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A GENERALIZATION OF PERSPECTIVE FUNCTION

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ABSTRACT. We study perspective of operator convex functions. In particular we give a generalization of perspective functions and establish its properties. We also give an operator extension of a classical inequality in information theory. As an application a refinement of the operator Jensen inequality is presented.

1. INTRODUCTION

Let $\mathbb{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$ and I denote the identity operator. An operator A is said to be positive (denoted by $A\geq 0$) if $\langle Ax,x\rangle\geq 0$ for all vectors $x\in \mathscr{H}$. If, in addition, A is invertible, then it is called strictly positive (denoted by $A>0$). By $A\geq B$ we mean that $A-B$ is positive, while $A>B$ means that $A-B$ is strictly positive. An operator C is called an isometry if $C^{*}C=I$, a contraction if $C^{*}C\leq I$ and an expansive operator if $C^{*}C\geq I$. A map Φ on $\mathbb{B}(\mathscr{H})$ is called positive if $\Phi(A)\geq 0$ for each $A\geq 0$.

A continuous real valued function f defined on an interval $[m, M]$ is said to be operator convex if

$$f(\lambda A+(1-\lambda)B)\leq\lambda f(A)+(1-\lambda)f(B),$$

for all self-adjoint operators A, B with spectra in $[m, M]$ and all $\lambda\in[0,1]$, where $f(A)$ is the functional calculus as usual. The Jensen operator inequality due to F. Hansen and G.K. Pedersen, which is a characterization of operator convex functions, states that f is operator convex on $[m, M]$ if and only if

$$f(C^{*}AC)\leq C^{*}f(A)C, \tag{1.1}$$

for any isometry C and any self-adjoint operator A with spectrum in $[m, M][12]$. If $f(0)\leq 0$, then f is operator convex on $[m, M]$ if and only if $f(C^{*}AC)\leq C^{*}f(A)C$ for any contraction C. Various characterizations of operator convex functions can be found in [11, 10]. Given in [17], the following result is a generalization of (1.1).

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Theorem A. Let f be an operator convex function on $[m, M]$ and $\Phi_1, \cdots, \Phi_n$ be positive linear maps on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^n \Phi_i(I) = I$. Then

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leq \sum_{i=1}^n \Phi_i(f(A_i)), \tag{1.2}$$

for all self-adjoint operators A_i $(i = 1, \cdots, n)$ with spectra in $[m, M]$.

Let f be a convex function on a convex set $K \subseteq \mathbb{R}^n$. The perspective function g associated to f is defined on the set $\{(x, y) : y > 0 \quad \text{and} \quad \frac{x}{y} \in K\}$ by

$$g(x, y) = yf\left(\frac{x}{y}\right).$$

(see [13]). As an operator extension of the perspective function, Effros [9] introduced the perspective function of an operator convex function f by

$$g(L, R) = Rf\left(\frac{L}{R}\right),$$

for commuting strictly positive operators L and R and showed that:

Theorem B.[9] If f is operator convex, when restricted to commuting strictly positive operators, then the perspective function $(L, R) \to g(L, R) = Rf\left(\frac{L}{R}\right)$ is jointly operator convex.

He also extended the generalized perspective function, defined by Maréchal [15, 16], to operators. Given continuous functions f and h and commuting strictly positive operators L and R, Effros defined the operator extension of the generalized perspective function by

$$(f\Delta h)(L, R) = h(R)f\left(\frac{L}{h(R)}\right),$$

and proved that:

Theorem C. If f is operator convex with $f(0) \leq 0$ and h is operator concave with $h > 0$ then $f\Delta h$ is jointly convex on commuting strictly positive operators.

The authors of [8] extended Effros's results by removing the restriction to commuting operators. They proved non-commutative versions of Theorem B and Theorem C.

A beautiful various study of such functions for operators was introduced by F. Kubo and T. Ando. They considered the case where f is an operator monotone function and established a relation between operator monotone functions and operator means (see [11]).

One of the most principal matters in applications of probability theory, is to find a suitable measure between two probability distributions. The theory of information divergence measures has been applied in several fields such as signal processing, genetics, economics and in pattern recognition. Many kinds of such measures have been studied. One of the most famous of such measures is the Csiszár f-divergence functional, which includes several measures.

For a convex function $f : [0, \infty) \to \mathbb{R}$, Csiszár [3, 4] introduced the f-divergence functional by

$$I_f(p, q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{1.3}$$

for probability distributions p and q, in which undefined expressions were interpreted by

$$\begin{aligned} f(0) &= \lim_{t \to 0^+} f(t), \qquad 0f\left(\frac{0}{0}\right) = 0, \\ 0f\left(\frac{p}{0}\right) &= \lim_{\epsilon \to 0^+} f\left(\frac{p}{\epsilon}\right) = p \lim_{t \to \infty} \frac{f(t)}{t}. \end{aligned}$$

Also Csiszár and Körner [5] obtained the following results.

Theorem D. If $f : [0, \infty) \to \mathbb{R}$ is convex, then $I_f(p, q)$ is jointly convex in p and q.

Theorem E. Let $f : [0, \infty) \to \mathbb{R}$ be convex. Then

$$\sum_{i=1}^{n} q_i f\left(\frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} q_i}\right) \le I_f(p, q), \tag{1.4}$$

for every $p, q \in \mathbb{R}_+^n$.

Definition of f-divergence functional was generalized to an n-tuple of vectors $x = (x_1, \cdots, x_n)$ and a probability distribution $q = (q_1, \cdots, q_n)$ as follows (see [6]). Let X be a vector space, K be a convex cone in X and $f : K \to \mathbb{R}$ be a convex function. For any n-tuple of vectors $x = (x_1, \cdots, x_n) \in K^n$ and a probability distribution $q = (q_1, \cdots, q_n)$, the f-divergence functional is defined by

$$I_f(x, q) = \sum_{i=1}^{n} q_i f\left(\frac{x_i}{q_i}\right).$$

A series of results and inequalities related to f-divergence functionals can be found in [1, 2, 6, 7, 14].

In section 2, we generalize the notion of operator perspective function and investigate some properties of generalized perspective function. In particular, an operator extension of (1.4) is presented. In section 3, we provide some applications for our results. More precisely, a refinement of the Jensen operator inequality is given in section 3.

2. Non-commutative f-divergence functionals

Let f be an operator convex function. The perspective function g associated to f is defined by

$$g(L,R)=R^{\frac{1}{2}}f(R^{-\frac{1}{2}}LR^{-\frac{1}{2}})R^{\frac{1}{2}},$$

for self-adjoint operator L and strictly positive operator R on a Hilbert space $\mathscr{H}$. It is known that [8] f is operator convex if and only if g is jointly operator convex. We consider a more general case. Let $\widetilde{L}=(L_1,\cdots,L_n)$ and $\widetilde{R}=(R_1,\cdots,R_n)$ be n-tuples of self-adjoint and strictly positive operators, respectively. Let us define the non-commutative f-divergence functional Θ by[18]

$$\Theta(\widetilde{L},\widetilde{R})=\sum_{i=1}^{n}R_i^{\frac{1}{2}}f(R_i^{-\frac{1}{2}}L_iR_i^{-\frac{1}{2}})R_i^{\frac{1}{2}}. \tag{2.1}$$

By the same argument as in [8], it is easy to see that Θ is jointly operator convex if and only if f is operator convex. In the sequel, we study some properties of Θ and establish some relations between Θ and g.

Theorem 2.1. *Let f be an operator convex function, and $\widetilde{L}=(L_1,\cdots,L_n)$ and $\widetilde{R}=(R_1,\cdots,R_n)$ be n-tuples of self-adjoint and strictly positive operators, respectively. Then*

$$g(L,R)\le\Theta(\widetilde{L},\widetilde{R}), \tag{2.2}$$

where $R=\sum_{i=1}^{n}R_i$ and $L=\sum_{i=1}^{n}L_i$.

Proof.

$$\begin{aligned}
f\left(R^{-\frac{1}{2}}LR^{-\frac{1}{2}}\right)&=f\left(\left(\sum_{j=1}^{n}R_j\right)^{-\frac{1}{2}}\left(\sum_{i=1}^{n}L_i\right)\left(\sum_{j=1}^{n}R_j\right)^{-\frac{1}{2}}\right)\\
&=f\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}R_j\right)^{-\frac{1}{2}}L_i\left(\sum_{j=1}^{n}R_j\right)^{-\frac{1}{2}}\right)\\
&=f\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}R_j\right)^{-\frac{1}{2}}R_i^{\frac{1}{2}}(R_i^{-\frac{1}{2}}L_iR_i^{-\frac{1}{2}})R_i^{\frac{1}{2}}\left(\sum_{j=1}^{n}R_j\right)^{-\frac{1}{2}}\right)\\
&\le\sum_{i=1}^{n}\left(\sum_{j=1}^{n}R_j\right)^{-\frac{1}{2}}R_i^{\frac{1}{2}}f(R_i^{-\frac{1}{2}}L_iR_i^{-\frac{1}{2}})R_i^{\frac{1}{2}}\left(\sum_{j=1}^{n}R_j\right)^{-\frac{1}{2}}
\end{aligned}$$

(by the Jensen operator inequality)

$$= \left(\sum_{j=1}^{n} R_j\right)^{-\frac{1}{2}} \sum_{i=1}^{n} R_i^{\frac{1}{2}} f(R_i^{-\frac{1}{2}} L_i R_i^{-\frac{1}{2}}) R_i^{\frac{1}{2}} \left(\sum_{j=1}^{n} R_j\right)^{-\frac{1}{2}}$$
$$= R^{-\frac{1}{2}} \Theta(\widetilde{L}, \widetilde{R}) R^{-\frac{1}{2}},$$

whence we have the desired inequality (2.2). □

Corollary 2.2. *The perspective function g of an operator convex function f is subadditive in the sense that*

$$g(L_1 + L_2, R_1 + R_2) \leq g(L_1, R_1) + g(L_2, R_2).$$

Corollary 2.3. *Under the same conditions of Theorem 2.1,*

$$f(L) \leq \Theta(\widetilde{L}, \widetilde{R}),$$

whenever $\sum_{i=1}^{n} R_i = I$.

For every positive integer n, let $J \subseteq \{1, \cdots, n\}$ and $\bar{J} = \{1, \cdots, n\} - J$. Then the following result holds true.

Corollary 2.4. *Let g be the perspective function of an operator convex function f, and $\widetilde{L} = (L_1, \cdots, L_n)$ and $\widetilde{R} = (R_1, \cdots, R_n)$ be n-tuples of self-adjoint and strictly positive operators, respectively. Then*

$$2g\left(\frac{1}{2}(L, R)\right) \leq g(L_J, R_J) + g(L_{\bar{J}}, R_{\bar{J}}) \leq \Theta(\widetilde{L}, \widetilde{R}), \tag{2.3}$$

where $R = \sum_{i=1}^{n} R_i$, $R_J = \sum_{i \in J} R_i$, $L = \sum_{i=1}^{n} L_i$ and $L_J = \sum_{i \in J} L_i$.

Proof. Since $(L, R) = (L_J, R_J) + (L_{\bar{J}}, R_{\bar{J}})$, the first inequality of (2.3) follows immediately from the joint convexity of g. Utilizing Theorem 2.1, we obtain

$$g(L_J, R_J) + g(L_{\bar{J}}, R_{\bar{J}}) \leq \sum_{i \in J} R_i^{\frac{1}{2}} f(R_i^{-\frac{1}{2}} L_i R_i^{-\frac{1}{2}}) R_i^{\frac{1}{2}} + \sum_{i \in \bar{J}} R_i^{\frac{1}{2}} f(R_i^{-\frac{1}{2}} L_i R_i^{-\frac{1}{2}}) R_i^{\frac{1}{2}} = \Theta(\widetilde{L}, \widetilde{R}).$$

□

Corollary 2.5. *Let L_{ij} and R_{ij} $(i, j = 1, \cdots, n)$ be self-adjoint and strictly positive operators, respectively, and let p_j $(j = 1, \cdots, n)$ be positive numbers. If f is operator convex, then*

$$\sum_{i=1}^{n} g(R_i, L_i) \leq \sum_{i=1}^{n} p_i \Theta(\widetilde{L}^i, \widetilde{R}^i),$$

where $R_i = \sum_{j=1}^{n} p_j R_{ij}$, $\widetilde{L}^i = (L_{i1}, \cdots, L_{in})$ and $p\widetilde{L}^i = (p_1 L_{i1}, \cdots, p_n L_{in})$.

Proof. Using Theorem 2.1 for each R_i and L_i we obtain

$$g(L_i, R_i) = R_i^{\frac{1}{2}} f\left(R_i^{-\frac{1}{2}} L_i R_i^{-\frac{1}{2}}\right) R_i^{\frac{1}{2}} \leq \Theta(p\widetilde{L}^i, p\widetilde{R}^i), \quad (1 \leq i \leq n). \tag{2.4}$$

In addition,

$$\begin{aligned} \Theta(p\widetilde{L}^i, p\widetilde{R}^i) &= \sum_{j=1}^{n} (p_j R_{ij})^{\frac{1}{2}} f\left((p_j R_{ij})^{-\frac{1}{2}} (p_j L_{ij})(p_j R_{ij})^{-\frac{1}{2}}\right) (p_j R_{ij})^{\frac{1}{2}} \\ &= \sum_{j=1}^{n} p_j R_{ij}^{\frac{1}{2}} f(R_{ij}^{-\frac{1}{2}} L_{ij} R_{ij}^{-\frac{1}{2}}) R_{ij}^{\frac{1}{2}}. \end{aligned} \tag{2.5}$$

Summing (2.4) over i we get

$$\begin{aligned} \sum_{i=1}^{n} g(L_i, R_i) &\leq \sum_{i=1}^{n} \Theta(p\widetilde{L}^i, p\widetilde{R}^i) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} p_j R_{ij}^{\frac{1}{2}} f(R_{ij}^{-\frac{1}{2}} L_{ij} R_{ij}^{-\frac{1}{2}}) R_{ij}^{\frac{1}{2}} \qquad \text{(by (2.5))} \\ &= \sum_{j=1}^{n} p_j \sum_{i=1}^{n} R_{ij}^{\frac{1}{2}} f(R_{ij}^{-\frac{1}{2}} L_{ij} R_{ij}^{-\frac{1}{2}}) R_{ij}^{\frac{1}{2}} \\ &= \sum_{j=1}^{n} p_j \Theta(\widetilde{L}^i, \widetilde{R}^i). \end{aligned}$$

$\square$

For continuous functions f and h and commuting matrices L and R, Effros [9] defined the function $(L, R) \to (f\Delta h)(L, R)$ by

$$(f\Delta h)(L, R) = f\left(\frac{L}{h(R)}\right) h(R).$$

He also proved that if f is operator convex with $f(0) \leq 0$ and h is operator concave with $h > 0$, then $f\Delta h$ is jointly operator convex. In [8], definition and properties of $f\Delta h$ was given for two not necessarily commuting self-adjoint operators L and R, by

$$(f\Delta h)(L, R) = h(R)^{\frac{1}{2}} f\left(h(R)^{-\frac{1}{2}} L h(R)^{-\frac{1}{2}}\right) h(R)^{\frac{1}{2}}.$$

Assume that f and h are continuous functions and $\widetilde{L} = (L_1, \cdots, L_n)$ and $\widetilde{R} = (R_1, \cdots, R_n)$ be n-tuples of self-adjoint operators. Let $\widetilde{p} = (p_1, \cdots, p_n)$ and $\widetilde{q} = (q_1, \cdots, q_n)$ be probability distributions. As a generalization of $f\Delta h$, we define $f\nabla h$ by

$$(f\nabla h)(\widetilde{L}, \widetilde{R}, \widetilde{p}, \widetilde{q}) = \sum_{i=1}^{n} p_i h(q_i R_i)^{\frac{1}{2}} f\left(h(q_i R_i)^{-\frac{1}{2}} L_i h(q_i R_i)^{-\frac{1}{2}}\right) h(q_i R_i)^{\frac{1}{2}}.$$

Note that with $p_1 = q_1 = 1$ and $p_i = 0$ $(i = 2, \cdots, n)$, $f\nabla h = f\Delta h$. It is not hard to see that f is operator convex with $f(0) < 0$ and h is operator concave with $h > 0$ if and only if $f\nabla h$ is jointly operator convex. The next result, is a Choi–Davis–Jensen type inequality for $f\Delta h$.

Theorem 2.6. [18] *Let f be an operator convex function with $f(0) \leq 0$, h be an operator concave function with $h > 0$ and $f\Delta h$ be the operator generalized perspective function. If Φ is a positive linear map on $\mathbb{B}(\mathscr{H})$ with $\Phi(I) \leq I$, then*

$$(f\Delta h)\left(\Phi(L), \Phi(R)\right) \leq \Phi((f\Delta h)(L, R)), \tag{2.6}$$

for all self-adjoint operators L, R. In particular, If g is the perspective function associated to f, then

$$g\left(\Phi(L), \Phi(R)\right) \leq \Phi(g(L, R)), \tag{2.7}$$

for all self-adjoint operator L and strictly positive operator R.

Proof. Let R be a self-adjoint operator. Define the positive linear map Ψ on $\mathbb{B}(\mathscr{H})$ by

$$\Psi(X) = h(\Phi(R))^{-\frac{1}{2}}\Phi\left(h(R)^{\frac{1}{2}}Xh(R)^{\frac{1}{2}}\right)h(\Phi(R))^{-\frac{1}{2}}.$$

Since h is operator concave, $h > 0$ and $\Phi(I) \leq I$, then $\Phi(h(R)) \leq h(\Phi(R))$. Therefore

$$\Psi(I) = h(\Phi(R))^{-\frac{1}{2}}\Phi(h(R))h(\Phi(R))^{-\frac{1}{2}} \leq I.$$

Hence

$$\begin{aligned}
(f\Delta h)(\Phi(L), \Phi(R)) &= h(\Phi(R))^{\frac{1}{2}}f\left(h(\Phi(R))^{-\frac{1}{2}}\Phi(L)h(\Phi(R))^{-\frac{1}{2}}\right)h(\Phi(R))^{\frac{1}{2}} \\
&= h(\Phi(R))^{\frac{1}{2}}f\left(\Psi\left(h(R)^{-\frac{1}{2}}Lh(R)^{-\frac{1}{2}}\right)\right)h(\Phi(R))^{\frac{1}{2}} \\
&\leq h(\Phi(R))^{\frac{1}{2}}\Psi\left(f\left(h(R)^{-\frac{1}{2}}Lh(R)^{-\frac{1}{2}}\right)\right)h(\Phi(R))^{\frac{1}{2}} \\
&\qquad \text{(by operator convexity of } f,\ f(0) \leq 0 \text{ and } \Psi(I) \leq I) \\
&= \Phi\left(h(R)^{\frac{1}{2}}f\left(h(R)^{-\frac{1}{2}}Lh(R)^{-\frac{1}{2}}\right)h(R)^{\frac{1}{2}}\right) \\
&= \Phi((f\Delta h)(L, R)).
\end{aligned}$$

Applying (2.6) for $h(t) = t$ gives (2.7). □

Corollary 2.7. *Let f be an operator convex function with $f(0) \leq 0$, h be an operator concave function with $h > 0$ and $f\Delta h$ be the operator generalized perspective function. Then*

$$(f\Delta h)\left(\langle Lx, x\rangle, \langle Rx, x\rangle\right) \leq \langle (f\Delta h)(L, R)x, x\rangle,$$

for all self-adjoint operators L and R and all unit vector $x \in \mathscr{H}$. In particular, if g is the perspective function of operator convex function f, then

$$g\left(\langle Lx, x\rangle, \langle Rx, x\rangle\right) \leq \langle g(L,R)x, x\rangle, \tag{2.8}$$

for all self-adjoint operator L and all strictly positive operator R and all unit vector $x \in \mathscr{H}$.

In the next theorem, we establish a relation between two functions $f\Delta h$ and $f\nabla h$.

Theorem 2.8. *Let f be an operator convex function with $f(0) < 0$ and h be an operator concave function with $h > 0$. If $\widetilde{p} = (p_1, \cdots, p_n)$ and $\widetilde{q} = (q_1, \cdots, q_n)$ are probability distributions, then*

$$(f\Delta h)(L, R) \leq (f\nabla h)(\widetilde{L}, \widetilde{R}, \widetilde{p}, \widetilde{q}), \tag{2.9}$$

for all n-tuples of self-adjoint operators $\widetilde{L} = (L_1, \cdots, L_n)$ and $\widetilde{R} = (R_1, \cdots, R_n)$, where $R = \sum_{i=1}^n q_i R_i$, $L = \sum_{i=1}^n p_i L_i$.

Proof.

$$\begin{aligned}
&f\left(h(R)^{-\frac{1}{2}} L h(R)^{-\frac{1}{2}}\right) \\
&= f\left(h\left(\sum_{j=1}^n q_j R_j\right)^{-\frac{1}{2}} \left(\sum_{i=1}^n p_i L_i\right) h\left(\sum_{j=1}^n q_j R_j\right)^{-\frac{1}{2}}\right) \\
&= f\left(\sum_{i=1}^n p_i h\left(\sum_{j=1}^n q_j R_j\right)^{-\frac{1}{2}} L_i h\left(\sum_{j=1}^n q_j R_j\right)^{-\frac{1}{2}}\right) \\
&= f\left(\sum_{i=1}^n p_i h\left(\sum_{j=1}^n q_j R_j\right)^{-\frac{1}{2}} h(q_iR_i)^{\frac{1}{2}} \left(h(q_iR_i)^{-\frac{1}{2}} L_i h(q_iR_i)^{-\frac{1}{2}}\right) h(q_iR_i)^{\frac{1}{2}} h\left(\sum_{j=1}^n q_j R_j\right)^{-\frac{1}{2}}\right).
\end{aligned} \tag{2.10}$$

Since h is operator concave and $h > 0$, it is operator monotone [11]. Hence

$$h(q_iR_i) \leq h\left(\sum_{j=1}^n q_j R_j\right), \qquad (i = 1, \cdots, n).$$

Therefore

$$\sum_{i=1}^n p_i h\left(\sum_{j=1}^n q_j R_j\right)^{-\frac{1}{2}} h(q_iR_i) h\left(\sum_j = 1^n q_j R_j\right)^{-\frac{1}{2}} \leq I.$$

So, it follows from (2.10), the operator convexity of f and $f(0) \leq 0$ that

$$
\begin{aligned}
&f\left(h(R)^{-\frac{1}{2}} L h(R)^{-\frac{1}{2}}\right) \\
&= f\left(\sum_{i=1}^{n} p_i h\left(\sum_{j=1}^{n} q_j R_j\right)^{-\frac{1}{2}} h(q_i R_i)^{\frac{1}{2}} \left(h(q_i R_i)^{-\frac{1}{2}} L_i h(q_i R_i)^{-\frac{1}{2}}\right) h(q_i R_i)^{\frac{1}{2}} h\left(\sum_{j=1}^{n} q_j R_j\right)^{-\frac{1}{2}}\right) \\
&\leq \sum_{i=1}^{n} p_i h\left(\sum_{j=1}^{n} q_j R_j\right)^{-\frac{1}{2}} h(q_i R_i)^{\frac{1}{2}} f\left(h(q_i R_i)^{-\frac{1}{2}} L_i h(q_i R_i)^{-\frac{1}{2}}\right) h(q_i R_i)^{\frac{1}{2}} h\left(\sum_{j=1}^{n} q_j R_j\right)^{-\frac{1}{2}} \\
&= h\left(\sum_{j=1}^{n} q_j R_j\right)^{-\frac{1}{2}} \sum_{i=1}^{n} p_i h(q_i R_i)^{\frac{1}{2}} f\left(h(q_i R_i)^{-\frac{1}{2}} L_i h(q_i R_i)^{-\frac{1}{2}}\right) h(q_i R_i)^{\frac{1}{2}} h\left(\sum_{j=1}^{n} q_j R_j\right)^{-\frac{1}{2}} \\
&= h(R)^{-\frac{1}{2}} (f \nabla h)(\widetilde{L}, \widetilde{R}, \widetilde{p}, \widetilde{q}) h(R)^{-\frac{1}{2}},
\end{aligned}
$$

whence we get the required inequality (2.9). □

Let $(\Phi_1, \cdots, \Phi_n)$ and $(\Psi_1, \cdots, \Psi_n)$ be n-tuples of positive linear maps on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^{n} \Phi_i(I) = I$ and $\sum_{i=1}^{n} \Psi_i(I) = I$, $(A_1, \cdots, A_n)$ and $(B_1, \cdots, B_n)$ be n-tuples of self-adjoint operators on $\mathscr{H}$ and g be a jointly operator convex function. Define the function $\Gamma : [0,1] \times [0,1] \to \mathbb{R}$ by

$$
\Gamma(t,s) = \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_i\left(\Psi_j\left(g\left(tA_i + (1-t)\sum_{i=1}^{n} \Phi_i(A_i), sB_j + (1-s)\sum_{j=1}^{n} \Psi_j(B_j)\right)\right)\right).
$$

We have the following result.

Theorem 2.9. *With the same assumption of above, Γ is jointly convex. Furthermore*

$$
g(A,B) \leq \Gamma(t,s),
$$

where $A = \sum_{i=1}^{n} \Phi_i(A_i)$ and $B = \sum_{j=1}^{n} \Psi_j(B_j)$.

Proof. It is easy to see that the joint convexity of Γ follows from the joint operator convexity of g. Also

$$
\begin{aligned}
\Gamma(t,s) &= \sum_{i=1}^{n} \Phi_i\left(\sum_{j=1}^{n} \Psi_j\left(g\left(tA_i + (1-t)\sum_{i=1}^{n} \Phi_i(A_i), sB_j + (1-s)\sum_{j=1}^{n} \Psi_j(B_j)\right)\right)\right) \\
&\geq \sum_{i=1}^{n} \Phi_i\left(g\left(tA_i + (1-t)\sum_{i=1}^{n} \Phi_i(A_i), \sum_{j=1}^{n} \Psi_j\left(sB_j + (1-s)\sum_{j=1}^{n} \Psi_j(B_j)\right)\right)\right) \\
&\geq g\left(\sum_{i=1}^{n} \Phi_i\left(tA_i + (1-t)\sum_{i=1}^{n} \Phi_i(A_i)\right), \sum_{j=1}^{n} \Psi_j\left(sB_j + (1-s)\sum_{j=1}^{n} \Psi_j(B_j)\right)\right) \\
&= g(A,B).
\end{aligned}
$$

□

3. Applications

In this section, we use the results of section 2 to derive some operator inequalities. Throughout this section, assume that $\widetilde{L} = (L_1, \cdots, L_n)$ and $\widetilde{R} = (R_1, \cdots, R_n)$ be n tuples of self-adjoint and strictly positive operators, respectively, and $p = (p_1, \cdots, p_n)$ and $q = (q_1, \cdots, q_n)$ be probability distributions.

For every positive integer n, Let $J \subseteq \{1, \cdots, n\}$ and $\bar{J} = \{1, \cdots, n\} - J$. As the first application of our result, we obtain the following refinement of the Jensen operator inequality.

Theorem 3.1. *Let f be an operator convex function, $\Phi_1, \cdots, \Phi_n$ be positive linear maps on $\mathbb{B}(\mathscr{H})$ such that $\sum_{i=1}^n \Phi_i(I) = I$ and $T_J = \sum_{i \in J} \Phi_i(I)$. Then*

$$
\begin{aligned}
(i)\ f\left(\sum_{i=1}^n \Phi_i(A_i)\right) &\leq T_J^{\frac{1}{2}} f\left(T_J^{-\frac{1}{2}} \sum_{i \in J} \Phi_i(A_i) T_J^{-\frac{1}{2}}\right) T_J^{\frac{1}{2}} + T_{\bar{J}}^{\frac{1}{2}} f\left(T_{\bar{J}}^{-\frac{1}{2}} \sum_{i \in \bar{J}} \Phi_i(A_i) T_{\bar{J}}^{-\frac{1}{2}}\right) T_{\bar{J}}^{\frac{1}{2}} \\
&\leq \sum_{i=1}^n \Phi_i(I)^{\frac{1}{2}} f\left(\Phi_i(I)^{-\frac{1}{2}} \Phi_i(A_i) \Phi_i(I)^{-\frac{1}{2}}\right) \Phi_i(I)^{\frac{1}{2}} \\
&\leq \sum_{i=1}^n \Phi_i(f(A_i)), \qquad (3.1)
\end{aligned}
$$

$$
\begin{aligned}
(ii)\ \sum_{i=1}^n \Phi_i(f(A_i)) - f\left(\sum_{i=1}^n \Phi_i(A_i)\right) &\geq \sum_{i \in J} \Phi_i(f(A_i)) - T_J^{\frac{1}{2}} f\left(T_J^{-\frac{1}{2}} \sum_{i \in J} \Phi_i(A_i) T_J^{-\frac{1}{2}}\right) T_J^{\frac{1}{2}} \\
&\geq 0. \qquad (3.2)
\end{aligned}
$$

for all self-adjoint operators A_i and all $J \subseteq \{1, \cdots, n\}$.

Proof. (i) Put $C = T_J^{\frac{1}{2}}$ and $D = T_{\bar{J}}^{\frac{1}{2}}$. Clearly $C^*C + D^*D = I$. It follows from the Jensen operator inequality that

$$
\begin{aligned}
&T_J^{\frac{1}{2}} f\left(T_J^{-\frac{1}{2}} \sum_{i \in J} \Phi_i(A_i) T_J^{-\frac{1}{2}}\right) T_J^{\frac{1}{2}} + T_{\bar{J}}^{\frac{1}{2}} f\left(T_{\bar{J}}^{-\frac{1}{2}} \sum_{i \in J} \Phi_i(A_i) T_{\bar{J}}^{-\frac{1}{2}}\right) T_{\bar{J}}^{\frac{1}{2}} \\
&= C^* f\left(C^{*-1} \sum_{i \in J} \Phi_i(A_i) C^{-1}\right) C + D^* f\left(D^{*-1} \sum_{i \in \bar{J}} \Phi_i(A_i) D^{-1}\right) D \\
&\geq f\left(\sum_{i \in J} \Phi_i(A_i) + \sum_{i \in \bar{J}} \Phi_i(A_i)\right) \\
&= f\left(\sum_{i=1}^n \Phi_i(A_i)\right),
\end{aligned}
$$

which is the first inequality of (3.1). Assume that g be the perspective function of f. It follows from Theorem 2.1 that

$$\begin{aligned}
&T_J^{\frac{1}{2}} f\left(T_J^{-\frac{1}{2}} \sum_{i\in J} \Phi_i(A_i) T_J^{-\frac{1}{2}}\right) T_J^{\frac{1}{2}} + T_{\bar{J}}^{\frac{1}{2}} f\left(T_{\bar{J}}^{-\frac{1}{2}} \sum_{i\in \bar{J}} \Phi_i(A_i) T_{\bar{J}}^{-\frac{1}{2}}\right) T_{\bar{J}}^{\frac{1}{2}} \\
&= g\left(\sum_{i\in J} \Phi_i(A_i), T_J\right) + g\left(\sum_{i\in \bar{J}} \Phi_i(A_i), T_{\bar{J}}\right) \\
&= g\left(\sum_{i\in J} \Phi_i(A_i), \sum_{i\in J} \Phi_i(I)\right) + g\left(\sum_{i\in \bar{J}} \Phi_i(A_i), \sum_{i\in \bar{J}} \Phi_i(I)\right) \\
&\le \sum_{i\in J} g(\Phi_i(A_i), \Phi_i(I)) + \sum_{i\in \bar{J}} g(\Phi_i(A_i), \Phi_i(I)) \qquad \text{(by (2.2))} \\
&= \sum_{i=1}^{n} g(\Phi_i(A_i), \Phi_i(I)) \\
&= \sum_{i=1}^{n} \Phi_i(I)^{\frac{1}{2}} f\left(\Phi_i(I)^{-\frac{1}{2}} \Phi_i(A_i) \Phi_i(I)^{-\frac{1}{2}}\right) \Phi_i(I)^{\frac{1}{2}},
\end{aligned}$$

whence we get the second inequality of (3.1). For each $i = 1, \cdots, n$, let the unital positive linear map Ψ_i be defined by

$$\Psi_i(X) = \Phi_i(I)^{-\frac{1}{2}} \Phi_i(X) \Phi_i(I)^{-\frac{1}{2}}.$$

Since f is operator convex, we have

$$\begin{aligned}
f\left(\Phi_i(I)^{-\frac{1}{2}} \Phi_i(A_i) \Phi_i(I)^{-\frac{1}{2}}\right) &= f(\Psi_i(A_i)) \\
&\le \Psi_i(f(A_i)) \\
&= \Phi_i(I)^{-\frac{1}{2}} \Phi_i(f(A_i)) \Phi_i(I)^{-\frac{1}{2}}. \qquad (3.3)
\end{aligned}$$

The last inequality of (3.1) now follows from (3.3).
(ii) Let Ψ be the unital positive linear map defined by $\Psi\left(\oplus_{i\in \bar{J}} A_i \oplus B\right) = \sum_{i\in \bar{J}} \Phi_i(A_i) + T_J^{\frac{1}{2}} B T_J^{\frac{1}{2}}$. Applying Choi–Davis–Jensen's inequality for Ψ we obtain

$$\begin{aligned}
f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right) &= f\left(\sum_{i\in \bar{J}} \Phi_i(A_i) + T_J^{\frac{1}{2}}\left(T_J^{-\frac{1}{2}} \sum_{i\in J} \Phi_i(A_i) T_J^{-\frac{1}{2}}\right) T_J^{\frac{1}{2}}\right) \\
&\le \sum_{i\in \bar{J}} \Phi_i(f(A_i)) + T_J^{\frac{1}{2}} f\left(T_J^{-\frac{1}{2}} \sum_{i\in J} \Phi_i(A_i) T_J^{-\frac{1}{2}}\right) T_J^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{i=1}^{n}\Phi_i(f(A_i)) - f\left(\sum_{i=1}^{n}\Phi_i(A_i)\right)\\
&\qquad \geq \sum_{i=1}^{n}\Phi_i(f(A_i)) - \sum_{i\in\bar{J}}\Phi_i(f(A_i)) - T_J^{\frac{1}{2}} f\left(T_J^{-\frac{1}{2}}\sum_{i\in J}\Phi_i(A_i)T_J^{-\frac{1}{2}}\right)T_J^{\frac{1}{2}}\\
&\qquad = T_J^{-\frac{1}{2}}\sum_{i\in J}\Phi_i(f(A_i))T_J^{\frac{1}{2}} - f\left(T_J^{-\frac{1}{2}}\sum_{i\in J}\Phi_i(A_i)T_J^{-\frac{1}{2}}\right)\\
&\qquad \geq 0.
\end{aligned}$$

The last inequality follows from the Choi–Davis–Jensen inequality. □

Example 3.2. Let $f(t) = t^2$ and $J = \{1\}$. Consider the positive linear maps $\Phi_1, \Phi_2, \Phi_3 : \mathcal{M}_3(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ defined by

$$\Phi_1(A) = \frac{1}{3}(a_{ij})_{1\leq i,j\leq 2}, \qquad \Phi_2(A) = \Phi_3(A) = \frac{1}{3}(a_{ij})_{2\leq i,j\leq 3},$$

for all $A \in \mathcal{M}_3(\mathbb{C})$. Then $\Phi_1(I_3) + \Phi_2(I_3) + \Phi_3(I_3) = I_2$, where I_3 and I_2 are the identity operators in $\mathcal{M}_3(\mathbb{C})$ and $\mathcal{M}_2(\mathbb{C})$, respectively. Also $T_J = \Phi_1(I_3) = \frac{1}{3}I_2$ and $T_{\bar{J}} = \Phi_2(I_3) + \Phi_3(I_3) = \frac{2}{3}I_2$. If

$$A_1 = 3\begin{pmatrix} 2 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad A_2 = 3\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad A_3 = 3\begin{pmatrix} 1 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 1\end{pmatrix},$$

then

$$(\Phi_1(A_1) + \Phi_2(A_2) + \Phi_3(A_3))^2 = \begin{pmatrix} 10 & 5\\ 5 & 5\end{pmatrix},$$

$$T_J^{\frac{1}{2}} f\left(T_J^{-\frac{1}{2}}\sum_{i\in J}\Phi_i(A_i)T_J^{-\frac{1}{2}}\right)T_J^{\frac{1}{2}} + T_{\bar{J}}^{\frac{1}{2}} f\left(T_{\bar{J}}^{-\frac{1}{2}}\sum_{i\in \bar{J}}\Phi_i(A_i)T_{\bar{J}}^{-\frac{1}{2}}\right)T_{\bar{J}}^{\frac{1}{2}} = \begin{pmatrix} 15 & 3\\ 3 & 6\end{pmatrix},$$

$$\begin{aligned}
&\Phi_1(I)^{\frac{1}{2}}\left(\Phi_1(I)^{-\frac{1}{2}}\Phi_1(A_1)\Phi_1(I)^{-\frac{1}{2}}\right)^2\Phi_1(I)^{\frac{1}{2}}\\
&\qquad + \Phi_2(I)^{\frac{1}{2}}\left(\Phi_2(I)^{-\frac{1}{2}}\Phi_2(A_2)\Phi_2(I)^{-\frac{1}{2}}\right)^2\Phi_2(I)^{\frac{1}{2}}\\
&\qquad + \Phi_3(I)^{\frac{1}{2}}\left(\Phi_3(I)^{-\frac{1}{2}}\Phi_3(A_3)\Phi_3(I)^{-\frac{1}{2}}\right)^2\Phi_3(I)^{\frac{1}{2}}\\
&\qquad = \begin{pmatrix} 18 & 3\\ 3 & 9\end{pmatrix},
\end{aligned}$$

$$\Phi_1(f(A_1)) + \Phi_2(f(A_2)) + \Phi_3(f(A_3)) = \begin{pmatrix} 21 & 3 \\ 3 & 15 \end{pmatrix}.$$

Now inequalities

$$\begin{pmatrix} 10 & 5 \\ 5 & 5 \end{pmatrix} \leq \begin{pmatrix} 15 & 3 \\ 3 & 6 \end{pmatrix} \leq \begin{pmatrix} 18 & 3 \\ 3 & 9 \end{pmatrix} \leq \begin{pmatrix} 21 & 3 \\ 3 & 15 \end{pmatrix},$$

show that all inequalities of (3.1) are strict. By the same computation, one can show that inequalities of (ii) are strict.

Corollary 3.3. *Let f be an operator convex function, $A_1, \cdots, A_n$ be self-adjoint operators and $C_1, \cdots, C_n$ be such that $\sum_{i=1}^n C_i^* C_i = I$. Then*

$$\begin{aligned} f\left(\sum_{i=1}^n C_i^* A_i C_i\right) &\leq T_J^{\frac{1}{2}} f\left(T_J^{-\frac{1}{2}} \sum_{i\in J} C_i^* A_i C_i T_J^{-\frac{1}{2}}\right) T_J^{\frac{1}{2}} + T_{\bar{J}}^{\frac{1}{2}} f\left(T_{\bar{J}}^{-\frac{1}{2}} \sum_{i\in \bar{J}} C_i^* A_i C_i T_{\bar{J}}^{-\frac{1}{2}}\right) T_{\bar{J}}^{\frac{1}{2}} \\ &\leq \sum_{i=1}^n (C_i^* C_i)^{\frac{1}{2}} f\left((C_i^* C_i)^{-\frac{1}{2}} (C_i^* A_i C_i)(C_i^* C_i)^{-\frac{1}{2}}\right) (C_i^* C_i)^{\frac{1}{2}} \\ &\leq \sum_{i=1}^n C_i^* f(A_i) C_i, \end{aligned}$$

where $T_J = \sum_{i\in J} C_i^* C_i$.

Proof. Apply Theorem 3.1 for $\Phi_i(A) = C_i^* A C_i$. □

The rest of this section is devoted to some operator inequalities derived from our results.

1°. For all self-adjoint operators C, D and strictly positive operators A, B,

$$(C + D)(A + B)^{-1}(C + D) \leq CA^{-1}C + DB^{-1}D. \tag{3.4}$$

Proof. Let $\widetilde{L} = (L_1, \cdots, L_n)$ and $\widetilde{R} = (R_1, \cdots, R_n)$ be n-tuples of self-adjoint and strictly positive operators, respectively. Applying Theorem 2.1 for operator convex function $f(t) = t^2$ we obtain

$$\left(\sum_{i=1}^n L_i\right)\left(\sum_{i=1}^n R_i\right)^{-1}\left(\sum_{i=1}^n L_i\right) \leq \sum_{i=1}^n L_i R_i^{-1} L_i. \tag{3.5}$$

Now (3.4) follows from (3.5) with $\widetilde{L} = (C, D)$ and $\widetilde{R} = (A, B)$. □

2°. Let Φ be a positive linear map on $\mathbb{B}(\mathscr{H})$. Applying Theorem 2.6 for the operator convex function $f(t) = t^\beta \quad (-1 \leq \beta \leq 0$ or $1 \leq \beta \leq 2)$, and the operator concave function $h(t)=t^{\alpha}\quad(0\leq\alpha\leq 1)$, we obtain

$$\Phi(R)^{\frac{\alpha}{2}}\left(\Phi(R)^{-\frac{\alpha}{2}}\Phi(L)\Phi(R)^{\frac{-\alpha}{2}}\right)^{\beta}\Phi(R)^{\frac{\alpha}{2}}\leq\Phi\left(R^{\frac{\alpha}{2}}\left(R^{-\frac{\alpha}{2}}LR^{-\frac{\alpha}{2}}\right)^{\beta}R^{\frac{\alpha}{2}}\right). \tag{3.6}$$

In particular, for $\alpha=\frac{1}{2}$ and $\beta=-1$, (3.6) gives rise to

$$\Phi(R)^{\frac{1}{2}}\Phi(L)^{-1}\Phi(R)^{\frac{1}{2}}\leq\Phi\left(R^{\frac{1}{2}}L^{-1}R^{\frac{1}{2}}\right).$$

Note that with $\alpha=1$ and $\beta=-1$ (3.6) gives the known inequality

$$\Phi(R)\Phi(L)^{-1}\Phi(R)\leq\Phi\left(RL^{-1}R\right).$$

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