END INVARIANTS OF HECKOID GROUPS FOR 2-BRIDGE LINKS
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1. INTRODUCTION

By extending the concept of a geometrically infinite end of a Kleinian group, Bowditch [4] introduced the notion of the end invariants of a type-preserving $SL(2, \mathbb{C})$-representation of the fundamental group $\pi_1(T)$ of the once-punctured torus $T$. Tan, Wong and Zhang [23, 24] extended this notion (with slight modification) to an arbitrary $SL(2, \mathbb{C})$-representation of $\pi_1(T)$. In [12], we gave an explicit description of the sets of end invariants of the $SL(2, \mathbb{C})$-characters of the once-punctured torus corresponding to the holonomy representation of hyperbolic 2-bridge link groups. The purpose of this note is to announce the result obtained in [14] which explicitly describes the sets of end invariants of the $SL(2, \mathbb{C})$-characters of the once-punctured torus corresponding to the holonomy representation of Heckoid groups (Theorem 4.1).

2. BOWDITCH, TAN-WONG-ZHANG END INVARIANTS

Motivated by the definition of the end of a geometrically infinite end of a Kleinian group, Bowditch [4] introduced the notion of the end invariants of an arbitrary type-preserving $PSL(2, \mathbb{C})$-representation of $\pi_1(T)$. Tan, Wong and Zhang [23, 24] extended this notion (with slight modification) to an arbitrary $PSL(2, \mathbb{C})$-representation of $\pi_1(T)$. To describe this, let $C$ be the set of free homotopy classes of essential simple loops on $T$. Then $C$ is identified with $\hat{\mathbb{Q}}$, the vertex set of the Farey tessellation $D$, by the following rule $s \mapsto \beta_s$, where $\beta_s$ is the image of a line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope $s$ in $T = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$. The projective lamination space $PLC$ of $T$ is then identified with $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and contains $C$ as the dense subset of rational points.

Definition 2.1. Let $\rho$ be a $PSL(2, \mathbb{C})$-representation of $\pi_1(T)$.

1. An element $X \in PLC$ is an end invariant of $\rho$ if there exists a sequence of distinct elements $X_n \in C$ such that $X_n \to X$ and that $\{ |\text{tr}\rho(X_n)| \}_n$ is bounded from above.

2. $E(\rho)$ denotes the set of end invariants of $\rho$.

In the above definition, it should be noted that $|\text{tr}\rho(X_n)|$ is well-defined though $\text{tr}\rho(X_n)$ is defined only up to sign. Note also that the condition that $\{ |\text{tr}\rho(X_n)| \}_n$ is bounded from above is equivalent to the condition that the (real) hyperbolic translation lengths of the isometries $\rho(X_n)$ of $\mathbb{H}^3$ are bounded from above. So, if $\rho$ is a faithful discrete type-preserving representation and $\nu$ is the end invariant of a geometrically infinite end of the quotient hyperbolic manifold, then $\nu$ is an end invariant of $\rho$ in the sense of the above definition.

Tan, Wong and Zhang [23, 24] showed that $E(\rho)$ is a closed subset of $PLC$ and proved various interesting properties of $E(\rho)$, including a characterization of those representations $\rho$ with $E(\rho) = \emptyset$ or $PLC$, generalizing results of Bowditch [4]. They also proposed an
interesting conjecture [24, Conjecture 1.8] concerning possible homeomorphism types of $E(\rho)$. The following is a modified version of the conjecture which Tan [22] informed to the authors.

**Conjecture 2.2.** Suppose $E(\rho)$ has at least two accumulation points. Then either $E(\rho) = PL$ or a Cantor set of $PL$.

They constructed a family of representations $\rho$ which have Cantor sets as $E(\rho)$, and proved the following supporting evidence to the conjecture (see [24, Theorem 1.7]).

**Theorem 2.3.** Let $\rho : \pi_1(T) \to SL(2, \mathbb{C})$ be discrete in the sense that the set $\{\text{tr}(\rho(X)) | X \in C\}$ is discrete in $C$. Then if $E(\rho)$ has at least three elements, then $E(\rho)$ is either a Cantor set of $PL$ or all of $PL$.

However, the above theorem does not describe the set $E(\rho)$ explicitly. In [12], we gave an explicit description of the sets of end invariants of the $SL(2, \mathbb{C})$-characters of the once-punctured torus corresponding to the holonomy representation of hyperbolic 2-bridge link groups. In this note, we announce a result obtained in [14] which explicitly describes the ends of end invariants of the $SL(2, \mathbb{C})$-characters of the once-punctured torus corresponding to the holonomy representation of Heckoid groups (Theorem 4.1). These give an infinite family of representations $\rho$ for which $E(\rho)$ are explicitly described Cantor sets.

3. Heckoid orbifold $S(r;n)$ and Heckoid group $G(r;n)$

For a rational number $r \in \hat{Q} := \mathbb{Q} \cup \{\infty\}$, let $K(r)$ be the 2-bridge link of slope $r$, which is defined as the sum $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ of rational tangles of slope $\infty$ and $r$. The common boundary $\partial(B^3, t(\infty)) = \partial(B^3, t(r))$ of the rational tangles is identified with the *Conway sphere* $(S^2, P) := (\mathbb{R}^2, \mathbb{Z}^2)/H$, where $H$ is the group of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the $\pi$-rotations around the points in the lattice $\mathbb{Z}^2$. Let $S$ be the 4-punctured sphere $S^2 - P$ in the link complement $S^3 - K(r)$. Any essential simple loop in $S$, up to isotopy, is obtained as the image of a line of slope $s \in \hat{Q}$ in $\mathbb{R}^2 - \mathbb{Z}^2$ by the covering projection onto $S$. The (unoriented) essential simple loop in $S$ so obtained is denoted by $\alpha_s$. We also denote by $\alpha_r$ the conjugacy class of an element of $\pi_1(S)$ represented by (a suitably oriented) $\alpha_r$. The loops $\alpha_\infty$ and $\alpha_r$ bound disks in $B^3 - t(\infty)$ and $B^3 - t(r)$, respectively. Thus the link group $G(K(r)) = \pi_1(S^3 - K(r))$ is obtained as follows:

$$G(K(r)) = \pi_1(S^3 - K(r)) \cong \pi_1(S)/(\langle \alpha_\infty, \alpha_r \rangle) \cong \pi_1(B^3 - t(\infty))/\langle \langle \alpha_r \rangle \rangle.$$  

For each rational number $r$ and an integer $n \geq 2$, the *even Heckoid orbifold of index $n$ for the 2-bridge link $K(r)$* is the 3-orbifold $S(r;n)$, such that the underlying space $[S(r;n)]$ is the exterior, $E(K(r)) = S^3 - \text{int} N(K(r))$, of $K(r)$, and that the singular set is the lower tunnel of $K(r)$ (i.e., the core tunnel of $(B^3, t(\infty))$ in the sense of [10, p.360]), where the index of the singularity is $n$ (see Figure 1). We call the orbifold fundamental group $\pi_1(S(r;n))$ the *Heckoid group of index $n$ for $K(r)$*, and denote it by $G(r;n)$. Since the loop $\alpha_r$ is isotopic to a meridional loop around the lower tunnel, the even Heckoid group $G(r;n) = \pi_1(S(r;n)) (n \geq 2)$ is obtained as follows:

$$G(r;n) = \pi_1(S(r;n)) \cong \pi_1(S)/(\langle \alpha_\infty, \alpha_r^n \rangle) \cong \pi_1(B^3 - t(\infty))/\langle \langle \alpha_r^n \rangle \rangle.$$  

The announcement by Agol [1] and the announcement made in the second author’s joint work with Akiyoshi, Wada and Yamashita in [2, Section 3 of Preface] suggest that
the group $G(r; n)$ makes sense even when $n$ is a half-integer greater than 1. We refer to [14, Definition 3.2] for the definition of the group $G(r; n)$ and the corresponding orbifold $S(r; n)$ when $n$ is a non-integral half-integer greater than 1. Roughly speaking, $S(r; n)$ is defined to be a $\mathbb{Z}/2\mathbb{Z}$-covering of a certain orbifold $O(r; m)$, with $m = 2n$, which is obtained from the quotient of $K(r)$ by the natural $(\mathbb{Z}/2\mathbb{Z})^3$-symmetry (see Figure 2 for the case when $K(r)$ is a knot). We call them the odd Heckoid orbifold and the odd Heckoid group, respectively, of index $n$ for $K(r)$. A topological description of an odd Heckoid orbifold is given by [14, Proposition 5.3 and Figures 5 and 6].

Remark 3.1. Our terminology is slightly different from that of Riley [20], where $G(r; n)$ is called the Heckoid group of index “$m$” for $K(r)$ with $m = 2n$. The Heckoid orbifold $S(r; n)$ and the Heckoid group $G(r; n)$ are even or odd according to whether Riley’s index $m = 2n$ is even or odd.

The following theorem was anticipated in [20] and is contained in [1] without proof.

**Theorem 3.2.** For a rational number $r$ and an integer or a half-integer $n > 1$, the Heckoid group $G(r; n)$ is isomorphic to a geometrically finite Kleinian group generated by two parabolic transformations.

A proof of this theorem is given in [14, Section 6] by using the orbifold theorem for pared orbifolds [3, Theorem 8.3.9] (cf. [5, 8]). As noted in [1], the proof is analogous to the arguments in [7, Proof of Theorem 9].

By this theorem and the topological description of odd Heckoid orbifolds ([14, Proposition 5.3]), we obtain the following proposition, which shows a significant difference between odd and even Heckoid groups (see [14, Section 6]).

**Proposition 3.3.** Any odd Heckoid group is not a one-relator group.

4. **END INVARIANTS OF EVEN HECKOID GROUPS**

For a rational number $r$ and an integer $n \geq 2$, let $\rho_{r,n}$ be the $\text{PSL}(2, \mathbb{C})$-representation of $\pi_1(S)$ obtained as the composition

$$\pi_1(S) \to \pi_1(S)/\langle\langle \alpha_\infty, \alpha_r^n \rangle\rangle \cong G(r; n) \to \text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C}),$$
The case when $K(r)$ is a knot and $m = 2n > 1$ is an odd integer. Here $r = 2/9 = [4, 2]$. The odd Heckoid orbifold $S(r; n)$ (middle right) is a $\mathbb{Z}/2\mathbb{Z}$-covering of $O(r; m)$ (lower left). The upper left figure is not an orbifold, but is a hyperbolic cone manifold. The odd Heckoid orbifold $S(r; n)$ is the quotient of the cone manifold by the $\pi$-rotation around the axis containing the singular set.

where the last homomorphism is the holonomy representation of the pared hyperbolic orbifold $S(r; n)$.

Now, let $O$ be the orbifold $(\mathbb{R}^2 - \mathbb{Z}^2)/\hat{H}$ where $\hat{H}$ is the group generated by $\pi$-rotations around the points in $(\frac{1}{2}\mathbb{Z})^2$. Note that $O$ is the orbifold with underlying space a once-punctured sphere and with three cone points of cone angle $\pi$. The surfaces $T$ and $S$, respectively, are $\mathbb{Z}/2\mathbb{Z}$-covering and $(\mathbb{Z}/2\mathbb{Z})^2$-covering of $O$, and hence their fundamental groups are identified with subgroups of the orbifold fundamental group $\pi_1(O)$ of indices 2 and 4, respectively. The $\text{PSL}(2, \mathbb{C})$-representation $\rho_{r,n}$ of $\pi_1(S)$ extends, in a unique way, to that of $\pi_1(O)$ (see [2, Proposition 2.2]), and so we obtain, in a unique way, a
PSL(2, C)-representation of π₁(𝕋) by restriction. We continue to denote it by ρᵣₙ. The following theorem, which determines the set $E(ρᵣₙ)$, is obtained by [16].

**Theorem 4.1.** For a non-integral rational number $r$ and an integer $n ≥ 2$, the set $E(ρᵣₙ)$ of end invariants of $ρᵣₙ$ is equal to the limit set $Λ(Γ(ᵣ; n))$ of the group $Γ(ᵣ; n)$.

5. **Simple Loops on Bridge Spheres of Heckoid Orbifolds**

Let $D$ be the Farey tessellation, that is, the tessellation of the upper half space $ℋ^2$ by ideal triangles which are obtained from the ideal triangle with the ideal vertices 0, 1, $∞ ∈ \mathbb{Q}$ by repeated reflection in the edges. Then $Q$ is identified with the set of the ideal vertices of $D$. For each $r ∈ Q$, let $Γ_r$ be the group of automorphisms of $D$ generated by reflections in the edges of $D$ with an endpoint $r$. It should be noted that $Γ_r$ is isomorphic to the infinite dihedral group and that the region bounded by two adjacent edges of $D$ with an endpoint $r$ is a fundamental domain for the action of $Γ_r$ on $ℋ^2$. For an integer $m$, let $C_r(m)$ be the group of automorphisms of $D$ generated by the parabolic transformation, centered on the vertex $r$, by $m$ units in the clockwise direction.

For $r$ a rational number and $n$ an integer or a half-integer greater than 1, let $Γ(r; n)$ be the group generated by $Γ_∞$ and $C_r(2n)$. Suppose that $r$ is not an integer. Then $Γ(r; n)$ is the free product $Γ_∞ * C_r(2n)$ having a fundamental domain, $R$, shown in Figure 3. Here, $R$ is obtained as the intersection of fundamental domains for $Γ_∞$ and $C_r(2n)$, and so $R$ is bounded by the following two pairs of Farey edges:

1. the pair of adjacent Farey edges with an endpoint $∞$ which cuts off a region in $ℋ^2$ containing $r$, and
2. a pair of Farey edges with an endpoint $r$ which cuts off a region in $ℋ^2$ containing $∞$, such that one edge is the image of the other by a generator of $C_r(2n)$.

Let $I(ᵣ; n)$ be the union of two closed intervals in $∂ℋ^2 = \hat{ℝ}$ obtained as the intersection of the closure of $R$ with $∂ℋ^2$. Note that there is a pair $\{r_1, r_2\}$ of boundary points of $I(ᵣ; n)$ such that $r_2$ is the image of $r_1$ by a generator of $C_r(2n)$. Set $I(ᵣ; n) = I(ᵣ; n) - \{r_i\}$ with $i = 1$ or 2. Note that $I(ᵣ; n)$ is the disjoint union of a closed interval and a half-open interval, except for the special case when $ᵣ ≡ ±1/p \ (mod \ Z)$.

![Figure 3](image.png)

**Figure 3.** A fundamental domain of $Γ(r; n)$ in the Farey tessellation (the shaded domain) for $r = \frac{3}{10} = \frac{1}{\frac{3}{2}} =: [3, 3]$ and $n = \frac{2}{3}$.
The following theorem proved in [14] is the starting point of all the results which we announce in this note.

**Theorem 5.1.** Suppose that $r$ is a non-integral rational number and that $n$ is an integer or a half integer greater than 1. Then, for any $s \in \bar{Q}$, there is a unique rational number $s_0 \in I(r; n) \cup \{\infty, r\}$ such that $s$ is contained in the $\Gamma(r; n)$-orbit of $s_0$. Moreover $\alpha_s$ is homotopic to $\alpha_{s_0}$ in $S(r; n)$. In particular, if $s_0 = \infty$, then $\alpha_s$ is null-homotopic in $S(r; n)$.

Theorem 5.2 is proved in [14], and Theorems 5.3 and 5.4 will be proved in [15].

**Theorem 5.2.** Suppose that $r$ is a non-integral rational number and that $n$ is an integer with $n \geq 2$. Then the loop $\alpha_s$ is null-homotopic in $S(r; n)$ if and only if $s$ belongs to the $\Gamma(r; n)$-orbit of $\infty$. In other words, if $s \in I(r; n) \cup \{r\}$, then $\alpha_s$ is not null-homotopic in $S(r; n)$.

**Theorem 5.3.** Suppose that $r$ is a non-integral rational number and that $n$ is an integer with $n \geq 2$. For two rational numbers $s$ and $s'$, the simple loops $\alpha_s$ and $\alpha_{s'}$ are homotopic in $S(r; n)$ if and only if $s$ and $s'$ belong to the same $\Gamma(r; n)$-orbit. In other words, for distinct $s, s' \in I(r; n) \cup \{\infty, r\}$, the simple loops $\alpha_s$ and $\alpha_{s'}$ are not homotopic in $S(r; n)$.

**Theorem 5.4.** Suppose that $r$ is a non-integral rational number and that $n$ is an integer with $n \geq 2$. Then the following hold.

1. The loop $\alpha_s$ is peripheral in $S(r; n)$ if and only if $s$ belongs to the $\Gamma(r; n)$-orbit of $\infty$.

2. The loop $\alpha_s$ is torsion in $S(r; n)$ if and only if $s$ belongs to the $\Gamma(r; n)$-orbit of $\infty$ or $r$.

In other words, there is no rational number $s \in I(r; n)$ for which the simple loop $\alpha_s$ is peripheral or torsion in $S(r; n)$.

In the above theorem, we say that $\alpha_s$ is peripheral or torsion if the conjugacy class $\alpha_s$ is represented by a (possibly trivial) parabolic or elliptic transformation, respectively, when we identify $G(r; n)$ with a Kleinian group generated by two parabolic transformations.

These theorems are proved by using the small cancellation theory [17]. Please see [13] for basic ideas of the proof. Theorem 4.1 is proved by using these theorems, Bowditch's results [4] and the discreteness of marked length spectrum of geometrically finite hyperbolic 3-manifolds, as in [12, Section 8].

**References**


[14] D. Lee and M. Sakuma, Epimorphisms from 2-bridge link groups onto Heckoid groups (I) and (II), to appear in Hiroshima J. Math..

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