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<td>KOKUBO, EIKI; KUWAE, KAZUHIRO</td>
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ON NON-LINEAR SPECTRAL GAP FOR SYMMETRIC MARKOV CHAINS WITH COARSE RICCI CURVATURES

EIKI KOKUBO AND KAZUHIRO KUWAE

ABSTRACT. In this note, we report the summary of [12] for the case that the target space is a complete separable CAT(0)-space. We prove an upper estimate of spectral radius for (non-linear) transition operator $P$ over $L^p$-maps in the framework of symmetric Markov chains on a Polish space with positive lower bound of $n$-step coarse Ricci curvatures without its proof. As consequences, strong $L^p$-Liouville property for $P$-harmonic maps, a global Poincaré inequality (spectral gaps) for energy functional over $L^2$-maps (or functions), and spectral bounds of $L^2$-generator of Markov chains are presented.

1. Coarse Ricci curvature

Throughout this note, let $(E, d)$ be a Polish space with complete distance $d$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Denote by $\mathcal{P}^p(E)$, the family of probability measures on $(E, d)$ with finite $p$-th moment. We consider a conservative Markov chain $X=(\Omega, X_k, \theta_k, \mathcal{F}_k, \mathcal{F}_\infty, P_x)_{x \in E}$ with state space $(E, d)$. Then the transition kernel $P(x, dy)$ (or $P_x(dy)$ in short) of $X$ defined by $P(x, dy) := P_x(X_1 \in dy)$, $x \in E$ satisfies

(P1) for each $x \in E$, $\mathcal{B}(E) \ni A \mapsto P(x, A)$ is a probability measure on $(E, \mathcal{B}(E))$.

(P2) for each $A \in \mathcal{B}(E)$, $E \ni x \mapsto P(x, A)$ is $\mathcal{B}(E)$-measurable.

Conversely, for $P(x, dy)$ satisfying (P1) and (P2), we can construct a conservative Markov chain $X$ such that $P(x, dy) = P_x(X_1 \in dy)$, $x \in E$. We set $Pf(x) := \int_X f(y)P(x, dy) = E_x[f(X_1)]$ for any non-negative or bounded $\mathcal{B}(E)$-measurable function $f$ on $E$. For $n \in \mathbb{N}$, if we set $P^n f(x) := P(P^{n-1} f)(x)$ inductively, then $P^n f(x) = E_x[f(X_n)]$

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and $P^n(x, A) := (P^n 1_A)(x) = P_x(X_n \in A)$. For any non-negative measure $\nu$ on $(E, \mathcal{B}(E))$ and $n \in \mathbb{N}$, we define a measure $\nu P^n$ by $\nu P^n(A) := \nu(P^n 1_A) := \int_E P^n(x, A) \nu(dx) = P_\nu(X_n \in A), A \in \mathcal{B}(E)$.

Note that $\delta_x P^n = P^n_x, x \in E$.

We further assume the following condition to $X$:

(P3) for each $x \in E, P_x \in \mathcal{P}^1(E)$.

For the given Markov chain $X$ as above and a fixed $n \in \mathbb{N}$, a Markov chain $X^n = (\Omega, X^n_k, \sigma^n_k, \mathcal{F}^n_k, \mathcal{F}^n_\infty, P^n_x)_{x \in E}$ with state space $(E, d)$ defined by the transition kernel $P^n(x, dy)$ is called an $n$-step Markov chain. Note that if $X$ satisfies (P3), then $X^n$ does so.

For $\mu, \nu \in \mathcal{P}^1(E)$, the $L^1$-Wasserstein/Kantorovich-Rubinstein distance $d_{W_1}(\mu, \nu)$ is defined by

$$d_{W_1}(\mu, \nu) := \inf \left\{ \int_{E \times E} d(x, y) \pi(dx, dy) \bigg| \pi \in \Pi(\mu, \nu) \right\},$$

where $\Pi(\mu, \nu) := \{ \pi \in \mathcal{P}(E \times E) \mid \pi(A \times E) = \mu(A), \pi(E \times B) = \nu(B) \text{ for any } A, B \in \mathcal{B}(E) \}$.

**Definition 1.1** (Coarse Ricci Curvature, [22]). For a pair of distinct points $x, y \in E$, the coarse Ricci curvature $\kappa(x, y)$ of $X$ along $(xy)$ is defined to be

$$\kappa(x, y) := 1 - \frac{d_{W_1}(P_x, P_y)}{d(x, y)} (> -\infty), \quad (x, y) \in E \times E \setminus \text{diag}$$

and $\kappa := \inf\{\kappa(x, y) \mid (x, y) \in E \times E \setminus \text{diag}\} \in [-\infty, 1]$ is said to be the lower bound of the coarse Ricci curvature. The $n$-step coarse Ricci curvature $\kappa_n(x, y)$ of $X$ along $(xy)$ is defined to be

$$\kappa_n(x, y) := 1 - \frac{d_{W_1}(P^n_x, P^n_y)}{d(x, y)} (\geq -\infty), \quad (x, y) \in E \times E \setminus \text{diag}$$

and $\kappa_n := \inf\{\kappa_n(x, y) \mid (x, y) \in E \times E \setminus \text{diag}\} \in [-\infty, 1]$ is said to be the lower bound of the $n$-step coarse Ricci curvature.

**Remark 1.2**. We denote the family of Lipschitz functions on $E$ by $\text{Lip}(E)$.

1. If $\kappa \in \mathbb{R}$, then $P^n f \in \text{Lip}(E)$ for any $f \in \text{Lip}(E)$ and $\text{Lip}(P^n f) \leq (1 - \kappa)^n \text{Lip}(f)$ by [22, Proposition 20], which implies that (P3) holds for all $X^n$ provided (P3) holds for $X$ and $\kappa \in \mathbb{R}$, in particular, $\kappa_n(x, y) > -\infty$ for all $n \in \mathbb{N}$ and $x \neq y$ under $\kappa \in \mathbb{R}$.

2. The $n$-step coarse Ricci curvature $\kappa_n(x, y)$ is nothing but the coarse Ricci curvature for $X^n$ and $\kappa_1(x, y) = \kappa(x, y)$ for $(x, y) \in E \times E \setminus \text{diag}$. In general, we have

$$(1 - \kappa_{k+\ell}) \leq (1 - \kappa_k)(1 - \kappa_\ell), \quad k, \ell \in \mathbb{N},$$

which implies $\lim_{\ell \to \infty} (1 - \kappa_\ell)^{1/\ell} = \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{1/n} \in [0, +\infty]$.

In particular, $\kappa \in \mathbb{R}$ implies $\kappa_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.
The following example due to [4] shows that the lower bound 0 for the coarse Ricci curvature does not necessarily mean the same bound for the n-step coarse Ricci curvature.

**Example 1.3** (Simple Random Walk on Cycle Graph, see [4]). Let $G = (V, E)$ be a cycle graph of size $N$, that is, $G$ is an unweighted finite graph with vertices $V := \{x_i\}_{i=1}^N$ and edges $E := \{x_ix_{i+1}\}_{i=1}^N$ by regarding $x_{N+i} = x_i (i \in \mathbb{N})$. The degree $d_x(G)$ for $x \in V$ is the number of edges starting from $x$ is given by $d_x(G) = 2$ for this cycle graph $G$. The weight $w_{xy}$ for $xy \in E$ is given by $w_{x_ix_{i+1}} = 1$ for $i = 1, 2, \cdots, N$.

Consider a symmetric Markov chain $X$ defined by the transition kernel $P_{x_i}(dy) := \frac{1}{2}\delta_{x_{i-1}}(dy) + \frac{1}{2}\delta_{x_{i+1}}(dy)$. As for the simple random walk on $\mathbb{Z}^1$, the coarse Ricci curvature $\kappa(x, y)$ on $X$ satisfies $\kappa(x, y) = 0$ for $(x, y) \in V \times V \setminus \text{diag}$ (by [8, Theorems 2,3,4 and 5], [4, Theorems 6 and 7]), hence the n-step coarse Ricci curvature $\kappa_n(x, y)$ satisfies $\kappa_n(x, y) \geq 0$ for $(x, y) \in V \times V \setminus \text{diag}$ by [22, Proposition 25]. $X$ (hence the n-step Markov chain $X^n$) is $m$-symmetric with respect to $m(dy) := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}(dy)$. For simplicity, hereafter, we assume $N = 5$. 3-step Markov chain $X^3$ is associated with the Cayley graph $G^3 := (V^3, E^3)$ defined by $V^3 := V$ and $E^3 := \{x_ix_j \mid i, j = 1, 2, 3, 4, 5 \text{ with } i \neq j\}$. $G^3$ is a weighted complete graph. The transition kernel $P^3_{x_i}(dy)$ of $X^3$ is given by $P^3_{x_i}(dy) = \frac{1}{2}\delta_{x_{i-2}}(dy) + \frac{3}{8}\delta_{x_{i-1}}(dy) + \frac{3}{8}\delta_{x_{i+1}}(dy) + \frac{1}{8}\delta_{x_{i+2}}(dy)$. $G^3$ is a weighted graph with no loop and the degree $d_x(G^3)$ for $x \in V$ is given by $d_x(G^3) = 4$. The weight $w_{xy}$ for $xy \in E^3$ is given by $w_{x_ix_i} = \frac{3}{2}$, $w_{x_{i-1}x_{i-1}} = w_{x_{i+1}x_{i+1}} = \frac{3}{4}$. Note here that our degree $d_x(G^3) = 4$ and the way for weighting on edges are different from those used in Section 6 of [4], but the conclusion is the same as we calculate below. The 3-step coarse Ricci curvature $\kappa_3(x, y)$ for $xy \in E^3$ can be estimated by use of [4, Theorems 6 and 7]:

$$\kappa_3(x_i, x_{i+1}) = \frac{3}{8}, \quad \frac{5}{8} \leq \kappa_3(x_i, x_{i+2}) \leq \frac{7}{8}.$$ 

Therefore, $\kappa_3(x, y) \geq \frac{3}{8}$ for all $(x, y) \in V \times V \setminus \text{diag}$.

**Remark 1.4.** For a continuous time parameter Markov process $M$, the notion of coarse Ricci curvature $\kappa(x, y)$ for $M$ is discussed in [22], [28]:

$$\kappa(x, y) := \lim_{t \to 0} \frac{1}{t} \left( 1 - \frac{d_{W_1}(P_t(x, \cdot), P_t(y, \cdot))}{d(x, y)} \right) \quad \text{for } (x, y) \in E \times E \setminus \text{diag}. $$

We can also define the n-step coarse Ricci curvature $\kappa_n(x, y)$:

$$\kappa_n(x, y) := \lim_{t \to 0} \frac{1}{t} \left( 1 - \frac{d_{W_1}(P_{nt}(x, \cdot), P_{nt}(y, \cdot))}{d(x, y)} \right) \quad \text{for } (x, y) \in E \times E \setminus \text{diag},$$

which is nothing but the coarse Ricci curvature for the time changed process $M^n = (\Omega, X_{nt}, P_x)$. Then, we easily see $\kappa_n(x, y) = n\kappa(x, y)$ for $(x, y) \in E \times E \setminus \text{diag}$, in particular, the positivity of lower bound
for curvature is equivalent to each other between both coarse Ricci curvatures. In our discrete setting, we have no such a relation.

**Example 1.5** (Riemannian manifold). Let \((M, g)\) be a \(d\)-dimensional complete smooth Riemannian manifold whose Ricci curvature is bound below by \(\kappa > 0\). In view of Bonnet-Myers theorem, \(M\) is compact. Consider a Brownian motion \(M = (\Omega, X_t, P_x)\) on \(M\) associated with the following Dirichlet energy form on \(L^2(M; m)\):

\[
\{\begin{array}{l}
D(\mathcal{E}) := \{u \in L^2(M; m) \mid \int_M g(\nabla f, \nabla f) dm < \infty\} \\
\mathcal{E}(f, g) := \int_M g(\nabla f, \nabla g) dm, \quad f, g \in D(\mathcal{E})
\end{array}\]

where \(m\) is the volume element of \((M, g)\). Let \(P_t(x, dy)\) be the transition kernel of \(M\). Under the Ricci curvature lower bound, \(M\) is a conservative process, that is, \(P_t(x, \cdot) \in \mathcal{P}(M)\) for any \(t > 0\). Moreover, we see \(P_t(x, \cdot) \in \mathcal{P}^1(M)\) for any \(t > 0\). We set \(P(x, dy) := P_1(x, dy)\) and consider an \(m\)-symmetric Markov chain \(X\) associated with \(P(x, dy)\). It is proved in [30] that

\[
d_{W_1}(P_t(x, \cdot), P_t(y, \cdot)) \leq e^{-\kappa t}d(x, y), \quad x, y \in M.
\]

So the coarse Ricci curvature \(\kappa_M(x, y)\) of \(M\) has the lower estimate

\[
\kappa_M(x, y) := \lim_{t \to 0} \frac{1}{t} \left(1 - \frac{d_{W_1}(P_t(x, \cdot), P_t(y, \cdot))}{d(x, y)}\right)
\]

\[
\geq \frac{d}{dt} (1 - e^{-\kappa t})\bigg|_{t=0} = \kappa > 0, \quad (x, y) \in M \times M \setminus \text{diag}.
\]

On the other hand, the \(n\)-step coarse Ricci curvature \(\kappa_n(x, y)\) of \(X\) has the lower estimate

\[
\kappa_n(x, y) = 1 - \frac{d_{W_1}(P^n_x, P^n_y)}{d(x, y)} \geq 1 - e^{-\kappa n} > 0, \quad (x, y) \in M \times M \setminus \text{diag}.
\]

Note that the same conclusion also holds for a Markov process whose coarse Ricci curvature is bounded below by \(\kappa > 0\).

2. **CAT(0)-spaces**

In this section, we summarize the notions of CAT(0)-space and its properties.

**Definition 2.1** (CAT(0)-space). A metric space \((Y, d)\) is called the **CAT(0)-space** (Hadamard space, or global NPC space) if for any pair of points \(\gamma_0, \gamma_1 \in Y\) and any \(t \in [0, 1]\) there exists a point \(\gamma_t \in Y\) such that for any \(z \in Y\)

\[
d_Y^2(z, \gamma_t) \leq (1 - t)d_Y^2(z, \gamma_0) + td_Y^2(z, \gamma_1) - t(1 - t)d_Y^2(\gamma_0, \gamma_1).
\]

By definition, \(\gamma := (\gamma_t)_{t\in[0,1]}\) is the minimal geodesic joining \(\gamma_0\) and \(\gamma_1\). Any CAT(0)-space is simply connected. Hadamard manifolds, Euclidean Bruhat-Tits buildings (e.g. metric tree), spiders, booklets and Hilbert spaces are typical examples of CAT(0)-spaces (cf. [25]).
Let $(Y, d_Y)$ be a CAT(0)-space. Then the distance function $d_Y : Y \times Y \to [0, \infty]$ is convex (Corollary 2.5 in [25]) and Jensen's inequality (Theorem 6.3 in [25]) can be applied to the convex function $Y \ni w \mapsto d_Y(w, z)$ for each $z \in Y$.

The inequality (2.1) yields the (strict) convexity of $Y \ni x \mapsto d_Y^2(z, x)$ for a fixed $z \in Y$. Any closed convex subset of a CAT(0)-space is again a CAT(0)-space.

The unique existence of projection (or foot-point) to closed convex set of CAT(0)-space is proved in [14] in more general setting.

**Lemma 2.2** (Projection Map to Convex Set, see [25]). Let $(Y, d_Y)$ be a complete CAT(0)-space. The following hold:

1. Let $F$ be a closed convex subset of $(Y, d_Y)$. Then, for each $x \in Y$, there exists a unique element $\pi_F(x) \in F$ such that $d_Y(x, F) = d_Y(\pi_F(x), x)$ holds. We call $\pi_F : Y \to F$ the projection map to $F$.

2. Let $F$ be as above. Then $\pi_F$ satisfies

   \[ d_Y^2(z, \pi_F(z)) + d_Y^2(\pi_F(z), w) \leq d_Y^2(z, w), \quad \text{for } z \in Y, w \in F, \]

   in particular, $d_Y(\pi_F(z), w) \leq d_Y(z, w)$ for $z \in Y, w \in F$.

Let $(Y, d_Y)$ be a metric space and $\mathcal{P}(Y)$ a family of Borel probability measures on $Y$. For $p \geq 1$, we set

\[ \mathcal{P}^p(Y) := \left\{ \mu \in \mathcal{P}(Y) \mid \int_Y d_Y^p(x, y)\mu(dy) < \infty \text{ for any/some } x \in Y \right\}. \]

Each element $\mu \in \mathcal{P}^p(Y)$ is called a probability measure with $p$-th moment.

**Definition 2.3** (Barycenter or Center of Mass, see [25]). For $\mu \in \mathcal{P}^2(Y)$, if $z \mapsto \int_Y d_Y^2(z, x)\mu(dx)$ has a minimizer $b(\mu) \in Y$, then we call $b(\mu)$ the barycenter, or center of mass of $\mu \in \mathcal{P}^2(Y)$. For $\mu \in \mathcal{P}^1(Y)$ and $w \in Y$, we consider the following function $F_w$:

\[ F_w(z) := \int_Y (d_Y^2(z, x) - d_Y^2(w, x))\mu(dx). \]

We easily see

\[ |F_w(z)| \leq 2d_Y(z, w)\int_Y (d_Y(z, x) + d_Y(w, x))\mu(dx) < \infty. \]

If $Y \ni z \mapsto F_w(z)$ admits a minimizer $b(\mu)$ independent of $w$ in the sense that $F_w(z) \geq F_w(b(\mu))$ if and only if $F_v(z) \geq F_v(b(\mu))$ for all $z, w, v \in Y$, we call it barycenter, or center of mass of $\mu \in \mathcal{P}^1(Y)$. If the barycenter of $\mu \in \mathcal{P}^2(Y)$ exists, then it is a barycenter of $\mu \in \mathcal{P}^1(Y)$.

Assume that $(Y, d_Y)$ is a geodesic space. For a subset $F$ of $Y$, denote by $C(F)$ the closed convex hull of $F$. That is, $C(F)$ is the smallest closed convex subset of $Y$ containing $F$. 

If \((Y, d_Y)\) is a complete \(\text{CAT}(0)\)-space, we can obtain the unique existence of barycenter of \(\mu \in \mathcal{P}^1(Y)\) proved in [25].

**Lemma 2.4** ([25], cf. [16],[21]). Let \((Y, d_Y)\) be a complete \(\text{CAT}(0)\)-space. Then \(\mu \in \mathcal{P}^1(Y)\) admits a unique barycenter.

For any metric space \((Y, d_Y)\), we easily see \(b(\delta_x) = x\) for \(x \in Y\).

The following proposition is proved in Proposition 5.5 in [25].

**Theorem 2.5** (Jensen’s Inequality, see [25, Theorem 6.3]). Let \((Y, d_Y)\) be a complete \(\text{CAT}(0)\)-space. Let \(\varphi\) be a lower semi-continuous convex function on \(Y\) and \(\mu \in \mathcal{P}^1(Y)\). Suppose \(\varphi \in L^1(Y; \mu)\). Then we have

\[
\varphi(b(\mu)) \leq \int_Y \varphi(x) \mu(dx).
\]

**Corollary 2.6** (Fundamental Contraction Property, see [25]). Let \((Y, d_Y)\) be a complete \(\text{CAT}(0)\)-space. Let \(\mu, \nu \in \mathcal{P}^1(Y)\). Then

\[
d_Y(b(\mu), b(\nu)) \leq d_{W^1}(\mu, \nu),
\]

where \(d_{W^1}(\mu, \nu)\) is the \(L^1\)-Wasserstein distance on \(\mathcal{P}^1(Y)\) defined by

\[
d_{W^1}(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{Y \times Y} d_Y(x, y) \pi(dx \, dy).
\]

Here \(\Pi(\mu, \nu) := \{\pi \in \mathcal{P}(Y \times Y) \mid \pi(A \times Y) = \mu(A), \pi(Y \times B) = \nu(B) \text{ for } A, B \in \mathcal{B}(Y)\}\).

### 3. \(L^p\)-Maps

Let \((E, \mathcal{E}, \mu)\) be a \(\sigma\)-finite measure space and \(\mathcal{E}^\mu\) a completion of \(\mathcal{E}\) with respect to \(\mu\). In what follows, we say measurable (resp. \(\mu\)-measurable) for \(\mathcal{E}\)-measurable (resp. \(\mathcal{E}^\mu\)-measurable). For function \(f : E \to [-\infty, \infty]\), we set \(\|f\|_p := (\int_E |f(x)|^p \mu(dx))^{1/p}, \|f\|_\infty := \inf\{\lambda > 0 \mid |f(x)| \leq \lambda \mu\text{-a.e. } x \in E\}\). For two \(\mathbb{R}\)-valued functions \(f, g\), they are said to be \(\mu\)-equivalent if \(f = g\ \mu\text{-a.e.}\).

For \(p \in [0, \infty]\), \(L^p(E; \mu)\) denotes the family of \(\mu\)-equivalence class of functions with finite \(\|f\|_p\)-norm. Also \(L^p(E; \mu)\) denotes the family of \(\mu\)-equivalence class of functions having finite value \(\mu\text{-a.e.}\). Fix a metric space \((Y, d_Y)\). For \(p \in [0, \infty]\) and measurable maps \(u, v : E \to Y\), the pseudo-distance \(d_{L^p}(u, v)\) is defined by \(d_{L^p}(u, v) := \|d_Y(u, v)\|_p\). More precisely, for \(p \in [0, \infty]\) we set

\[
d_{L^p}(u, v) := \left(\int_E d_Y^p(u(x), v(x)) \mu(dx)\right)^{1/p},
\]

and for \(p = \infty, d_\infty(u, v)\) is the \(\mu\)-essentially supremum of \(x \mapsto d_Y(u(x), v(x))\).

We say that \(u\) and \(v\) are \(\mu\)-equivalent (\(u \sim^\mu v\) in short) if

\[u(x) = v(x)\ \mu\text{-a.e. } x \in E\]
For a fixed measurable map $h : E \to Y$, we set

$$L^p_{h}(E, Y; \mu) := \{ f \in \mathcal{E}(\mathcal{B}(Y)) \mid d_Y(f, h) \in L^p(E; \mu) \}/\sim.$$  

Such a map $h : E \to Y$ is called a base map of $L^p_{h}(E, Y; \mu)$. If $\mu(E) < \infty$ and the image of $h : E \to Y$ is bounded, $L^p_{h}(E, Y; \mu)$ is independent of the choice of such a base map $h$. In this case, we can assume $h \equiv 0$ for some fixed point $o \in Y$.

**Proposition 3.1** ([23, Proposition 3.3]). Let $(Y, d_Y)$ be a metric space and $h : E \to Y$ a measurable map. Take $p \in [1, \infty]$. Then we have the following:

1. If $(Y, d_Y)$ is complete, then $(L^p_{h}(E, Y; \mu), d_{L^p})$ is so.
2. If $(Y, d_Y)$ is a geodesic space and any point $\gamma_{t}$ of the constant speed geodesic $\gamma : [0, 1] \to Y$ joining $\gamma_{0}$ to $\gamma_{1}$ is a continuous map with respect to $(\gamma_{0}, \gamma_{1})$, then $(L^p_{h}(E, Y; \mu), d_{L^p})$ is also a geodesic space.

In what follows, we assume $m(E) < \infty$. Let $L^p(E, Y; m)$ be the space of $L^p$-maps with bounded base maps, that is,

$$L^p(E, Y; m) := \left\{ u : E \to Y \mid u \text{ is } m\text{-measurable, } \int_{E} d_{Y}^{p}(u,o)dm < \infty \text{ for some } o \in Y \right\}/\sim.$$  

**Definition 3.2** (Lipschitz Maps). Let $(Y, d_Y)$ be a geodesic space and $(E, d)$ a metric space. For a map $u : E \to Y$, we set $\text{Lip}(u) := \sup_{x \neq y} \frac{d_Y(u(x), u(y))}{d(x,y)}$ and

$$\text{Lip}(E, Y) := \{ u : E \to Y \mid \text{Lip}(u) < \infty \}.$$  

**Lemma 3.3.** Let $(Y, d_Y)$ be a geodesic space and $(E, d)$ a metric space. Suppose that $m$ has a $p$-th moment, that is, $\int_{E} d^{p}(x, x_0)m(dx) < \infty$ for some/any point $x_0 \in E$. Then $\text{Lip}(E, Y) \subset L^p(E, Y; m)$.

Let $S(E, Y)$ be a space of finite valued maps from $E$ to $Y$. Any element of $S(E, Y)$ is called a step map or a simple map. Since $m(E) < \infty$, $S(E, Y)$ (more precisely $S(E, Y)/\sim$) is a subset of $L^p(E, Y; m)$.

**Theorem 3.4.** Suppose that $(E, d)$ is a Polish space and $(Y, d_Y)$ is a separable geodesic space. Take $p \in [1, \infty[$. Then any element of $L^p(E, Y; m)$ can be $L^p$-approximated by elements in $S(E, Y)$. In particular, if $E = \text{supp}[m]$, then $(L^p(E, Y; m), d_{L^p})$ is a separable metric space. Moreover, if $m$ has a finite $p$-th moment, then $L^p(E, Y; m)$ can be $L^p$-approximated by elements in $\text{Lip}(E, Y)$, if further $E = \text{supp}[m]$, then $\text{Lip}(E, Y)$ is a dense subset of $L^p(E, Y; m)$.

In what follows, $(E, d)$ denotes a Polish space with complete distance $d$. 
Definition 3.5 ($P^tu$ for Borel Map $u$). Let $X$ be a conservative Markov chain on $(E, d)$. Suppose that $(Y, d_Y)$ is a complete CAT(0)-space and a $B(E)/B(Y)$-measurable map $u : E \to Y$ satisfies $u_\ell P_x^\ell \in \mathcal{P}^1(Y)$ for $\ell \in \mathbb{N}$. Then we set

$$P^tu(x) := b(u_\#P_x^\ell).$$

Here $u_\#P_x^\ell$ is a push-forward measure of $P(x, \cdot)$ by $u$; $u_\#P_x^\ell(A) := P^\ell(x, u^{-1}(A))$, $A \in \mathcal{B}(Y)$.

Remark 3.6. Note that any $u \in S(E, Y)$ satisfies $u_\#P_x \in \mathcal{P}^1(Y)$. Indeed, for $u \in S(E, Y)$, $u$ is a constant on each Borel set $A_i$, where $\{A_i\}_{i=1}^l$ is a finite family of disjoint Borel sets satisfying $E = \bigcup_{i=1}^l A_i$, hence $\int_E d_Y(z_0, z)u_\#P_x(dz) = \sum_{i=1}^l \|d_Y(z_0, u)\|_{\infty, A_i} P_x(A_i) < \infty$. For $u \in Lip(E, Y)$, we have

$$\int_E d_Y(z_0, z)u_\#P_x(dz) = \int_E d_Y(z_0, u(y)) P_x(dy) \leq d_Y(z_0, u(y_0)) + Lip(u) \int_E d(y_0, y) P_x(dy) < \infty.$$

Lemma 3.7 (Lemma 6.4 in [23]). Let $X$ be a conservative Markov chain on $(E, d)$. Suppose that $(Y, d_Y)$ is a complete separable CAT(0)-space. Then, for any Borel map $u : E \to Y$ satisfying $u_\#P_x \in \mathcal{P}^1(Y)$ for all $x \in E$, $Pu : E \to Y$ is $B(E)/B(Y)$-measurable.

Definition 3.8 ($Pu$ for $L^p$-map $u$). Fix $p \geq 1$. Let $X$ be an $m$-symmetric conservative Markov chain on $(E, d)$. Suppose that $(Y, d_Y)$ is a complete CAT(0)-space and $u \in L^p(E, Y; m)$, we can define $Pu \in L^p(E, Y; m)$ in the following way: Let $\{u_k\} \subset S(E, Y)$ be an $L^p$-approximating sequence to $u$. Applying the Jensen’s inequality to the convex function $d_Y^p$ on $Y \times Y$ and the $m$-symmetry, we have the following inequality for any maps $v, w \in S(E, Y)$.

$$(3.1) \quad d_{L^p}^p(Pv, Pw) = \int_E d_Y^p(Pv(x), Pw(x))m(dx) \leq \int_E Pd_Y^p(v, w)dm \leq d_{L^p}^p(v, w).$$

These mean that $\{Pu_k\}$ forms an $L^p$-Cauchy sequence. We set $Pu := \lim_k Pu_k \in L^p(E, Y; m)$. The well-definedness of $Pu$ is clear from (3.1) and this is valid for any $v, w \in L^p(E, Y; m)$.

Definition 3.9 ($P$-harmonic Map, [16],[15]). A (lower or upper) bounded Borel function $f : E \to \mathbb{R}$ is said to be $P$-subharmonic if $f \leq Pf$ on $E$ and it is said to be $P$-harmonic if both $f$ and $-f$ are $P$-subharmonic. A Borel map $u : E \to Y$ is said to be $P$-harmonic if $u = Pu$ on $E$ holds under that $u_\#P_x \in \mathcal{P}^1(Y)$ for all $x \in E$. 

94
Lemma 3.10. Let $X$ be a Markov chain on $(E,d)$. Fix $n \in \mathbb{N}$ and assume $\kappa \in \mathbb{R}$. Suppose that $(Y,d_Y)$ is a complete CAT(0)-space. Then for $u \in \mathrm{Lip}(E,Y)$ and $\ell \in \mathbb{N}$, we have $P^{\ell}u \in \mathrm{Lip}(E,Y)$ and
\[
\mathrm{Lip}(P^{\ell}u) \leq (1 - \kappa_{n})^{\ell}\mathrm{Lip}(u),
\]
in particular,
\[
\mathrm{Lip}(P^\ell u) \leq (1 - \kappa)^\ell \mathrm{Lip}(u).
\]

Corollary 3.11 (Strong Liouville Property for Lipschitz Maps). Let $X$ be a Markov chain on $(E,d)$. Assume that $\kappa \in \mathbb{R}$ and there exists $n \in \mathbb{N}$ such that $\kappa_n > 0$. Suppose that $(Y,d_Y)$ is a complete $\text{CAT}(0)$-space. Then for $u \in \text{Lip}(E,Y)$ and $\ell \in \mathbb{N}$, we have $P^{\ell}u \in \text{Lip}(E,Y)$ and $\text{Lip}(P^{\ell}u) \leq (1 - \kappa_{n})^{\ell} \text{Lip}(u)$, in particular, $\text{Lip}(P^{\ell}u) \leq (1 - \kappa)^{\ell} \text{Lip}(u)$.

Definition 3.12 (Variance). Fix $p \geq 1$, $\mu \in \mathcal{P}(E)$, a metric space $(Y,d_Y)$ and $u \in L^{p}(E, Y; \mu)$. The $p$-variance $\text{Var}_{\mu}^{p}(u)$ of $u$ is defined by
\[
\text{Var}_{\mu}^{p}(u) := \inf_{y \in Y} \int_{E} d_{Y}^{p}(u(x), y) \mu(dx) (< \infty).
\]
The quasi $p$-variance $\overline{\text{Var}}_{\mu}^{p}(u)$ is defined by
\[
\overline{\text{Var}}_{\mu}^{p}(u) := \frac{1}{2} \int_{E} \int_{E} d_{Y}^{p}(u(y), u(x)) \mu(dx) \mu(dy) (< \infty).
\]
We easily see $\text{Var}_{\mu}^{p}(u) \leq 2 \overline{\text{Var}}_{\mu}^{p}(u)$. When $p = 2$, we write $\text{Var}_{\mu}(u) := \text{Var}_{\mu}^{2}(u)$ and $\overline{\text{Var}}_{\mu}(u) := \overline{\text{Var}}_{\mu}^{2}(u)$, and call them simply variance, quasi variance, respectively. Let $(Y,d_Y)$ be a complete CAT(0)-space. If $u \in L^{2}(E,Y; \mu)$, then $\text{Var}_{\mu}(u) = \int_{E} d_{Y}^{2}(u(x), b(u_{\#}\mu))) \mu(dx)$ holds. For $u \in L^{2}(E,Y; \mu)$, we have $\text{Var}_{\mu}(u) \leq \overline{\text{Var}}_{\mu}(u)$. If $(Y,d_Y)$ is a Hilbert space $H$, then we have $\text{Var}_{\mu}(u) = \overline{\text{Var}}_{\mu}(u)$. In this case we can define the covariance $\text{Cov}_{\mu}(f,g)$ for $f,g \in L^{2}(E,H; \mu)$ by
\[
\text{Cov}_{\mu}(f,g) := \int_{E} \langle f(x) - \langle \mu, f \rangle, g(x) - \langle \mu, g \rangle \rangle_{H} \mu(dx)
\]
\[
= \langle \mu, (f,g)_{H} \rangle - \langle \langle \mu, f \rangle, \langle \mu, g \rangle \rangle_{H}
\]
\[
= \frac{1}{2} \int_{E} \int_{E} \langle f(y) - f(x), g(y) - g(x) \rangle_{H} \mu(dx) \mu(dy),
\]
where $\langle \mu, f \rangle := \int_{H} f(x) \mu(dx) \in H$ is the barycenter of $f_{\#} \mu \in \mathcal{P}^{2}(Y)$.

Definition 3.13 (Energy of Maps). Take $m \in \mathcal{P}(E)$ and let $X$ be an $m$-symmetric Markov chain and $(Y,d_Y)$ is a metric space. For $u \in L^{p}(E,Y;m)$,
\[
E^{p}(u) := \frac{1}{2} \int_{E} \int_{E} d_{Y}^{p}(u(y), u(x)) P(x, dy)m(dx)
\]
is said to be $p$-energy of $u$ with respect to $X$ and
\[
E_{*}^{p}(u) := \frac{1}{2} \int_{E} d_{Y}^{p}(Pu(x), u(x))m(dx) = \frac{1}{2} d_{L^{P}}^{p}(Pu, u)
\]
is said to be quasi $p$-energy of $u$ with respect to $X$ for $p \geq 1$ when $(Y, d_Y)$ is a complete separable CAT(0)-space.

When $p = 2$, we simply say energy (resp. quasi 2-energy) and write $E(u) := E^2(u)$ (resp. $E_*(u) := E_*^2(u)$). Since

$$\text{Var}_P^p(u) \leq \int_E d_Y^p(u(y), u(x))P(x, dy),$$

we see

$$\frac{1}{2} \int_E \text{Var}_P^p(u) m(dx) \leq E^p(u).$$

We use

$$\left\{ \begin{array}{ll} D(E^p) := \{ u \in L^p(E, Y ; m) \mid E^p(u) < \infty \} \\ E^p(u) := \frac{1}{2} \int_E \int_E d_Y^p(u(y), u(x))P(x, dy)m(dx), & u \in D(E^p). \end{array} \right.$$ 

When $(Y, d_Y)$ is a Hilbert space $H$, we use the symbol $\mathcal{E}$ instead of $E$ for the (2-)energy on $L^2(E, H ; m)$ and set

$$\mathcal{E}(f, g) := \frac{1}{2} \int_{E \times E} \langle f(y) - f(x), g(y) - g(x) \rangle_{H} P_x(dy)m(dx)$$

for $f, g \in D(\mathcal{E})$. We see $\mathcal{E}(f) = \mathcal{E}(f, f)$ for $f \in L^2(E, H ; m)$.

**Proposition 3.14.** Let $X$ be an $m$-symmetric Markov chain on $(E, d)$ and $(Y, d_Y)$ is a metric space. Fix $p \in [1, \infty]$. For measurable maps $u, v : E \to Y$, the following inequalities hold:

$$E^p(u)^{\frac{1}{p}} \leq E^p(v)^{\frac{1}{p}} + 2^{1-\frac{1}{p}}d_{L^p}(u, v),$$

$$\text{Var}_m^p(u)^{\frac{1}{p}} \leq \text{Var}_m^p(v)^{\frac{1}{p}} + d_{L^p}(u, v),$$

$$\overline{\text{Var}}_m^p(u)^{\frac{1}{p}} \leq \overline{\text{Var}}_m^p(v)^{\frac{1}{p}} + 2^{1-\frac{1}{p}}d_{L^p}(u, v).$$

**Corollary 3.15.** Let $X$ be an $m$-symmetric Markov chain on $(E, d)$. Suppose that $(Y, d_Y)$ is a complete separable CAT(0)-space. For $p \geq 1$ and $u \in L^p(E, Y ; m)$, the following inequalities hold:

$$\text{Var}_m^p(u)^{\frac{1}{p}} \leq \text{Var}_m^p(Pu)^{\frac{1}{p}} + 2^\frac{1}{p} E^p(u)^{\frac{1}{p}},$$

$$\overline{\text{Var}}_m^p(u)^{\frac{1}{p}} \leq \overline{\text{Var}}_m^p(Pu)^{\frac{1}{p}} + 2E^p(u)^{\frac{1}{p}}.$$

If $u \in L^2(E, Y ; m)$, we have

$$E_*^2(u) \leq 4E^2(u).$$

**Corollary 3.16 (Lower Semi Continuity of Energy).** Let $X$ be an $m$-symmetric Markov chain with $m \in \mathcal{P}(E)$ and $(Y, d_Y)$ a metric space. Take $p \geq 1$ and let $(E^p, D(E^p))$ be the $p$-energy on $L^p(E, Y ; m)$ associated with $X$. We set $E^p(u) := \infty$ for $u \in L^p(E, Y ; m) \setminus D(E^p)$. Then $E^p$ is a $[0, \infty]$-valued lower semi continuous functional on $L^p(E, Y ; m)$. 

Remark 3.17. When $p = 2$, $Y = \mathbb{R}$, the lower semi continuity of energy is equivalent to the completeness of $D(\mathcal{E})$ with respect to the norm $\|\cdot\|_{\mathcal{E}_1}$ defined by $\|f\|_{\mathcal{E}_1} := \sqrt{\mathcal{E}_1(f, f)}$. Here $\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + (f, g)_{m}$, $f, g \in D(\mathcal{E})$.

**Lemma 3.18 (Contraction Property).** Let $X$ be an $m$-symmetric Markov chain on $(E, d)$ with $m \in \mathcal{P}(E)$. Fix $p \geq 1$. Let $(Y, d_Y)$ be a complete separable $CAT(0)$-space. Then, for any $u \in L^p(E, Y; m)$, we have

\begin{align}
(3.9) \quad \Var_{m}^{p}(Pu) & \leq \Var_{m}^{p}(u), \\
(3.10) \quad \overline{\Var}_{m}^{p}(Pu) & \leq \overline{\Var}_{m}^{p}(u).
\end{align}

4. **Main Results**

In this section, we fix $p \geq 1$ and assume $m \in \mathcal{P}(E)$ and $\text{supp}[m] = E$.

**Theorem 4.1 (Non-linear Spectral Radius of $P$ on $L^p(E, Y; m)/\{\text{const}\}$).** Let $X$ be an $m$-symmetric Markov chain on $(E, d)$ with $m \in \mathcal{P}^p(E)$ and assume $\kappa \in \mathbb{R}$. Let $(Y, d_Y)$ be a complete separable $CAT(0)$-space. Then, we have

\begin{align}
(4.1) \quad \lim_{\ell \to \infty} \left( \sup_{u \in L^p(E, Y; m)} \frac{\Var_{m}^{p}(P^\ell u)}{\Var_{m}^{p}(u)} \right)^{\frac{1}{p\ell}} & \leq \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1,
\end{align}

\begin{align}
(4.2) \quad \lim_{\ell \to \infty} \left( \sup_{u \in L^p(E, Y; m)} \frac{\overline{\Var}_{m}^{p}(P^\ell u)}{\Var_{m}^{p}(u)} \right)^{\frac{1}{p\ell}} & \leq \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1.
\end{align}

**Corollary 4.2 (Linear Spectral Radius of $P$ on $L^2(E, H; m)/\{\text{const}\}$).** Let $X$ be an $m$-symmetric Markov chain on $(E, d)$ and $H$ a real separable Hilbert space. Assume $m \in \mathcal{P}^2(E)$ and $\kappa \in \mathbb{R}$. Then, we have

\begin{align}
(4.3) \quad \lim_{\ell \to \infty} \left( \sup_{f \in L^2(E, H; m)} \frac{\Var_{m}(P^\ell f)}{\Var_{m}(f)} \right)^{\frac{1}{2\ell}} & \leq \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1.
\end{align}

Consequently, $P$ is an $\inf_{n \in \mathbb{N}}(1 - \kappa_n)^{\frac{1}{n}} \wedge 1$-contraction operator on $L^2(E, H; m)/\{\text{const}\}$. In particular, for $f \in L^2(E, H; m)/\{\text{const}\}$, the following hold:

\begin{align}
(4.4) \quad \Var_{m}(Pf) & \leq (\inf_{n \in \mathbb{N}}(1 - \kappa_n)^{\frac{2}{n}} \wedge 1)\Var_{m}(f), \\
(4.5) \quad |\Cov_{m}(Pf, f)| & \leq (\inf_{n \in \mathbb{N}}(1 - \kappa_n)^{\frac{1}{n}} \wedge 1)\Var_{m}(f).
\end{align}

The main part of the following theorem is a slight generalization of [22, Corollary 31], and its proof is similar as in [22] based on Corollary 4.2 above.
Theorem 4.3 (Poincaré Inequality, cf. Corollary 31 in [22]). Assume $m \in \mathcal{P}^2(E)$ and $\kappa \in \mathbb{R}$. Let $X$ be an $m$-symmetric Markov chain on $(E,d)$ and $H$ a real separable Hilbert space. Then, for $f \in L^2(E, H;m)$

\begin{align}
(4.6) \quad (1 - \inf_{n\in\mathbb{N}} (1 - \kappa_n)^{\frac{2}{n}} \wedge 1) \text{Var}_m(f) \leq \int_E \text{Var}_{P_x}(f)m(dx), \\
(4.7) \quad 1 - \inf_{n\in\mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1 \leq \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \leq 1 + \inf_{n\in\mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1.
\end{align}

In particular, if $\kappa_n > 0$ for some $n \in \mathbb{N}$, we have a global Poincaré inequality:

\[ 0 < 1 - \inf_{n\in\mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \leq \inf_{f\in L^2(E, H;m)} \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \leq 1 + \inf_{n\in\mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} < 2. \]

Moreover, if $X$ is an even step Markov chain obtained from an $m$-symmetric Markov chain, then

\[ \sup_{f\in L^2(E, H;m)} \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \leq 1. \]

Corollary 4.4 (Estimates of Eigenvalues). Let $X$ be an $m$-symmetric Markov chain on $(E,d)$ with $m \in \mathcal{P}^2(E)$ and assume that $\kappa \in \mathbb{R}$ and the embedding $D(\mathcal{E}) \subset L^2(E;m)$ is compact. Then any non-zero eigenvalue $\lambda$ of the $L^2$-operator $-\Delta = I - P$ on $L^2(E;m)$ satisfies

\begin{align}
(4.9) \quad 1 - \inf_{n\in\mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1 \leq \lambda \leq 1 + \inf_{n\in\mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1.
\end{align}

Moreover, if $X$ is an even step Markov chain obtained from an $m$-symmetric Markov chain, then any eigenvalue $\lambda$ satisfies $\lambda \leq 1$.

Corollary 4.5 (Recurrence). Assume $m \in \mathcal{P}^2(E)$ and $\kappa \in \mathbb{R}$. Let $X$ be an $m$-symmetric Markov chain on $(E,d)$. Suppose that there exists $n \in \mathbb{N}$ such that $\kappa_n > 0$. Then $X$ is recurrent, that is, for any non-trivial $f \in L^1_+(E;m)$, we have $Gf = \infty$ m-a.e. Here $Gf := \sum_{i=0}^{\infty} P^i f$.

Remark 4.6. (1) When $X$ is an $m$-symmetric random walk on a finite undirected weighted connected graph $G = (V,E)$ with $m\{x\} := d_x(G)$, the degree of $G$ at $x \in V$, Bauer-Jost-Liu [4] proved (4.9) for any $n \in \mathbb{N}$.

(2) If we assume the existence of non-constant Lipschitz eigenfunction of $-\Delta := I - P$, then we can directly prove the estimate for the associated real eigenvalue $\lambda$;

\begin{align}
(4.10) \quad 1 - (1 - \kappa_n)^{\frac{1}{n}} \leq \lambda \leq 1 + (1 - \kappa_n)^{\frac{1}{n}}
\end{align}

under $\kappa_n(x,y) \geq \kappa_n(\in \mathbb{R})$ for $(x,y) \in E \times E \setminus \text{diag}$ without assuming the $m$-symmetricity of $X$. If $\kappa_n > 0$, (4.10) is equivalent to (4.9). We show (4.10) as mentioned above. Let $f$
be a non-constant Lipschitz eigenfunction and assume that \( \lambda \) is a real eigenvalue of \( f \) with respect to \(-\Delta\). Then, we have 
\[(f - P)f = \lambda f, \text{ equivalently, } P^k f = (1 - \lambda)^k f \text{ for any } k \in \mathbb{N}.\]
By scaling, we may assume that the Lipschitz constant of \( f \) is 1. Kantorovich-Rubinstein duality formula yields
\[
d(x, y)(1 - \kappa_n) \geq d_{W_1}(P^n_x, P^n_y) \geq P^n f(x) - P^n f(y) \\
= (1 - \lambda)^n(f(x) - f(y))
\]
for \((x, y) \in E \times E\), which implies \((1 - \kappa_n) \geq |1 - \lambda|^n\), that is, we obtain (4.10).

**Theorem 4.7** (Strong \( L^p\)-Liouville Property). Assume \( m \in \mathcal{P}^p(E) \). Let \( X \) be an \( m \)-symmetric Markov chain on \((E, d)\). Suppose that \( \kappa \in \mathbb{R} \) and there exists \( n \in \mathbb{N} \) such that \( \kappa_n > 0 \). Let \((Y, d_Y)\) be a complete separable \( \text{CAT}(0)\)-space. Suppose that \( u \in L^p(E, Y; m) \) satisfies \( Pu = u \) \( m \)-a.e. on \( E \). Then \( u \) is a constant map \( m \)-a.e. In particular, if \( u \in \text{Lip}(E, Y) \) is \( P\)-harmonic, then \( u \) is a constant map.

**Corollary 4.8** (Ergodicity). Let \( X \) be an \( m \)-symmetric Markov chain on \((E, d)\). Suppose that \( \kappa \in \mathbb{R} \) and there exists \( n \in \mathbb{N} \) such that \( \kappa_n > 0 \). Then \( X \) is ergodic, that is, for any \( P \)-invariant Borel set \( A \), \( m(A) = 0 \) or \( m(A^c) = 0 \).

**Theorem 4.9** (Poincaré Inequality). Assume \( m \in \mathcal{P}^2(E) \) and \( \kappa \in \mathbb{R} \). Let \( X \) be an \( m \)-symmetric Markov chain on \((E, d)\). Suppose that there exists \( n \in \mathbb{N} \) such that \( \kappa_n > 0 \). Let \((Y, d_Y)\) be a complete separable \( \text{CAT}(0)\)-space. Then for any \( \epsilon \in ]0, 1 - (1 - \kappa_n)^{\frac{1}{2}}[\), there exists \( \ell_0 \in \mathbb{N} \) depending on \( \epsilon, \kappa_n, (E, d, m, X) \) and \((Y, d_Y)\) such that
\[
\inf_{u \in L^2(E, Y; m)} \frac{E(u)}{\text{Var}_m(u)} \geq \frac{(1 - (1 - \kappa_n)^{\frac{1}{2}} \wedge 1 - \epsilon)^2}{8\ell_0^2} > 0.
\]

**Remark 4.10.** (1) For the random walk on an undirected weighted finite graph \( G = (V, E) \) with \( N := |V| \), Bauer-Jost-Liu [4] proved the equivalence among the following:
(i) \( G \) is non-bipartite.
(ii) \( \lambda_{N-1} < 2 \).
(iii) There exists \( n \in \mathbb{N} \) such that \( \kappa_n > 0 \).
Since \( G \) is connected, we have \( \lambda_1 > 0 \). Here \( \lambda_1 \) (resp. \( \lambda_{N-1} \)) is the smallest non-zero (resp. maximum) eigenvalue of the Laplace operator on \( G \). Under the equivalent conditions (i)-(iii), we have a positivity of non-linear spectral gap as in Theorem 4.9. Remark that the positivity of non-linear spectral gap on the finite connected weighted graph \( G \) (having no loop and no multi-edges) with graph distance is already proved by Izeki-Kondo-Nayatani [7] for \( \text{CAT}(0)\)-space target.
(2) Our Theorem 4.9 covers the case for the random walk derived from the Brownian motion on Riemannian manifolds with positive Ricci curvature as in Example 1.5.

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EIKI KOKUBO
HINODE-CHO 7-37, KASUGA CITY
FUKUOKA, 816-0873
JAPAN
E-mail address: city-hunter-xxyyzz@hotmail.co.jp

KAZUHIRO KUWAE
DEPARTMENT OF MATHEMATICS AND ENGINEERING
GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY
KUMAMOTO UNIVERSITY
KUMAMOTO, 860-8555
JAPAN
E-mail address: kuwae@gpo.kumamoto-u.ac.jp