Title: UNIQUENESS OF POSITIVE RADIAL SOLUTIONS OF

\[ \Delta u + g(r)u + h(r)u^p = 0 \]

and its applications (Global qualitative theory of ordinary differential equations and its applications)

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UNIQUENESS OF POSITIVE RADIAL SOLUTIONS OF
\[\Delta u + g(r)u + h(r)u^p = 0\]
AND ITS APPLICATIONS

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1. INTRODUCTION AND MAIN RESULTS

We consider the problem
\[
\begin{cases}
  u_{rr} + \frac{n-1}{r} u_r + g(r)u + h(r)u^p = 0, & 0 < r < R,
  \\
  u(0) \in (0, \infty), & u(R) = 0,
\end{cases}
\]
where \( n \geq 2, R \in (0, \infty), p \in (1, \infty) \) and \( g, h : (0, R) \to \mathbb{R} \) are appropriate functions.
Here, \( u(R) = 0 \) in the case \( R = \infty \) means \( u(x) \to 0 \) as \( |x| \to \infty \). Such a problem has been studied by many researchers; see [1, 3, 5, 8, 9, 12-18, 20-27, 30, 32-36] and others.

In this note, we introduce a result obtained in [28].

**Theorem 1.** Let \( 0 < R \leq \infty, n \in \mathbb{R} \) with \( n \geq 2 \) and \( p \in (1, \infty) \). Let \( g \in C([0, R]) \cap C^1((0, R)) \) and \( h \in C^2([0, R]) \cap C^3((0, R)) \) such that \( h \) is positive on \((0, R)\). Assume the following.

(i) In the case of \( R < \infty \), \( g \in C([0, R]), h \in C^2([0, R]) \) and \( h(R) > 0 \) are also satisfied.

(ii) There exists \( \kappa \in [0, R] \) such that
\[
G(r) \geq 0 \text{ on } (0, \kappa) \quad \text{and} \quad G(r) \leq 0 \text{ on } (\kappa, R),
\]
where
\[
G(r) = \frac{2(n-1)(p+1-3)^{-3}}{2(p+3)^3 h(r)^{p+3}} \left( 4(n-1)[n+2-(n-2)p][n-4+(n-2)p]h(r)^3 
+ [2(n-1)(p-1)(p+3)^2 r^2 h(r)^3 - 4(p+3)^2 r^3 h(r)^2 h_r(r)]g(r)
+ (p+3)^3 r^3 g_r(r)h(r)^3 
+ (n-1)[(2n-3)p(6-p) + 6n - 33] rh(r)^2 h_r(r) 
+ 3(n-1)(p-1)(p+5)r^2 h(r)h_r(r)^2 - 2(p+4)(p+5)r^3 h_r(r)^3 \right).
\]
\[ -3(n-1)(p-1)(p+3)r^2 h(r)^2 h_{rr}(r) \]
\[ +3(p+3)(p+5)r^3 h(r)h_r(r)h_{rr}(r) - (p+3)^2 r^2 h(r)^2 h_{rrr}(r) \].

(iii) In the case of \( R = \infty \), \( G^- \not\equiv 0 \) is satisfied.

Then in the case of \( R < \infty \), problem (1.1) has at most one positive solution, and in the case of \( R = \infty \), problem (1.1) has at most one positive solution \( u \) which satisfies \( J(r; u) \to 0 \) as \( r \to \infty \), where
\[
J(r; u) = \frac{1}{2} a(r) u_r(r)^2 + b(r) u_r(r) u(r) + \frac{1}{2} c(r) u(r)^2 + \frac{1}{2} a(r) g(r) u(r)^2 + \frac{1}{p+1} a(r) h(r) u(r)^{p+1}.
\]

Remark 1. In [32, Theorems 2.1 and 2.2], Yanagida obtained a closely related result.

By the theorem above, we can obtain the following; see [13, Theorem 0.1].

Corollary 1 (Kabeya-Tanaka). Let \( n \in \mathbb{N} \) with \( n \geq 2 \). Let \( p > 1 \) and \( g \in C^2([0, \infty)) \) such that \( -\infty < \inf_{r \in [0, \infty)} g(r) \leq \sup_{r \in [0, \infty)} g(r) < 0 \), and set
\[
L = \frac{2(n-1)[(n-2)p+n-4]}{(p+3)^2} \quad \text{and} \quad \beta = \frac{2(n-1)(p-1)}{p+3}.
\]
Assume that
\[
g_r(r)^2 + \beta g(r) r^2 - (\beta - 2)L < 0 \quad \text{for each} \quad r \geq 0
\]
in the case of \( n = 2 \), and that \( p < (n+2)/(n-2) \) and
\[
\sup_{r>0} \left( g_{rr}(r) r^2 + (3 + \beta) g_r(r) r + 2 \beta g(r) \right) < 0
\]
in the case of \( n \geq 3 \). Then the problem
\[
(1.2) \quad u \in H^1(\mathbb{R}^n), \quad \Delta u(x) + g(|x|) u(x) + u(x)^p = 0 \quad \text{in} \ \mathbb{R}^n
\]
has a unique positive radial solution.
Next, we consider the problem
\begin{equation}
\begin{cases}
u_{rr}(r) + \frac{n-1}{r}u_r + g(r)u(r) + h(r)u(r)^p = 0, \quad R' < r < R, \\
u(R') = 0, \quad u(R) = 0.
\end{cases}
\end{equation}
(1.3)

The uniqueness of a positive solution of such a problem was studied in [4, 6, 7, 10, 11, 19, 24, 29–31].

The following is also obtained in [28].

**Theorem 2.** Let $0 < R' < R \leq \infty$, $n \in \mathbb{R}$, $p \in (1, \infty)$, $g \in C([R', R]) \cap C^1((R', R))$, $h \in C^2([R', R]) \cap C^3((R', R))$ such that $h$ is positive on $[R', R)$. Let $a$, $b$, $c$, $G$ and $J$ be the functions given in Theorem 1. Assume the following.

(i) In the case of $R < \infty$, $g \in C([R', R])$, $h \in C^2([R', R])$ and $h(R) > 0$ are also satisfied.

(ii) There exists $\kappa \in [R', R]$ such that

\[
G(r) \geq 0 \text{ on } (R', \kappa) \quad \text{and} \quad G(r) \leq 0 \text{ on } (\kappa, R).
\]

Then in the case of $R < \infty$, problem (1.3) has at most one positive solution, and in the case of $R = \infty$, problem (1.3) has at most one positive solution $u$ which satisfies $J(r; u) \to 0$ as $r \to \infty$.

**Remark 2.** For the case $h(r) \equiv 1$, a similar result is obtained by Felmer-Martínez-Tanaka; see [10, Theorem 1.1].

2. Applications

In this section, we give examples of Theorem 1. First, we give a comment on the scalar field equation
\[
\Delta u(x) - u(x) + u(x)^p = 0 \quad \text{in } \mathbb{R}^n, \quad u(x) \to 0 \quad \text{as } |x| \to \infty.
\]

The unique existence of its positive solution was established by Kwong [18]. Since the uniqueness of its positive solution can be derived from Corollary 1, of course, it can be also done by Theorem 1.

Next, we consider the following Brezis-Nirenberg problem.
\begin{equation}
\begin{cases}
\Delta_{S^n} u + \lambda u + u^p = 0 \quad \text{in } D, \\
u = 0 \quad \text{on } \partial D.
\end{cases}
\end{equation}
(2.1)

Here, $n$ is a natural number with $n \geq 3$, $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$, $\Delta_{S^n}$ is the Laplace-Beltrami operator on $S^n$, $D = \{X \in S^n : X_{n+1} > \cos \theta_1\}$ with $\theta_1 \in (0, \pi)$,
1 < p \leq (n + 2)/(n - 2) and \lambda < \lambda_1, where \lambda_1 is the first eigenvalue of \(-\Delta_{S^n}\) on D with the Dirichlet boundary condition.

Let \(P : S^n \setminus \{(0, \ldots, 0, -1)\} \rightarrow \mathbb{R}^n\) be the stereographic projection defined by

\[ P(X_1, \ldots, X_n, X_{n+1}) = \frac{1}{X_{n+1} + 1}(X_1, \ldots, X_n) \quad \text{for} \quad X \in S^n \setminus \{(0, \ldots, 0, -1)\}. \]

Then we can see \(P(D) = B_R\), where \(B_R = \{x \in \mathbb{R}^n : |x| < R\}\) with

\[ R = \frac{\sin \theta_1}{1 + \cos \theta_1}. \]

Let \(u\) be a positive solution of (2.1) and define \(v : \overline{B_R} \rightarrow \mathbb{R}\) by \(u(P^{-1}x) = (1 + |x|^2)^{\frac{n-2}{2}}v(x)\) for \(x \in \overline{B_R}\). Then we see that \(v\) is a positive solution of

\[ \begin{cases} \Delta v + \frac{n(n-2)+4\lambda}{(1+|x|^2)^2}v + 4(1+|x|^2)^{(n-2)p-(n+2)\frac{1}{2}}v^p = 0 & \text{in} \ B_R, \\ v = 0 & \text{on} \ \partial B_R. \end{cases} \]

We set

\[ g(r) = \frac{n(n-2)+4\lambda}{(1+r^2)^2} \quad \text{and} \quad h(r) = 4(1+r^2)^{(n-2)p-(n+2)\frac{1}{2}} \quad \text{for} \ r \geq 0. \]

We can see that \(G\) in Theorem 1 is given by

\[ G(r) = \frac{2^{p+1}}{(p+3)^3} \frac{(n-1)}{r} \frac{2(n-1)^2}{(p+1)^3} - 3(1+r^2)^{n+2-(n-2)p-(n+2)\frac{1}{2}}(1-r^2)(Ar^4 + Br^2 + A), \]

where

\[ A = (n-2)^2 \left( \frac{n+2}{n-2} - p \right) \left( p + \frac{n-4}{n-2} \right) = (p+3)[3n^2 - 6n - (n^2 - 4n + 4)p] - 8(n-1)^2, \]

\[ B = (p+3)[-6n^2 + 12n + (2n^2 + 4\lambda - 4)p + 2\lambda p^2 - 6\lambda - 12] + 16(n-1)^2. \]

Then we can infer the following. For the details, see [28].

**Theorem 3.** Let \(n \in \mathbb{N}\) with \(n \geq 3\), \(1 < p \leq (n + 2)/(n - 2)\) and \(\theta_1 \in (0, \pi)\). Assume that one of the following conditions:

(i) \(\theta_1 \in (0, \pi/2]\) and \(\lambda < \lambda_1\),

(ii) \(\theta_1 \in (\pi/2, \pi)\) and

\[ \frac{6 + (6-4n)p}{(p+3)(p-1)} \leq \lambda < \lambda_1. \]

Then (2.1) has at most one positive radial solution. Moreover, if \(\lambda \geq -n(n-2)/4\) is also satisfied, then (2.1) has at most one positive solution.
Remark 3. It holds that
\[
\frac{6 + (6 - 4n)p}{(p + 3)(p - 1)} \leq -\frac{n(n - 2)}{4},
\]
and if \(p = (n + 2)/(n - 2)\) then the constants in the both sides in the inequality above coincide.

Remark 4. In the case of \(n = 3\), Bandle-Benguria obtained a sharper result. For the details, see [2].

Remark 5. In the case of \(R > 1\), we cannot apply Yanagida’s uniqueness theorem [32, Theorem 2.1]. Indeed, by his notation, we have
\[
G(r; n - 2) = \frac{2(4\lambda + n(n - 2))r^{n-1}(1-r^2)}{(r^2 + 1)^3}.
\]
So one of his assumptions \(G(r; n - 2) \leq 0\) on \((0, R)\) is not satisfied even if \(\lambda > -n(n - 2)/4\).

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