\pi_{1}$-injective immersions $\iota : \Sigma_{i} \rightarrow N$ of surfaces $\Sigma_{i}$ such that the following hold:

1. For each $i$, the group $\iota_{*}(\pi_{1}(\Sigma_{i}))$ is a quasifuchsian surface group.
2. The sequence $\partial \Sigma_{i} \subset \partial \mathbb{H}^{3}$ converges to $C$ in the Hausdorff metric.

**Theorem 3.9** (Kahn–Markovic [14]). Every closed hyperbolic 3-manifold contains a dense set of quasifuchsian surface groups.

### 3.6. Constructing cube complex

Let $G$ be a finitely generated group with Cayley graph $\text{Cay}(G)$. A subgroup $H \subset G$ is codimension-1 if it has a finite neighborhood $N_{r}(H)$ such that $\text{Cay}(G) \setminus N_{r}(H)$ contains at least two components that are deep in the sense that they do not lie in any $N_{s}(H)$.

**Example 3.10.** (1) $\mathbb{Z}^{n}$ in $\mathbb{Z}^{n+1}$.

(2) Any infinite cyclic subgroup of a closed surface subgroup.

Let $H_{1}, \ldots, H_{k}$ be a collection of codimension-1 subgroups. The wall associated to $H_{i}$ is a fixed partition $\{\tilde{N}_{i}, \overline{N}_{i}\}$ of $\text{Cay}(G)$. The translated wall associated to $gH_{i}$ is the partition $\{g\tilde{N}_{i}, g\overline{N}_{i}\}$.

The (1-skeleton of) “dual cube complex” due to Sageev is defined as follows:

1. A 0-cube is a choice of one halfspace from each wall such that each element of $G$ lies in all but finitely many of these chosen halfspaces.
2. Two 0-cubes are joined by a 1-cube precisely when their choices differ on exactly one wall.
Let $G$ be a word-hyperbolic group, and $H_1, \ldots, H_k$ be a collection of quasiconvex codimension-1 subgroups. Then the action of $G$ on the dual cube complex is cocompact. (See Sageev [16], [17].)

**Theorem 3.11** ([7]). Let $G$ be a word-hyperbolic group. Suppose that for each pair of distinct points $(u,v) \in (\partial G)^2$ there exists a quasiconvex codimension-1 subgroup $H$ such that $u$ and $v$ lie in distinct components of $\partial G \setminus \partial H$. Then there is a finite collection $H_1, \ldots, H_k$ of quasiconvex codimension-1 subgroups such that $G$ acts properly and cocompactly on the resulting dual $\text{CAT}(0)$ cube complex.

Combining theorem 3.9 and theorem 3.11, Bergeron and Wise showed that

**Theorem 3.12** ([7]). Let $M$ be a closed hyperbolic 3-manifold. Then $\pi_1(M)$ acts properly and cocompactly on a $\text{CAT}(0)$ cube complex.

### 3.7. RFRS and virtual fiber

For a group $G$, set $G_{r}^{(1)} = \{x \in G \mid \exists k \neq 0, x^k \in [G,G]\}$.

**Definition 3.13** (RFRS [4]). A group $G$ is residually finite $\mathbb{Q}$-solvable (RFRS) if there is a sequence of subgroups $G = G_0 > G_1 > G_2 > \ldots$ such that $G \triangleright G_i$, $[G : G_i] < \infty$, $\cap G_i = \{1\}$ and $G_{i+1} \geq (G_i)^{(1)}$.

**Theorem 3.14** (Agol [4]). The following hold:
- If $G$ is RFRS, then any subgroup $H < G$ is as well.
- Right angled Artin groups are virtually RFRS. (Agol [4])

**Examples 3.15.** The following are other examples of (virtually) RFRS:
- surface groups,
- reflection groups,
- arithmetic hyperbolic groups defined by a quadratic.

**Theorem 3.16** (Agol [4]). If $M$ is aspherical and $\pi_1(M)$ is RFRS, then $M$ virtually fibers.

### 4. The final step

This is what Agol showed for the final step of the conjectures.

**Theorem 4.1** (Agol [3]). Let $G$ be a word-hyperbolic group acting properly and cocompactly on a $\text{CAT}(0)$ cube complex $X$. Then $G$ has a finite index subgroup $F$ acting specially on $X$.

Combining and theorem 3.8, 3.12, 4.1 and other cases, the situation can be described in a very nice way.

**Theorem 4.2** (Virtually Compact Special Theorem). If $N$ is a hyperbolic 3-manifold, then $\pi_1(N)$ is virtually compact special.

If $\pi_1(N)$ is virtually (compact) special, then it is a subgroup of a RAAG (theorem3.3). By theorem 3.14 and 3.16, $N$ virtually fibers.

To show that $\pi_1(M)$ is LERF, we need the next theorem.

**Theorem 4.3** (Tameness [2],[9]). Let $N$ be a hyperbolic 3-manifold, not necessarily of finite volume. If $\pi_1(N)$ is finitely generated, then $N$ is topologically tame, i.e., $N$ is homeomorphic to the interior of a compact 3-manifold.
A 3-manifold $N$ is fibered if there exists a fibration $N \to S^1$. A surface fiber in a 3-manifold $N$ is the fiber of a fibration $N \to S^1$. $\Gamma \subset \pi_1(N)$ is a surface fiber subgroup if there exists a surface fiber $\Sigma$ such that $\Gamma = \pi_1(\Sigma)$. $\Gamma \subset \pi_1(N)$ is a virtual surface fiber subgroup if $N$ admits a finite cover $N' \to N$ such that $\Gamma \subset \pi_1(N')$ and such that $\Gamma$ is a surface fiber subgroup of $N'$.

**Theorem 4.4** (Subgroup Tameness Theorem). Let $N$ be a hyperbolic 3-manifold and let $\Gamma \subset \pi_1(N)$ be a finitely generated subgroup. Then either

(1) $\Gamma$ is a virtual surface fiber group, or

(2) $\Gamma$ is geometrically finite.

For the proof of the theorem, theorem 4.3 and the covering theorem (due to Canary) is used.

Theorem 3.2 and 4.4 are used to show the next theorem.

**Theorem 4.5** (Agol). Let $M$ be a closed hyperbolic 3-manifold. Then there is a finite-sheeted cover $\bar{M} \to M$ such that $\bar{M}$ fibers over the circle. Moreover, $\pi_1(M)$ is LERF and large.

Then, a standard argument in 3-manifold theory shows the next theorem.

**Theorem 4.6** (Agol). Let $M$ be a closed aspherical 3-manifold. Then there is a finite-sheeted cover $\bar{M} \to M$ such that $\bar{M}$ is Haken.

**References**


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