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Semilinear elliptic equations in symmetric domains

佐賀大学・理工学部　梶木屋 龍治
Ryuji Kajikiya
Faculty of Science and Engineering,
Saga University

Abstract

In this note, we review the author's recent result in [12] on the existence of asymmetric positive solutions for semilinear elliptic equations in symmetric domains.

1 Introduction

We prove the existence of positive solutions without symmetry for the generalized Hénon equation in symmetric domains

$$-\Delta u = h(x)u^p, \quad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with piecewise smooth boundary. We assume that $1 < p < \infty$ for $N = 2$, $1 < p < (N+2)/(N-2)$ for $N \geq 3$, $h \in L^\infty(\Omega)$ and $h(x)$ may or may not change its sign. Let G be a closed subgroup of the orthogonal group $O(N)$ such that $G \neq \{I\}$, where I is the unit matrix. We call Ω a G *invariant domain* if $g(\Omega) = \Omega$ for any $g \in G$ and $h(x)$ a G *invariant function* if $h(gx) = h(x)$ for any $g \in G$ and $x \in \Omega$. In the same way, a G invariant solution is defined. Assume that $h_+(x) := \max(h(x), 0) \not\equiv 0$ in Ω. Then (1.1) has a G invariant positive solution. However, we are looking for a solution without G invariance. To this end, we define the Rayleigh quotient $R(u)$ with the definition domain $D(R)$ by

$$R(u) := \left(\int_\Omega |\nabla u|^2 dx\right) \Big/ \left(\int_\Omega h(x)|u|^{p+1} dx\right)^{2/(p+1)},$$

$$D(R) := \{u \in H_0^1(\Omega) : \int_\Omega h(x)|u|^{p+1}dx > 0\}.$$

Moreover, we define the Nehari manifold $\mathcal{N}$ by

$$\mathcal{N} := \{u \in H_0^1(\Omega) \setminus \{0\} : \int_\Omega (|\nabla u|^2 - h(x)|u|^{p+1})dx = 0\}.$$

The *least energy* R_0 is defined by

$$R_0 := \inf\{R(u) : u \in D(R)\} = \inf\{R(u) : u \in \mathcal{N}\}. \tag{1.2}$$

We call u a *least energy solution* if $u \in \mathcal{N}$ and $R(u) = R_0$. It becomes a positive or negative solution of (1.1). We choose a positive one as a least energy solution after replacing u by $-u$, if necessary.

To explain our purpose, we introduce the Hénon equation

$$-\Delta u = |x|^\lambda u^p, \quad u > 0 \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B, \tag{1.3}$$

where B is the unit ball in $\mathbb{R}^N$. Smets, Willem and Su [17] have proved that if λ is large enough, then a least energy solution of (1.3) is not radially symmetric. It is known that there exists a radial positive solution. Therefore (1.3) has both a radial positive solution and a nonradial positive solution. There are many papers which have studied the Hénon equation ([1, 2, 3, 4, 5, 6, 7, 8, 9, 15, 16]).

On the other hand, Moore and Nehari [13, pp.32–33] have studied the two point boundary value problem of the ordinary differential equation

$$u''(t) + h(t)u^p = 0, \quad u > 0 \quad \text{in } (-1, 1), \qquad u(-1) = u(1) = 0, \tag{1.4}$$

where $h(t) = 0$ for $|t| < a$ and $h(t) = 1$ for $a < |t| < 1$. When $a (< 1)$ is sufficiently close to 1, they have constructed at least three positive solutions of (1.4): an even solution $u(t)$, a non-even solution $v(t)$ and the reflection $v(-t)$. The purpose of this paper is to extend the results above to more general symmetric domains Ω and to more general weight functions $h(x)$.

2 Main result

In this section, we state main results and give several examples of Ω and $h(x)$. We first define the *fixed point set* of G by

$$F = \text{Fix}(G) := \{x \in \mathbb{R}^N : gx = x \text{ for all } g \in G\}.$$

Then F is a linear subspace of $\mathbb{R}^N$. Since $G \neq \{I\}$ is assumed with the unit matrix I, it holds that $0 \leq \dim F \leq N - 1$.

Definition 2.1. Let $F^{\perp}$ be the orthogonal complement of F in $\mathbb{R}^N$. We denote by $x = x' + x''$ the orthogonal decomposition of x into $x' \in F$ and $x'' \in F^{\perp}$. We define

$$\operatorname{dist}(x, F) := \inf\{|x - y| : y \in F\} = |x''|,$$

$$\Omega(|x''| < a) := \{x' + x'' \in \Omega : \ |x''| < a\} \quad \text{for } a > 0.$$

Put

$$L := \max\{\operatorname{dist}(x, F) : x \in \overline{\Omega}\} = \max\{|x''| : x' + x'' \in \overline{\Omega}\}.$$

We denote the set of the farthest points in $\overline{\Omega}$ from F by $\partial\Omega_0$, i.e.,

$$\partial\Omega_0 := \{x \in \partial\Omega : \operatorname{dist}(x, F) = L\}.$$

Assumption 2.2. Assume that $h(x)$ satisfies either (A) or (B) below.

(A) Let $h(x)$ take the form $h(x) = f(x)^{\lambda}$ with $\lambda > 0$ large enough, where $f(x)$ is a G invariant continuous function on $\overline{\Omega}$ such that

$$0 \le f(x) < \max_{y \in \partial\Omega_0} f(y) \quad \text{for } x \in \overline{\Omega} \setminus \partial\Omega_0.$$

(B) $h(x) \le 0$ in $\Omega(|x''| < a)$, $h_+(x) \not\equiv 0$ in $\Omega(a < |x''| < L)$ and $a \in (0, L)$ is sufficiently close to L.

We state our main result in the following.

Theorem 2.3. *Let Ω and h be G invariant and h satisfy either* (A) *or* (B). *Then a least energy solution of* (1.1) *is not G invariant. Therefore* (1.1) *has both a G invariant positive solution and a G non-invariant positive solution.*

When $\operatorname{Fix}(G) = \{0\}$ and $h(r)$ is radial, conditions (A) and (B) reduce to the following conditions.

(A)' $h(r) = f(r)^{\lambda}$ with λ large enough and $0 \le f(r) < f(L)$ for $0 \le r < L$.

(B)' $h(r) \le 0$ in $(0, a)$, $h_+(r) \not\equiv 0$ in (a, L) and a is sufficiently close to L.

Examples of $h(x)$ satisfying (A)' are $h(|x|) = |x|^{\lambda}$, $e^{\lambda|x|}$, $(|x|/(1 + |x|))^{\lambda}$. A simple example of h satisfying (B)' is $h(|x|) = (|x| - a)/(L - a)$.

Corollary 2.4. *Suppose that* $\operatorname{Fix}(G) = \{0\}$ *and $h(r)$ satisfies either* (A)' *or* (B)'. *Then the same conclusion as in Theorem 2.3 holds.*

Example 2.5. Let $G = O(N)$ and Ω be a ball with radius L. Then $\mathrm{Fix}(G) = \{0\}$. Let $h(r)$ satisfy either (A)' or (B)'. Then a least energy solution is not radially symmetric. This example extends the result by Smets, Willem and Su [17] to more general $h(x)$.

Example 2.6. Let Ω be a convex regular polytope with center origin in $\mathbb{R}^N$. We define the regular polytope group $G(\Omega)$ by

$$G(\Omega) := \{g \in SO(N) : g(\Omega) = \Omega\},$$

where $SO(N)$ denotes the rotation group. Then it holds that $\mathrm{Fix}(G(\Omega)) = \{0\}$ for any regular polytope Ω. Let $h(r)$ satisfy either (A)' or (B)', where L is the radius of a circumscribed sphere of Ω. Then a least energy solution of (1.1) is not invariant under the action of the regular polytope group $G(\Omega)$.

Example 2.7. Let Ω be a cylinder in $\mathbb{R}^3$, which is defined by

$$\Omega := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < \alpha^2,\ |x_3| < \beta\},$$

with $\alpha, \beta > 0$. Put $L := \sqrt{\alpha^2 + \beta^2}$ and let $h(r)$ be a radially symmetric function satisfying (A)' or (B)'. Then a least energy solution is not even, not rotationally symmetric around the x_3-axis and not reflectionally symmetric with respect to the plane $x_3 = 0$.

We shall prove this assertion. First, we choose $G := \{I, -I\}$. Then $\mathrm{Fix}(G) = \{0\}$. By Corollary 2.4, a least energy solution is not even.

Next, we choose

$$G := \left\{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} : g \in O(2) \right\}.$$

Then G invariance means the rotational invariance around the x_3-axis. By Theorem 2.3, a least energy solution is not rotationally invariant around the x_3-axis.

Lastly, we choose

$$G = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} I_2 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

where I_2 denotes the 2×2 unit matrix. By Theorem 2.3, a least energy solution is not reflectionally symmetric with respect to the plane $x_3 = 0$.

Example 2.8. Let

$$\Omega := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i > 0,\ (1 \le i \le 3), \quad x_1 + x_2 + x_3 < 1\}.$$

Then Ω is a triangular pyramid. Let $h(r)$ satisfy either (A)' or (B)' with $L = 1$. Then a least energy solution is not invariant under the rotation by angle $2\pi/3$ around the line $x_1 = x_2 = x_3$.

3 Proof of the main theorem

We give a sketch of proof of Theorem 2.3. The next lemma is known, but we give a proof for the reader's convenience.

Lemma 3.1. *Let u be a positive solution of* (1.1). *Then we have*

$$0 < \int_\Omega |\nabla u|^2 dx = \int_\Omega h u^{p+1} dx, \tag{3.1}$$

$$R(u) = \left(\int_\Omega |\nabla u|^2 dx\right)^{(p-1)/(p+1)} = \left(\int_\Omega h u^{p+1} dx\right)^{(p-1)/(p+1)}. \tag{3.2}$$

Proof. Multiplying (1.1) by u and integrating it over Ω, we obtain (3.1), which leads to (3.2). □

To prove Theorem 2.3, we define

$$H_0^1(\Omega, G) := \{u \in H_0^1(\Omega) : \ u \text{ is } G \text{ invariant}\},$$

$$D(R, G) := D(R) \cap H_0^1(\Omega, G), \quad \mathcal{N}(G) := \mathcal{N} \cap H_0^1(\Omega, G).$$

We define a G invariant least energy

$$R_G := \inf\{R(u) : \ u \in D(R, G)\} = \inf\{R(u) : \ u \in \mathcal{N}(G)\}.$$

We call u a G invariant least energy solution if $u \in \mathcal{N}(G)$ and $R(u) = R_G$. We call R_0 a global least energy, which has already been defined by (1.2). To prove the theorem, it is enough to show that $R_0 < R_G$. Indeed, this inequality ensures that a global least energy solution corresponding to R_0 cannot be G invariant because R_G is the infimum of $R(u)$ for all G invariant solutions u.

Let us show $R_0 < R_G$. Let u be a G invariant least energy solution. We shall define $\phi(x)$ later on, which satisfies

$$R((1 + \varepsilon\phi)u) < R(u) = R_G \quad \text{for } \varepsilon > 0 \text{ small enough.} \tag{3.3}$$

Putting $v(x) := (1 + \varepsilon\phi)u$, we obtain

$$R_0 \leq R(v) < R(u) = R_G.$$

We shall construct a function $\phi(x)$ satisfying (3.3). For simplicity of discussion, we consider the case where $N = 2$ and Ω is a regular triangle in $\mathbb{R}^2$ and G is given by

$$G := \{g(2j\pi/3) : j = 0, 1, 2\}, \quad g(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.4}$$

Let $B(x,r)$ denote the ball centered at x with radius r. Let x_0 and x_1 be vertices of the regular triangle Ω such that $g(2\pi/3)(x_0) = x_1$, where $g(2\pi/3)$ is defined by (3.4). We take two small balls $B_0 = B(x_0, 2r_0)$ and $B_1 = B(x_1, 2r_0)$ with the same radius $2r_0$ small enough. Therefore $g(2\pi/3)(B_0) = B_1$. Let $\phi_0 \in C_0^\infty(\mathbb{R}^2)$ be a radially symmetric function such that $0 \le \phi_0(x) \le 1$ in $\mathbb{R}^2$ and

$$\phi_0(x) = 1 \quad \text{for } |x| < r_0, \qquad \mathrm{supp}\phi_0 \subset B(0, 2r_0).$$

Here $\mathrm{supp}\phi_0$ denotes the support of ϕ_0.

Definition 3.2. We define $\phi(x)$ in the whole space $\mathbb{R}^2$ by

$$\phi(x) := \begin{cases} \phi_0(x - x_0) & \text{if } x \in B(x_0, 2r_0), \\ -\phi_0(x - x_1) & \text{if } x \in B(x_1, 2r_0), \\ 0 & \text{otherwise.} \end{cases}$$

We define the inner product in $H_0^1(\Omega)$ by

$$(u,v)_{H_0^1} := \int_\Omega \nabla u \nabla v dx.$$

The orthogonal complement of $H_0^1(\Omega, G)$ in $H_0^1(\Omega)$ is denoted by $H_0^1(\Omega, G)^\perp$, i.e.,

$$H_0^1(\Omega, G)^\perp := \{u \in H_0^1(\Omega) : (u,v)_{H_0^1} = 0 \quad \text{for all } v \in H_0^1(\Omega, G)\}.$$

Then $\phi(x)$ defined above satisfies

$$\phi \in C_0^\infty(\Omega_1) \cap H_0^1(\Omega_1, G)^\perp \cap L^2(\Omega_1, G)^\perp, \tag{3.5}$$

where Ω_1 is defined by

$$\Omega_1 := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \Omega) < 1\}.$$

Proposition 3.3 ([12]). *Let u be a G invariant least energy solution and ϕ be defined by Definition 3.2. Then we have*

$$\int_\Omega |\nabla\phi|^2 u^2 dx - 2(p-1)\int_\Omega u\phi\nabla u \nabla\phi dx < (p-1)\int_\Omega |\nabla u|^2\phi^2 dx. \tag{3.6}$$

Using the proposition above, we prove Theorem 2.3 in the following.

Proof of Theorem 2.3. We compute $R(v)$ for $v := (1+\varepsilon\phi)u$. Multiplying (1.1) by $\phi^2 u$ and integrating it over Ω, we have

$$\int_\Omega \left(|\nabla u|^2\phi^2 + 2u\phi\nabla u\nabla\phi\right) dx = \int_\Omega hu^{p+1}\phi^2 dx. \tag{3.7}$$

Combining (3.6) with (3.7), we have

$$\frac{1}{p-1}\int_\Omega |\nabla\phi|^2 u^2 dx < \int_\Omega \left(|\nabla u|^2\phi^2 + 2u\phi\nabla u\nabla\phi\right) dx = \int_\Omega hu^{p+1}\phi^2 dx.$$

Hence

$$\int_\Omega hu^{p+1}\phi^2 dx > 0. \tag{3.8}$$

Using (3.6) and (3.7), we obtain

$$\begin{aligned}&\int_\Omega (|\nabla u|^2\phi^2 + 2u\phi\nabla u\nabla\phi + |\nabla\phi|^2u^2)dx\\ &\qquad < p\int_\Omega (|\nabla u|^2\phi^2 + 2u\phi\nabla u\nabla\phi)dx = p\int_\Omega hu^{p+1}\phi^2 dx.\end{aligned}$$

Choose $q \in (1,p)$ slightly less than p such that

$$\int_\Omega (|\nabla u|^2\phi^2 + 2u\phi\nabla u\nabla\phi + |\nabla\phi|^2u^2)dx < q\int_\Omega hu^{p+1}\phi^2 dx. \tag{3.9}$$

We expand $|\nabla v|^2$ as

$$\begin{aligned}|\nabla v|^2 &= (1+2\varepsilon\phi+\varepsilon^2\phi^2)|\nabla u|^2 + 2\varepsilon u\nabla u\nabla\phi\\ &\quad +2\varepsilon^2 u\phi\nabla u\nabla\phi + \varepsilon^2|\nabla\phi|^2u^2.\end{aligned} \tag{3.10}$$

From now on, we extend u onto the whole space $\mathbb{R}^2$ by putting $u=0$ outside of Ω. Then $u \in H^1(\Omega_1)$. By (3.5), ϕ and $|\nabla u|^2$ are orthogonal in $L^2(\Omega_1)$, i.e., the integral of $\phi|\nabla u|^2$ on Ω_1, or equivalently on Ω, is zero. By the same reason, the integral of $2u\nabla u\nabla\phi$ on Ω vanishes. Integrating both sides of (3.10) over Ω, we get

$$\begin{aligned}\int_\Omega |\nabla v|^2 dx &= \int_\Omega (1+\varepsilon^2\phi^2)|\nabla u|^2 dx + 2\varepsilon^2\int_\Omega u\phi\nabla u\nabla\phi dx\\ &\quad +\varepsilon^2\int_\Omega |\nabla\phi|^2u^2 dx\\ &< \int_\Omega |\nabla u|^2 dx + \varepsilon^2 q\int_\Omega hu^{p+1}\phi^2 dx,\end{aligned}$$

where we have used (3.9). Observing Lemma 3.1, we put

$$A := \int_\Omega |\nabla u|^2 dx = \int_\Omega h u^{p+1} dx, \quad B := \frac{1}{A}\int_\Omega h u^{p+1}\phi^2 dx.$$

Then

$$\int_\Omega |\nabla v|^2 dx < A(1+\varepsilon^2 qB). \tag{3.11}$$

Next, we shall compute the denominator of $R(v)$. Expanding $(1+\varepsilon\phi)^{p+1}$ by the Taylor theorem, we get

$$\begin{aligned}\int_\Omega h(x)v^{p+1}dx &= \int_\Omega h(x)(1+\varepsilon\phi)^{p+1}u^{p+1}dx \\ &= \int_\Omega h(x)u^{p+1}dx + \varepsilon(p+1)\int_\Omega h(x)\phi u^{p+1}dx \\ &\quad + \frac{\varepsilon^2 p(p+1)}{2}\int_\Omega h(x)u^{p+1}\phi^2(1+\psi)dx.\end{aligned}$$

Here $\psi(x,\varepsilon)$ in the last integral is a remainder term, which converges to zero uniformly on $\overline{\Omega}_1$ as $\varepsilon \to 0$. Since ϕ is orthogonal to any G invariant function, we have

$$\int_\Omega h\phi u^{p+1}dx = 0.$$

Therefore

$$\begin{aligned}\int_\Omega h(x)v^{p+1}dx &= \int_\Omega h(x)u^{p+1}dx \\ &\quad + \frac{\varepsilon^2 p(p+1)}{2}\int_\Omega h(x)u^{p+1}\phi^2(1+\psi)dx \\ &= A(1+\varepsilon^2 C_\varepsilon),\end{aligned}$$

where

$$C_\varepsilon := \frac{p(p+1)}{2A}\int_\Omega h(x)u^{p+1}\phi^2(1+\psi(x,\varepsilon))dx.$$

Observe the easy inequality

$$(1+t)^{-q} \le 1 - qt(1+t)^{-q-1} \quad \text{for } t \ge 0, \ q > 0.$$

Substituting $t = \varepsilon^2 C_\varepsilon$ and $q = 2/(p+1)$, we have

$$\begin{aligned}&\left(\int_\Omega h(x)v^{p+1}dx\right)^{-2/(p+1)} \\ &= A^{-2/(p+1)}(1+\varepsilon^2 C_\varepsilon)^{-2/(p+1)} \\ &\le A^{-2/(p+1)}\left\{1 - \frac{2\varepsilon^2}{p+1}(1+\varepsilon^2 C_\varepsilon)^{-(p+3)/(p+1)}C_\varepsilon\right\} \\ &= A^{-2/(p+1)}(1-\varepsilon^2 p\theta_\varepsilon D_\varepsilon),\end{aligned} \tag{3.12}$$

where we have put

$$\theta_\varepsilon := (1+\varepsilon^2 C_\varepsilon)^{-(p+3)/(p+1)},$$

$$D_\varepsilon := \frac{1}{A}\int_\Omega hu^{p+1}\phi^2(1+\psi(x,\varepsilon))dx.$$

Combining (3.11) with (3.12), we get

$$\begin{aligned} R(v) &= \left(\int_\Omega |\nabla v|^2 dx\right)\left(\int_\Omega hv^{p+1}dx\right)^{-2/(p+1)} \\ &< A^{(p-1)/(p+1)}\left\{1+\varepsilon^2(qB - p\theta_\varepsilon D_\varepsilon)\right\}. \end{aligned}$$

By (3.8), we have

$$\lim_{\varepsilon\to 0} p\theta_\varepsilon D_\varepsilon = \frac{p}{A}\int_\Omega hu^{p+1}\phi^2 dx > 0.$$

Since $q < p$, it holds that

$$qB = \frac{q}{A}\int_\Omega hu^{p+1}\phi^2 dx < \frac{p}{A}\int_\Omega hu^{p+1}\phi^2 dx.$$

Therefore $qB < p\theta_\varepsilon D_\varepsilon$ for $\varepsilon > 0$ small enough. Then we have $R(v) < A^{(p-1)/(p+1)} = R(u)$, where the last equation follows from Lemma 3.1. The proof is complete. □

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