MORE ON OPERATOR MONOTONE FUNCTIONS
(Research on structures of operators via methods in geometry and probability theory)

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MORE ON OPERATOR MONOTONE FUNCTIONS

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ABSTRACT. We investigate some properties of operator monotone functions. In particular, we show that if $f$ is a non-constant operator monotone function on an interval $J$ and $A, B$ are self-adjoint operators with spectra in $J$ such that $A > B$, then $f(A) > f(B)$. As an application we extend the celebrated Löwner–Heinz inequality.

1. INTRODUCTION

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$ equipped with the operator norm $\| \cdot \|$. An operator $A \in \mathbb{B}(\mathcal{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ holds for every $x \in \mathcal{H}$ and then we write $A \geq 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$, we say $A \leq B$ if $B - A \geq 0$. Also for self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$, we say $A \succ B$ if $\langle Ax, x \rangle > \langle Bx, x \rangle$ holds for all non-zero elements $x \in \mathcal{H}$. Also $A > B$ if $A \geq B$ and $A - B$ is invertible.

A continuous real valued function $f$ defined on an interval $J$ is called operator monotone if $A \geq B$ implies $f(A) \geq f(B)$ for all self adjoint operators $A, B$ acting on a Hilbert space with spectra in $J$.

The Löwner theorem says that a function $f$ is operator monotone on an interval $J$ if and only if $f$ has an analytic continuation to the upper half plane $\Pi_+$ such that $f$ maps $\Pi_+$ into itself. If $f(t)$ is an operator monotone function on $(a, b)$, then clearly $f \left( \frac{2t-a-b}{b-a} \right)$ is operator monotone on $(-1, 1)$, so in this paper we study the family of operator monotone functions on $(-1, 1)$.

Let $\mathcal{K}$ denote the family of all operator monotone functions on $(-1, 1)$ such that $f(0) = 0$ and $f'(0) = 1$. Hansen and Pedersen [8] showed that $\mathcal{K}$ is a compact convex subset of the space of all bounded functions on $(-1, 1)$ with pointwise convergence topology and that the extreme points of $\mathcal{K}$ are of the form $f_\lambda(t) = \frac{t}{1-\lambda t}$ with $|\lambda| < 1$. They [8] also proved that every $f \in \mathcal{K}$ can be represented as

$$f(t) = \int_{-1}^{1} \frac{t}{1-\lambda t} d\mu(\lambda),$$

where $\mu$ is a positive measure on $(-1, 1)$, see also [3].

The Löwner–Heinz inequality says that, $f(x) = x^r$ $(0 < r \leq 1)$ is operator monotone on $[0, \infty)$. Löwner proved the inequality for matrices. Heinz proved it for positive
operators acting on a Hilbert space of arbitrary dimension. Based on the $C^*$-algebra theory, Pedersen [14] gave a shorter proof of the inequality.

There exist several operator norm inequalities each of which is equivalent to the Löwner-Heinz inequality. One of them is $\|A^r B^r\| \leq \|AB\|^r$, called the Côrdes inequality in the literature, in which $A$ and $B$ are positive operators and $0 < r \leq 1$. A generalization of the Côrdes inequality for operator monotone functions is given in [5]. It is shown in [2] that this norm inequality is related to the Finsler structure of the space of positive invertible elements.

Kwong [10] sowed that if $A > B$ ($A \succ B$, resp.), then $A^r > B^r$ ($A^r \succ B^r$, ressp.) for $0 < r \leq 1$. Uchiyama [15] showed that for every non-constant operator monotone function $f$ on an interval $J$, $A > B$ implies $f(A) > f(B)$ for all self-adjoint operators $A, B$ with spectra in $J$.

There are several extensions of the Löwner–Heinz inequality. The Furuta inequality [6], which states that if $A \geq B \geq 0$, then for $r \geq 0$, $(A^{r/2} A^p A^{r/2})^{1/q} \geq (A^{r/2} B^p A^{r/2})^{1/q}$ holds for $p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$, is known as an exquisite extension of the Löwner–Heinz inequality; Also Ando [1] extended the Löwner Heinz inequality for a pair of $J$-selfadjoint matrices.

Let $\Omega$ be a open subset of $\mathbb{C}$. A set $\mathcal{F} \subseteq C(\Omega)$ is bounded if for each compact subset $K \subseteq \Omega$, sup$\{\|f\|_K : f \in \mathcal{F}\} < \infty$. The Montel theorem states that if $\mathcal{F}$ is a bounded subset of the set $A(\Omega)$ of all analytic functions on $\Omega$, then $\mathcal{F}$ is a normal family, i.e, each sequence $\{f_n\}$ in $\mathcal{F}$ has a subsequence $\{f_{n_j}\}$ converging uniformly on each compact subset of $\Omega$.

2. THE RESULTS

Throughout this note, let $\Omega = \Pi_+ \cup \Pi_- \cup (-1, 1)$, where $\Pi_-$ is the lower half plan.

**Theorem 2.1.** The family $\mathcal{K}$ is bounded in $A(\Omega)$, so it is a normal family.

**Proof.** Let $S$ be the convex hull of $\{f_\lambda : |\lambda| < 1\}$ where $f_\lambda(t) = \frac{t}{1-\lambda t}$. By Krein–Millman’s theorem, $\mathcal{K}$ is the closed convex hull of it’s extreme points, so $\overline{S} = \mathcal{K}$. Fix $K \subseteq \Omega$ as a compact set. Then $h(\lambda, z) = |1 - \lambda z|$ is continuous on $[-1,1] \times K$ and so takes its minimum value. It should be noticed that the minimum value $m$ of $h$ $[-1,1] \times K$ is nonzero. Put $M_K := \sup\{|z| : z \in K\}$. Then

$$|f_\lambda(z)| = \frac{|z|}{|1-\lambda z|} \leq \frac{M_K}{m}$$

If $g = \sum_{i=1}^n c_i f_{\lambda_i} \in S$, then

$$|g(z)| = \left| \sum_{i=1}^n c_i f_{\lambda_i}(z) \right| \leq \sum_{i=1}^n c_i |f_{\lambda_i}(z)| \leq \sum_{i=1}^n c_i \frac{M_k}{m} = \frac{M_k}{m},$$

whence $\|g\|_K \leq M_K$. Now assume that $g \in \mathcal{K}$ is arbitrary. There exists $\{f_n\}$ in $S$ such that $f_n(t) \to g(t)$ for each $t \in (-1,1)$. Since $S$ is bounded, the sequence $\{f_n\}$ is bounded. By Montel’s theorem there exists a subsequence $\{f_{n_j}\}$ converging to $g'$.
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in uniform compact convergence topology on $\Omega$. Since $g = g'$ on $(-1,1)$, we have $g(z) = g'(z)$ for each $z \in \Omega$. Hence

$$|g(z)| = |g'(z)| = \lim_{n_j \to \infty} |f_{n_j}(z)| \leq \frac{M_K}{m}.$$ 

Therefore $\mathcal{K}$ is a normal family.

**Proposition 2.2.** Let $f \in \mathcal{K}$ and $f(-1,1) \subseteq (-1,1)$. Then $f(t) = t$ for each $t \in (-1,1)$.

**Proof.** Since $f(-1,1) \subseteq (-1,1)$, so $f^n = f \circ f \cdots \circ f \in \mathcal{K}$. Hence by Theorem (2.11), $f^n$ has a convergent subsequence that converges to a function $h \in \mathcal{K}$. Assume that $f(t_0) < t_0$ for some $t_0 \in (-1,1)$. Hence $\{f^{(n)}(t_0)\}$ is an increasing sequence converging to $h(t_0)$. Thus

$$h(f(t_0)) = \lim_{n \to \infty} f^n(f(t_0)) = \lim_{n \to \infty} f^{n+1}(t_0) = h(t_0)$$

Since $h$ is one-one, we infer that $f(t_0) = t_0$, which is a contradiction and this completes the proof.

**Remark 2.3.** We can prove Proposition 2.2 directly as follows.

It follows from

$$f(t) = \int_{-1}^{1} \frac{t}{1-\lambda t} d\mu(\lambda),$$

that

$$-1 \leq \int_{-1}^{1} \frac{t}{1-\lambda t} d\mu(\lambda) \leq 1 \quad (-1 < t < 1).$$

Since for each $\lambda$ the integrand $\frac{t}{1-\lambda t}$ is positive and increasing on $0 < t < 1$, by the Lebesgue's monotone convergence theorem

$$\int_{-1}^{1} \frac{1}{1-\lambda} d\mu(\lambda) = \lim_{t \to 1-} \int_{-1}^{1} \frac{t}{1-\lambda t} \leq 1.$$ 

Similarly we have

$$\int_{-1}^{1} \frac{-1}{1+\lambda} d\mu(\lambda) = \lim_{t \to 1+} \int_{-1}^{1} \frac{t}{1-\lambda t} \geq -1.$$ 

Thus we have

$$\int_{-1}^{1} \frac{1}{1-\lambda^2} d\mu(\lambda) = \frac{1}{2} \int_{-1}^{1} \left( \frac{1}{1-\lambda} + \frac{1}{1+\lambda} \right) d\mu(\lambda) \leq 1 = \int_{-1}^{1} 1 d\mu(\lambda).$$

From this it follows that $\frac{1}{1-\lambda^2} = 1$ almost everywhere with respect to $\mu$, Thus $\mu\{0\} = 1$, which implies $f(t) = t$. 

Corollary 2.4. Let $f$ be an odd operator monotone function on $(-1,1)$ and $A$ is a bounded linear operator on a Hilbert space with spectrum in $(-1,1)$. Then $f(|A|) \geq f'(0)|A|$.

Proof. If $f(t_0) < f'(0)t_0$ for some $t_0 \in (0,1)$, then $f_1(t) = \frac{1}{f'(0)t_0} f(t_0 t) \in \mathcal{K}$ and $f_1(-1,1) \subseteq (-1,1)$, so, by Proposition (2.2), we have $f_1(1) = 1$, which is a contradiction. Hence

$$f(|t|) \geq f'(0)|t|, \quad t \in (-1,1) \quad (2.1)$$

Therefore $f(|A|) \geq f'(0)|A|$.

Remark 2.5. A direct proof of (2.1) reads as follows. Notice that $f(0) = 0$. Hence

$$f(t) = f'(0) \int_{-1}^{t} \frac{t}{1 - \lambda t} d\mu(\lambda). \quad (2.2)$$

Since $f(t) = -f(-t)$, we obtain

$$\int_{-1}^{1} \frac{1}{1 - \lambda t} d\mu(\lambda) = \int_{-1}^{1} \frac{1}{1 + \lambda t} d\mu(\lambda).$$

Thus

$$\int_{-1}^{1} \frac{1}{1 - \lambda t} d\mu(\lambda) = \frac{1}{2} \int_{-1}^{1} \left(\frac{1}{1 - \lambda t} + \frac{1}{1 + \lambda t}\right) d\mu(\lambda)$$

$$= \int_{-1}^{1} \frac{1}{1 - (\lambda t)^2} d\mu(\lambda) \geq \int_{-1}^{1} \frac{1}{1 - (\lambda t)^2} d\mu(\lambda) = 1.$$

(2.2) yields $|f(t)| \geq f'(0)|t|$.

In the sequel we need the following lemma.

Lemma 2.6. [3, Lemma 2.4] If $f$ is an operator monotone function on an interval $(a,b)$, then $f^{2p+1}(t) \geq 0$ for all $p = 0,1,2, \cdots$ and all $a < t < b$.

Corollary 2.7. Let $f$ be an odd operator monotone function on $(-1,1)$. Then $f$ is concave on $(-1,0)$ and convex on $(0,1)$.

Proof. Without loss of generality we may assume that $f \in \mathcal{K}$. We shall show that $f$ is convex on $(0,1)$. The proof of Lemma 4.1 of [8] shows that $f'(t) \geq \frac{f(0)^2}{t^2}$. It follows from Corollary (2.4) that $f'(t) \geq 1$ for each $t \in (0,1)$. Therefore

$$f''(0) = \lim_{t \to 0^+} \frac{f'(t) - f'(0)}{t} = \lim_{t \to 0^+} \frac{f'(t) - 1}{t} \geq 0.$$

By Lemma (2.6), $f^{(3)}(t) \geq 0$ for all $t \in (-1,1)$, so $f''(t) \geq 0$ for all $t \in (0,1)$ since $f''$ is monotone. Hence $f$ is a convex function on $(0,1)$. Since $f$ is an odd function, $f$ is concave on $(-1,0)$.
**Theorem 2.8.** An odd operator monotone function on $(-1,1)$ is of the form

$$f(t) = f'(0) \int_{-1}^{1} \frac{t}{1 - (\lambda t)^2} d\mu(\lambda),$$

where $\mu$ is a probability measure on $(-1,1)$.

**Proof.** As before, we may assume that $f \in \mathcal{K}$. The function $f$ can be represented as a power series $f(t) = \sum_{n=1}^{\infty} a_n t^n$, which is convergent for $|t| < 1$, cf. [3]. Since $f$ is odd, $a_{2n} = 0$ for all $n$. Due to $f$ is operator monotone, there is a probability measure $\mu$ on $(-1,1)$ such that

$$f(t) = \int_{-1}^{1} \sum_{n=1}^{\infty} t(\lambda t)^n d\mu(\lambda) = \sum_{n=1}^{\infty} t^{n+1} \int_{-1}^{1} \lambda^n d\mu(\lambda)$$

Therefore $a_{2n} = \int_{-1}^{1} \lambda^{2n-1} = 0$ and so

$$f(t) = \int_{-1}^{1} \frac{t}{1 - \lambda t} d\mu(\lambda) = \frac{1}{2} \int_{-1}^{1} \frac{t}{1 - \lambda t} + \frac{t}{1 + \lambda t} d\mu(\lambda) = \frac{1}{2}(g(t) - g(-t)),$$

where $g(t) = \int_{-1}^{1} \frac{t}{1 - \lambda t} d\mu(\lambda)$. Hence $f$ is an odd operator monotone function on $(-1,1)$. \qed

We start main results with the following useful lemma.

**Lemma 2.9.** Let $A, B \in \mathbb{B}(\mathcal{H})$ be invertible positive operators such that $A - B \geq m > 0$. Then

$$B^{-1} - A^{-1} \geq \frac{m}{(||A|| - m)||A||}.$$  

**(2.4)**

**Proof.** Since $f(t) = \frac{1}{t}$ is a decreasing operator monotone function on $[0, \infty)$ we have $B^{-1} \geq (A - m)^{-1}$. On the other hand

$$(A - m)^{-1} \geq A^{-1} + \frac{m}{(||A|| - m)||A||}$$

$\iff (A^{-1} + \frac{m}{(||A|| - m)||A||})(A - m) \leq 1$

$\iff \frac{A^2}{(||A|| - m)||A||} - \frac{mA}{(||A|| - m)||A||} \leq 1$

$\iff A^2 - mA \leq (||A|| - m)||A||$

$\iff ||A^2 - mA|| \leq (||A|| - m)||A||.$
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There exists $\lambda_0 \in \text{sp}(A)$ such that $||A|| = \lambda_0$. Since $A \geq m > 0$, we have

$$||A^2 - mA|| = \max\{\lambda : \lambda \in \text{sp}(A^2 - mA)\}$$
$$= \max\{\lambda^2 - m\lambda : \lambda \in \text{sp}(A)\}$$
$$= \lambda_0^2 - m\lambda_0$$
$$= (||A|| - m)||A||.$$

So $B^{-1} \geq (A - m)^{-1} \geq A^{-1} + \frac{m}{(||A|| - m)||A||}$. \qed

**Proposition 2.10.** Let $f$ be a non-constant operator monotone function on an interval $J$ and $A, B$ be self-adjoint operators with spectra in $J$ such that $A > B$. Then $f(A) > f(B)$.

**Proof.** Without loss of generality we assume that $J = (-1, 1)$. Let $A, B \in B(\mathcal{H})$ be self-adjoint operators with spectra in $(-1, 1)$ and $A - B$ is positive and invertible. So there exists $m > 0$ such that $A - B \geq m > 0$. Put $f_\lambda(t) = \frac{t}{1 - \lambda t}$ for each $\lambda$ with $|\lambda| < 1$. We shall show that $f_\lambda(A) - f_\lambda(B)$ is bounded and so invertible. It is clear that the claim is true for $\lambda = 0$. If $0 < \lambda < 1$, then $(1 - \lambda B) - (1 - \lambda A) = \lambda(A - B) > \lambda m > 0$ as well as $1 - \lambda B$ and $1 - \lambda A$ are positive invertible operators. Since

$$\frac{t}{1 - \lambda t} = \frac{-1}{\lambda} + \frac{1}{\lambda} \left( \frac{1}{1 - \lambda t} \right),$$

by Lemma 2.9, we have

$$f_\lambda(A) - f_\lambda(B) = \frac{1}{\lambda} \left( \frac{1}{1 - \lambda A} - \frac{1}{1 - \lambda B} \right)$$
$$\geq \frac{1}{\lambda} \left( \frac{\lambda m}{||1 - \lambda B|| - \lambda m ||1 - \lambda B||} \right)$$
$$= \frac{\lambda m}{||1 - \lambda B|| - \lambda m ||1 - \lambda B||} > 0 \quad (\text{by } (2.9))$$

A similar argument shows that

$$f_\lambda(A) - f_\lambda(B) \geq \frac{m}{||1 - \lambda A|| + \lambda m ||1 - \lambda A||} > 0$$

for each $-1 < \lambda < 0$. Since $f$ is operator monotone on $(-1, 1)$, it can be represented as

$$f(t) = f(0) + f'(0) \int_{-1}^{1} f_\lambda(t)d\mu(\lambda),$$
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where $\mu$ is a nonzero positive measure on $(-1, 1)$. Since $f$ is nonconstant, $f'(0) > 0$, [3, Lemma 2.3]. Hence

\[
f(A) - f(B) = f'(0) \int_{-1}^{1} \left( \frac{A}{1 - \lambda A} - \frac{B}{1 - \lambda B} \right) d\mu(\lambda)
\]

\[
= f'(0) \int_{-1}^{1} \left( f_\lambda(A) - f_\lambda(B) \right) d\mu(\lambda)
\]

\[
\geq f'(0) \int_{-1}^{1} m_\lambda d\mu(\lambda),
\]

where

\[
m_\lambda = \frac{m}{(||1 - \lambda B|| - \lambda m)||1 - \lambda B||}
\]

if $0 \leq \lambda < 1$, and

\[
m_\lambda = \frac{m}{(||1 - \lambda A|| + \lambda m)||1 - \lambda A||}
\]

if $-1 < \lambda < 0$. Since $\mu$ is a nonzero positive measure and $m_\lambda > 0$, we have

\[
f(A) - f(B) \geq f'(0) \int_{-1}^{1} m_\lambda d\mu(\lambda) > 0.
\]

Therefore $f(A) > f(B)$. \hfill \square

**Theorem 2.11.** Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators such that $A - B \geq m > 0$ and $0 < r \leq 1$. Then

\[
A^r - B^r \geq ||A||^r - (||A|| - m)^r.
\]

**Proof.** Let $0 < r < 1$. It is known that

\[
t^r = \frac{\sin(r\pi)}{\pi} \int_{0}^{\infty} \frac{t}{\lambda + t} \lambda^{r-1} d\lambda,
\]

in which $0 < r < 1$, see e.g. [1, Chapter V]. First note that,

\[
\frac{A}{\lambda + A} - \frac{B}{\lambda + B} = \lambda \left( \frac{1}{\lambda + B} - \frac{1}{\lambda + A} \right)
\]

\[
\geq \frac{\lambda m}{(||A + \lambda|| - m)||A + \lambda||} \text{ by (2.4)}
\]

\[
= \frac{\lambda m}{(||A|| + \lambda - m)(||A|| + \lambda)}
\]

for each $\lambda > 0$. By using (2.5) we have

\[
A^r - B^r
\]

\[
= \frac{\sin(r\pi)}{\pi} \int_{0}^{\infty} \lambda^{r-1} \left( \frac{A}{\lambda + A} - \frac{B}{\lambda + B} \right) d\lambda
\]

\[
\geq \frac{\sin(r\pi)}{\pi} \int_{0}^{\infty} \frac{m\lambda^r}{(||A|| + \lambda - m)(||A|| + \lambda)} d\lambda,
\]
We need to compute
\[ I = \int_{0}^{\infty} \frac{\lambda^{r}}{\left(\lambda + ||A||\right)\left(\lambda + (||A|| - m)\right)} d\lambda \]
where \(0 < r < 1\). We will need the branch cut for \(z^{r} = \rho^{r}e^{ir\theta}\), in which \(z = \rho e^{i\theta}\) and \(0 \leq \theta \leq 2\pi\). Consider
\[ \int_{C} \frac{z^{r}}{(z + ||A||)(z + (||A|| - m))} \, dz, \]
where the keyhole contour \(C\) consists of a large circle \(C_{R}\) of radius \(R\), a small circle \(C_{\epsilon}\) of radius \(\epsilon\) and two lines just above and below the branch cuts \(\theta = 0\); see Figure 1. The contribution from \(C_{R}\) is \(O(R^{r-2})2\pi R = O(R^{r-1}) = 0\) as \(R \to \infty\). Similarly the contribution from \(C_{\epsilon}\) is zero as \(\epsilon \to 0\). The contribution from just above the branch cut and from just below the branch cut is \(I\) and \(-e^{2r\pi i}I\), respectively, as \(\epsilon \to 0\) and \(R \to \infty\). Hence, taking the limits as \(\epsilon \to 0\) and \(R \to \infty\),
\[ (1 - e^{2r\pi i})I = \int_{C} \frac{z^{r}}{(z + ||A||)(z + (||A|| - m))} \, dz \]
by the Cauchy residue theorem. So
\[ I = \frac{\pi}{m \sin(r\pi)} (||A||^{r} - (||A|| - m)^{r}) \, . \]
Therefore
\[ A^{r} - B^{r} \geq \frac{\sin(r\pi)}{\pi} \int_{0}^{\infty} \frac{m\lambda^{r}}{(||A|| + \lambda - m)(||A|| + \lambda)} d\lambda = ||A||^{r} - (||A|| - m)^{r}. \]

**Figure 1.** Keyhole contour
Corollary 2.12. Let $A, B \in \mathbb{B}(\mathcal{H})$ be positive operators such that $A - B \geq m > 0$. Then
\[
\log A - \log B \geq \log \|A\| - \log(\|A\| - m).
\]

Proof. Put $f_n(t) = n(t^{\frac{1}{n}} - 1)$ on $[0, \infty)$. Then the sequence $\{f_n\}$ uniformly converges to $\log t$ on any compact subset of $(0, \infty)$. Hence
\[
\log A - \log B = \lim_{n \to \infty} f_n(A) - f_n(B) \\
\geq \lim_{n \to \infty} n(\|A\|^{\frac{1}{n}} - (\|A\| - m)^{\frac{1}{n}}) \\
= \log \|A\| - \log(\|A\| - m).
\]

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6. T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1 + r)q \geq p + 2r$, Proc. Amer. Math. Soc., 101 (1987), 85-88.