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UNIQUENESS OF POSITIVE RADIAL SOLUTIONS OF
$\Delta u + g(r)u + h(r)u^p = 0$ AND ITS APPLICATIONS

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1. INTRODUCTION AND MAIN RESULTS

We consider the problem

\[ \begin{cases} u_{rr} + \frac{n-1}{r} u_r + g(r)u + h(r)u^p = 0, & 0 < r < R, \\
 u(0) \in (0, \infty), & u(R) = 0, \end{cases} \tag{1.1} \]

where $n \geq 2$, $R \in (0, \infty]$, $p \in (1, \infty)$ and $g, h : (0, R) \to \mathbb{R}$ are appropriate functions. Here, $u(R) = 0$ in the case $R = \infty$ means $u(x) \to 0$ as $|x| \to \infty$. Such a problem has been studied by many researchers; see [1, 3, 5, 8, 9, 12-18, 20-27, 30, 32-36] and others.

In this note, we introduce a result obtained in [28].

**Theorem 1.** Let $0 < R \leq \infty$, $n \in \mathbb{R}$ with $n \geq 2$ and $p \in (1, \infty)$. Let $g \in C([0, R]) \cap C^1((0, R))$ and $h \in C^2([0, R]) \cap C^3((0, R))$ such that $h$ is positive on $[0, R)$. Assume the following.

(i) In the case of $R < \infty$, $g \in C([0, R])$, $h \in C^2([0, R])$ and $h(R) > 0$ are also satisfied.

(ii) There exists $\kappa \in [0, R]$ such that

\[ G(r) \geq 0 \text{ on } (0, \kappa) \quad \text{and} \quad G(r) \leq 0 \text{ on } (\kappa, R), \]

where

\[ G(r) = \frac{2(n-1)(p+1)(n+2-(n-2)p)[n-4+(n-2)p]}{2(p+3)^3h(r)^{2/3}+3} \left(4(n-1)[n+2-(n-2)p][n-4+(n-2)p]h(r)^3 \right. \]
\[ + \left[2(n-1)(p-1)(p+3)^2r^2h(r)^3 - 4(p+3)^2r^3h(r)^2h_r(r) \right]g(r) \]
\[ + (p+3)^3r^3g_r(r)h(r)^3 \]
\[ + (n-1)[(2n-3)p(6-p)+6n-33]rh(r)^2h_r(r) \]
\[ + 3(n-1)(p-1)(p+5)r^2h(r)h_r(r)^2 - 2(p+4)(p+5)r^3h_r(r)^3 \]
\[-3(n-1)(p-1)(p+3)r^2h(r)^2h_{rr}(r)\]
\[+3(p+3)(p+5)r^3h(r)h_{r}(r)h_{rr}(r) - (p+3)^2r^2h(r)^2h_{rrr}(r)\).

(iii) In the case of $R = \infty$, $G^- \neq 0$ is satisfied.

Then in the case of $R < \infty$, problem (1.1) has at most one positive solution, and in the case of $R = \infty$, problem (1.1) has at most one positive solution $u$ which satisfies $J(r; u) \to 0$ as $r \to \infty$, where

\[
a(r) = r \frac{2(n-1)(p+1)}{p+3} h(r)^{\frac{2}{p+3}},
\]
\[b(r) = \frac{2(n-1)(p+1)}{(p+3)h(r)^{\frac{2}{p+3}}} \left(2(n-1)h(r) + rh_{r}(r)\right),
\]
\[c(r) = \frac{2(n-1)(p+1)}{(p+3)^2h(r)^{\frac{2(p+4)}{p+3}}} \left(2(n-1)[n^2 + (n-2)p]h(r)^2 + (p+5)r^2h_{r}(r)^2
\]
\[\quad - (n-1)(p-5)rh(r)h_{r}(r) - (p+3)r^2h(r)h_{rr}(r)\right),
\]
\[J(r; u) = \frac{1}{2} a(r)u_{r}(r)^2 + b(r)u_{r}(r)u(r) + \frac{1}{2} c(r)u(r)^2
\]
\[+ \frac{1}{2} a(r)g(r)u(r)^2 + \frac{1}{p+1} a(r)h(r)u(r)^{p+1}.
\]

Remark 1. In [32, Theorems 2.1 and 2.2], Yanagida obtained a closely related result.

By the theorem above, we can obtain the following; see [13, Theorem 0.1].

Corollary 1 (Kabeya-Tanaka). Let $n \in \mathbb{N}$ with $n \geq 2$. Let $p > 1$ and $g \in C^2([0, \infty))$ such that $-\infty < \inf_{r \in [0, \infty)} g(r) \leq \sup_{r \in [0, \infty)} g(r) < 0$, and set

\[L = \frac{2(n-1)(n-2)p + n + 4}{(p+3)^2} \quad \text{and} \quad \beta = \frac{2(n-1)(p-1)}{p+3}.
\]

Assume that

\[g_r(r)r^3 + \beta g(r)r^2 - (\beta - 2)L < 0 \quad \text{for each} \ r \geq 0
\]

in the case of $n = 2$, and that $p < (n+2)/(n-2)$ and

\[
\sup_{r > 0} \left(g_{rr}(r)r^2 + (3 + \beta)g_{r}(r)r + 2\beta g(r)\right) < 0
\]

in the case of $n \geq 3$. Then the problem

\[
(1.2) \quad u \in H^1(\mathbb{R}^n), \quad \Delta u(x) + g(|x|)u(x) + u(x)^p = 0 \quad \text{in} \ \mathbb{R}^n
\]

has a unique positive radial solution.
Next, we consider the problem

\[
\begin{cases}
  u_{rr}(r) + \frac{n-1}{r} u_r + g(r)u(r) + h(r)u(r)^p = 0, & R' < r < R, \\
  u(R') = 0, & u(R) = 0.
\end{cases}
\] (1.3)

The uniqueness of a positive solution of such a problem was studied in [4, 6, 7, 10, 11, 19, 24, 29–31].

The following is also obtained in [28].

**Theorem 2.** Let \(0 < R' < R \leq \infty, n \in \mathbb{R}, p \in (1, \infty), g \in C([R', R]) \cap C^1((R', R)), h \in C^2([R', R]) \cap C^3((R', R))\) such that \(h\) is positive on \([R', R)\). Let \(a, b, c, G\) and \(J\) be the functions given in Theorem 1. Assume the following.

(i) In the case of \(R < \infty, g \in C([R', R]), h \in C^2([R', R])\) and \(h(R) > 0\) are also satisfied.

(ii) There exists \(\kappa \in [R', R]\) such that

\[G(r) \geq 0 \text{ on } (R', \kappa) \quad \text{and} \quad G(r) \leq 0 \text{ on } (\kappa, R).\]

Then in the case of \(R < \infty\), problem (1.3) has at most one positive solution, and in the case of \(R = \infty\), problem (1.3) has at most one positive solution \(u\) which satisfies \(J(r;u) \to 0\) as \(r \to \infty\).

**Remark 2.** For the case \(h(r) \equiv 1\), a similar result is obtained by Felmer-Martínez-Tanaka; see [10, Theorem 1.1].

2. Applications

In this section, we give examples of Theorem 1. First, we give a comment on the scalar field equation

\[\Delta u(x) - u(x) + u(x)^p = 0 \quad \text{in } \mathbb{R}^n, \quad u(x) \to 0 \quad \text{as } |x| \to \infty.\]

The unique existence of its positive solution was established by Kwong [18]. Since the uniqueness of its positive solution can be derived from Corollary 1, of course, it can be also done by Theorem 1.

Next, we consider the following Brezis-Nirenberg problem.

\[
\begin{cases}
  \Delta_{S^n} u + \lambda u + u^p = 0 & \text{in } D, \\
  u = 0 & \text{on } \partial D.
\end{cases}
\] (2.1)

Here, \(n\) is a natural number with \(n \geq 3\), \(S^n\) is the unit sphere in \(\mathbb{R}^{n+1}\), \(\Delta_{S^n}\) is the Laplace-Beltrami operator on \(S^n\), \(D = \{X \in S^n : X_{n+1} > \cos \theta_1\}\) with \(\theta_1 \in (0, \pi),\)
1 < p \leq \frac{(n+2)/(n-2)}{\lambda < \lambda_1},\text{ where } \lambda_1 \text{ is the first eigenvalue of } -\Delta_{S^n} \text{ on } D \text{ with the Dirichlet boundary condition.}

Let \( P : S^n \setminus \{(0, \ldots, 0, -1)\} \rightarrow \mathbb{R}^n \) be the stereographic projection defined by

\[
P(X_1, \ldots, X_n, X_{n+1}) = \frac{1}{X_{n+1} + 1}(X_1, \ldots, X_n) \quad \text{for } X \in S^n \setminus \{(0, \ldots, 0, -1)\}.
\]

Then we can see \( P(D) = B_R \), where \( B_R = \{x \in \mathbb{R}^n : |x| < R\} \) with

\[
R = \frac{\sin \theta_1}{1 + \cos \theta_1}.
\]

Let \( u \) be a positive solution of (2.1) and define \( v : \overline{B_R} \rightarrow \mathbb{R} \) by \( u(P^{-1}x) = (1+|x|^2)^{(n-2)/2} v(x) \) for \( x \in \overline{B_R} \). Then we can see that \( v \) is a positive solution of

\[
\begin{cases}
\Delta v + \frac{n(n-2) + 4\lambda}{(1+|x|^2)^2} v + 4(1+|x|^2)^{\frac{(n-2)p-(n+2)}{2}} v^p = 0 \quad \text{in } B_R, \\
v = 0 \quad \text{on } \partial B_R.
\end{cases}
\]

We set

\[
g(r) = \frac{n(n-2) + 4\lambda}{(1+r^2)^2} \quad \text{and} \quad h(r) = 4(1+r^2)^{\frac{(n-2)p-(n+2)}{2}} \quad \text{for } r \geq 0.
\]

We can see that \( G \) in Theorem 1 is given by

\[
G(r) = \frac{2^{\frac{n-1}{p+3}}(n-1)r^{2(n-1)(p+1)-3}(1+r^2)^{\frac{n+2-(n-2)p}{p+3}-3}(1-r^2)(Ar^4 + Br^2 + A)},
\]

where

\[
A = (n-2)^2\left(\frac{n+2}{n-2} - p\right)\left(\frac{n + 4}{n - 2}\right),
\]
\[
B = (p+3)[-6n^2 + 12n + (2n^2 + 4\lambda - 4)p + 2\lambda p^2 - 6\lambda - 12] + 16(n-1)^2.
\]

Then we can infer the following. For the details, see [28].

**Theorem 3.** Let \( n \in \mathbb{N} \) with \( n \geq 3 \), \( 1 < p \leq \frac{(n+2)/(n-2)}{\theta_1 \in (0, \pi/2]} \) and \( \lambda < \lambda_1 \), \( \theta_1 \in (0, \pi) \). Assume that one of the following conditions:

(i) \( \theta_1 \in (0, \pi/2] \) and \( \lambda < \lambda_1 \),

(ii) \( \theta_1 \in (\pi/2, \pi) \) and

\[
\frac{6 + (6 - 4n)p}{(p + 3)(p - 1)} \leq \lambda < \lambda_1.
\]

Then (2.1) has at most one positive radial solution. Moreover, if \( \lambda \geq -n(n-2)/4 \) is also satisfied, then (2.1) has at most one positive solution.
Remark 3. It holds that
\[
\frac{6 + (6 - 4n)p}{(p + 3)(p - 1)} \leq \frac{-n(n - 2)}{4},
\]
and if \( p = (n + 2)/(n - 2) \) then the constants in the both sides in the inequality above coincide.

Remark 4. In the case of \( n = 3 \), Bandle-Benguria obtained a sharper result. For the details, see [2].

Remark 5. In the case of \( R > 1 \), we cannot apply Yanagida’s uniqueness theorem [32, Theorem 2.1]. Indeed, by his notation, we have
\[
G(r; n - 2) = \frac{2(4\lambda + n(n - 2))r^{n-1}(1-r^2)}{(r^2 + 1)^3}.
\]
So one of his assumptions \( G(r; n - 2) \leq 0 \) on \((0, R)\) is not satisfied even if \( \lambda > -n(n-2)/4 \).

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