数理解析研究所講究録
第 1839 巻 2013 年 67-80

OPERATOR MEANS AS UNIQUE SOLUTIONS OF OPERATOR EQUATIONS

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ABSTRACT. In this article we consider means of positive bounded linear operators on a Hilbert space and announce the new results of [31]. We extend the theory of matrix power means to arbitrary operator means in the sense of Kubo-Ando. We consider generalized Karcher equations for positive operators and show that such equations admit unique positive solutions that can be obtained as a limit of one-parameter families of operator means called induced operator means. These means are themselves unique fixed points of one parameter families of strict contractions induced, through Kubo-Ando theory of operator means, by semigroups of holomorphic functions mapping the upper half-plane into itself. These semigroups of holomorphic functions are considered with Koenigs function corresponding to Schroeder's functional equation. Koenigs function in this setting provides us with a logarithm map corresponding to every 2-variable operator mean. The unique solutions of these generalized Karcher equations are called lambda extensions and have numerous desirable properties which are inherited from the induced operator means themselves.

1. INTRODUCTION

Let E be Hilbert space and $S(E)$ denote the Banach space of bounded linear self-adjoint operators. Let $\mathbb{P} \subseteq S(E)$ denote the cone of positive definite operators on E. Recently the theory of means of n-tuple of operators in $\mathbb{P}$ received increased interest, motivated by practical applications, that require the averaging of positive definite matrices, see for example [2, 12, 20, 3, 4]. On the other hand from the theoretical point of view, means of two positive operators are relatively well understood, considering some required basic properties, one can characterize all such functions by positive operator monotone functions on the real half line $(0, \infty)$ [5]. This characterization was achieved by Kubo and Ando some time ago in 1980 [22]. Operator monotonictiy here means monotonicity with respect to the positive definite order on $\mathbb{P}$.

However the several variable theory of operator means still remains subtle, since one does not have the required nice and complete theory of operator monotone functions of several variables. One of the driving forces behind developing a multivariable theory is, that though some simple means like the arithmetic and harmonic are trivially defined for several positive operators, even the case of the geometric mean is non-trivial. The first extension of the geometric mean which received wider interest is due to Ando, Li and Mathias in 2004 [1]. Their extension where based on a so called symmetrization procedure that inductively defines the mean, starting

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Date: March 3, 2013.

Key words and phrases. operator monotone function, starlike function, operator mean, matrix mean.

with the 2-variable geometric mean, as the limit of this process. Later several other such procedures where considered [9, 30] for the geometric mean and later also for other operator means in the sense of Kubo-Ando [29, 30, 32]. These extensions were in general different, but seemed to be identically useful.

Later from the applied side emerged a new geometric mean, which was based on not a symmetrization procedure, but an external geometric characterization, which defined the new mean, at least in the first step, in the finite dimensional case of positive matrices [28]. In the finite dimensional case $\mathbb{P}$ is just the cone of positive definite matrices and this set carries a natural Riemannian metric, which turns the set $\mathbb{P}$ into a Riemannian symmetric space [6]. The Riemannian metric is given as

$$\langle X,Y\rangle_p = Tr\left\{p^{-1}Xp^{-1}Y\right\},$$

where $p \in \mathbb{P}$ is the base point and X, Y are Hermitian matrices. The metric distance induced by the Riemannian structure is

$$d^2(A,B) = Tr\log^2\left(A^{-1/2}BA^{-1/2}\right)$$

for $A, B \in \mathbb{P}$. The new geometric mean is then defined as

$$\Lambda(\omega;\mathbb{A}) = \arg\min_{X\in\mathbb{P}}\sum_{i=1}^{k} w_i d^2(X,A_i), \tag{1}$$

for a positive probability vector $\omega = (w_1,\ldots,w_k)$, $w_i > 0$, $\sum_{i=1}^{k} w_i = 1$ and $\mathbb{A} = (A_1,\ldots,A_k) \in \mathbb{P}^k$. The minimum in (1) exist and is unique due to the convexity of the squared distances appearing on the right hand side [28]. Though several properties which were satisfied by the other geometric means defined by symmetrization procedures did not automatically followed for this new mean (for example operator monotonicity), Bhatia and Holbrook [7] noticed the importance of the mean and concluded that this mean differs from the others. Later also the monotonicity of the mean were proved by Lawson and Lim in [25]. By considering the gradient of the cost function on the right hand side in (1) one can write up the following operator equation

$$\sum_{i=1}^{n} w_i \log_X(A_i) = 0 \tag{2}$$

where $\log_X(B) = X^{1/2}\log(X^{-1/2}BX^{-1/2})X^{1/2}$, and the geometric mean $\Lambda(\omega;\mathbb{A})$ is its unique solution in $\mathbb{P}$. The mean $\Lambda(\omega;\mathbb{A})$ is usually referred to as the *Karcher mean* and the corresponding gradient equation (2) that it uniquely satisfies is the *Karcher equation.*

The property that makes the Karcher mean *the* geometric mean to choose among the other extensions is its importance in applications and its external characterization due to (1) and (2). No other geometric mean admits such external characterization. However one runs into obstacles if wishes to extend this to the case of infinite dimensions. If E is an infinite dimensional Hilbert space, then the cone $\mathbb{P}$ no longer admits Riemannian structures. So one of the characterizations (1) in inevitably lost. However if we stick to instead (2) very recently Lawson and Lim [26] showed that the equation (2) still admits a unique solution in $\mathbb{P}$ even in the infinite dimensional case. Their proof is based on another family of operator means, the matrix power means defined in [27]. The power means $P_s(\omega;\mathbb{A})$ are defined as the unique solutions in $\mathbb{P}$ of the operator equations

$$X=\sum_{i=1}^{k} w_i G_s(X,A_i) \tag{3}$$

where $s\in[-1,1]$ and $G_s(A,B) = A^{1/2}(A^{-1/2}BA^{-1/2})^sA^{1/2}$ is the 2-variable weighted geometric mean. Existence and uniqueness of the solutions follow from the fact that the function

$$f(X)=\sum_{i=1}^{k} w_i G_s(X,A_i)$$

is a strict contraction for $s\in[-1,1], s\neq 0$ with respect to Thompson's part metric [27] on the cone $\mathbb{P}$. The Thompson's part metric [38] is defined as

$$d_\infty(A,B)=\max\{\log M(A/B), M(B/A)\} \tag{4}$$

for any $A,B\in\mathbb{P}$, where $M(A/B)=\inf\{\alpha : A\leq\alpha B\}$. It is a very useful complete metric distance on $\mathbb{P}$, moreover the topology induced by it coincides with the norm topology. Lawson and Lim in [26] use the property of the power means that for $s=0$ the family of power means $P_s(\omega;\mathbb{A})$ are not defined due to the lack of the strict contraction property of $f(X)$, but this singularity is removable at $s=0$ and as $s\to 0$, $P_s(\omega;\mathbb{A})\to\Lambda(\omega;\mathbb{A})$. This property still holds in the infinite dimensional setting and the limit is the unique solution of (2) in $\mathbb{P}$. Also the nice properties of the mean is still fulfilled, like operator monotonicity and others.

The results that we are announcing here is basically the extension of this external characterization of means related to generalized versions of the Karcher equation (2). The results worked out in detail can be found in [31]. The idea is to first generalize the power means by considering operator equations of the form

$$X=\sum_{i=1}^{k} w_i M(X,A_i) \tag{5}$$

where M is an operator mean in the sense of Kubo-Ando. Then we show that the function

$$f_M(X)=\sum_{i=1}^{k} w_i M(X,A_i)$$

is a strict contraction on arbitrary bounded subsets of $\mathbb{P}$ with respect to Thompson's part metric. This is essentially achieved by showing that the function $g(X) = M(X,A)$ itself is a strict contraction on arbitrary bounded subsets of the cone $\mathbb{P}$. To show this we prove new integral characterizations for positive operator monotone functions on $(0,\infty)$ and also new inequalities for Thompson's part metric. Then we conclude that (5) admits a unique positive solution in $\mathbb{P}$, that we denote by $M(\omega;\mathbb{A})$ and call the *induced operator mean* of the original 2-variable operator mean $M(\cdot,\cdot)$. Also we prove several crucial properties fulfilled by $M(\omega;\mathbb{A})$, for example operator monotonicity.

At the same time further representations are also proved for positive operator monotone functions on $(0,\infty)$. These functions are exactly the ones that occur in Kubo-Ando theory and characterize 2-variable operator means in the form

$$M(A,B)=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}, \tag{6}$$

where $f : (0,\infty) \to (0,\infty)$ is a normalized operator monotone function. Normalization means $f(1) = 1$. For such functions f we prove that there exists a unique operator monotone function $\log_I$ on $(0,\infty)$ such that $\log_I(1) = 0$ and $\log_I'(1) = 1$ and fulfilling the functional equation

$$\log_I(f(x)) = f'(1)\log_I(x) \tag{7}$$

for all $x > 0$. We call the function $\log_I$ the corresponding *logarithm map* to f and its inverse $\exp_I$ the *exponential map* corresponding to f. Then we prove that the function $f_t(x)$ defined as

$$f_t(x) = \exp_I(t\log_I(x)), \tag{8}$$

where $\exp_I$ is the inverse of $\log_I$, is operator monotone for all $0 \le t \le f'(1)$. This then makes it possible for us to consider one parameter families of 2-variable operator means in the form

$$M_t(A,B) = A^{1/2} f_t\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \tag{9}$$

corresponding to any 2-variable operator mean $M(A,B)$.

With such one parameter families $M_t(A,B)$ at hand we consider the corresponding one parameter families of induced operator means $M_t(\omega;\mathbb{A})$. Then we conclude that the limit $M_t(\omega;\mathbb{A})$ as $t \to 0+$ exists and define

$$\Lambda_M(\omega;\mathbb{A}) = \lim_{t\to 0+} M_t(\omega;\mathbb{A}) \tag{10}$$

where $\Lambda_M(\omega;\mathbb{A})$ is called the *lambda extension of M*. We then prove that actually $\Lambda_M(\omega;\mathbb{A})$ is the unique solution of the operator equation

$$\sum_{i=1}^{n} w_i \log_X(A_i) = 0 \tag{11}$$

where $\log_X(B) = X^{1/2}\log_I(X^{-1/2}BX^{-1/2})X^{1/2}$ and $\log_I$ is the logarithm map corresponding to f. We call an equation of the form (11) a *generalized Karcher equation.* Then we show that the nice properties, like operator monotonicity of the induced means $M_t(\omega;\mathbb{A})$ are preserved in the limit as $t \to 0+$. Hence $\Lambda_M(\omega;\mathbb{A})$ also have several of these nice properties.

Later then we characterize the set of 2-variable operator means that are actually occur as lambda extensions at the same time. This is important, since a lambda extension is defined for any number of operators, so the multivariable forms occur naturally as extensions of the 2-variable ones, like in the case of the geometric mean $\Lambda(\omega;\mathbb{A})$ and actually all matrix power means $P_s(\omega;\mathbb{A})$.

2. Operator means and representations

In this section we shortly review the new results in [31] related to representing functions of 2-variable operator means.

Let E be Hilbert space and $S(E)$ denote the Banach space of bounded linear self-adjoint operators. Let $\mathbb{P} \subseteq S(E)$ denote the cone of positive definite operators on E.

Let us recall the family of matrix (or operator) means [22]:

Definition 2.1. A two-variable function M: $\mathbb{P} \times \mathbb{P} \mapsto \mathbb{P}$ is called a matrix or operator mean if

(i) $M(I,I) = I$ where I denotes the identity,

(ii) if $A \leq A'$ and $B \leq B'$, then $M(A,B) \leq M(A',B')$,
(iii) $CM(A,B)C \leq M(CAC, CBC)$ for all Hermitian C,
(iv) if $A_n \downarrow A$ and $B_n \downarrow B$ then $M(A_n, B_n) \downarrow M(A,B)$,

where $\downarrow$ denotes the convergence in the strong operator topology of a monotone decreasing net.

In property (ii), (iii), (iv) the partial order being used is the positive definite order, i.e. $A \leq B$ if and only if $B - A$ is positive semidefinite. An important consequence of these properties is [22] that every matrix mean can be uniquely represented by a normalized, operator monotone function $f(t)$ in the following form

$$M(A,B) = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}. \tag{12}$$

This unique $f(t)$ is said to be the representing function of the matrix mean $M(A,B)$. So actually matrix means are in one to one correspondence with normalized operator monotone functions, the above characterization provides an order-isomorphism between them. Normalization means that $f(1) = 1$. For symmetric means, i.e. for means $M(A,B) = M(B,A)$, we have $f(t) = tf(1/t)$ which implies that $f'(1) = 1/2$. Operator monotone functions have strong continuity properties, namely all of them are analytic functions and can be analytically continued to the upper complex half-plane. This is the consequence of the integral characterization of an operator monotone function $f(t)$, which is given over the interval $(0, \infty)$:

$$f(t) = \alpha + \beta t + \int_0^\infty \left(\frac{\lambda}{\lambda^2+1} - \frac{1}{\lambda+t}\right) d\mu(\lambda), \tag{13}$$

where α is a real number, $\beta \geq 0$ and μ is a unique positive measure on $(0, \infty)$ such that

$$\int_0^\infty \frac{1}{\lambda^2+1} d\mu(\lambda) < \infty. \tag{14}$$

The set of all matrix means is denoted by $\mathfrak{M}$, i.e.

$$\mathfrak{M} = \{M(\cdot,\cdot) : M(A,B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},\ f \text{ operator monotone on } (0,\infty), f(1) = 1\}.$$

Similarly $\mathfrak{m} = \{f(x) : f \text{ is a representing function of an } M \in \mathfrak{M}\}$.

We will use the notation $\mathfrak{P}(t)$ to denote the set of all operator monotone functions f on $(0, \infty)$ such that $f(x) > 0$ for all $x \in (0, \infty)$ and $f(1) = 1, f'(1) = t$. We can find the minimal and maximal elements of $\mathfrak{P}(t)$ for all $t \in (0,1)$ easily.

Lemma 2.1. *For all $f(x) \in \mathfrak{P}(t)$ we have*

$$\left((1-t) + tx^{-1}\right)^{-1} \leq f(x) \leq (1-t) + tx. \tag{15}$$

Lemma 2.2. *All $f(x) \in \mathfrak{P}(t)$ for $t \in (0,1)$ has only one fixed point in $(0,\infty)$ which is 1 and 1 is an attractive fixed point on $(0, \infty)$.*

Proposition 2.3. *Let $f(x)$ be a representing function of a matrix mean in $\mathfrak{M}$. Then*

$$f(x) = \int_{[0,1]} [(1-s) + sx^{-1}]^{-1} d\nu(s) \tag{16}$$

where ν is a probability measure over the closed interval $[0,1]$.

Remark 2.1. The above results gives us that all matrix means are uniquely represented as convex combinations of weighted harmonic means, since the normalized operator monotone function

$$f_t(x) = [(1-t) + tx^{-1}]^{-1}$$

is the representing function of the weighted harmonic mean, see Lemma 2.1. Also the extreme points of this set are these weighted harmonic means.

Corollary 2.4. *The property $d\nu(s) = d\nu(1-s)$ characterizes symmetric means.*

There are two degenerate cases of matrix means induced by a ν which are supported only over the single points 0 or 1. One of them is the left trivial mean

$$l(x) = 1$$

with represented matrix mean $M(A, B) = A$ and the right trivial mean

$$r(x) = x$$

with represented matrix mean $M(A, B) = B$.

Proposition 2.5. *Let $M \in \mathfrak{M}$ with representing function $f(x)$. Then $0 \leq f'(1) \leq 1$. Moreover if M is not the left or right trivial mean ($f(x) \neq 1$ or x), then $f \in \mathfrak{P}(t)$.*

3. Exponential and logarithm map of operator means

In this section we define the logarithm map $\log_I$ and its inverse the exponential map corresponding to a normalized operator monotone function and hence to 2-variable operator means.

Theorem 3.1. *Let $M(A, B)$ be an operator mean with representing function $f \in \mathfrak{P}(t)$. Then*

$$\lim_{n\to\infty} \frac{M(A,B)^{\circ n} - A}{f'(1)^n} = A^{1/2} \log_I \left(A^{-1/2} B A^{-1/2}\right) A^{1/2} \tag{17}$$

where the limit exists and is uniform for all $A, B \in \mathbb{P}$ and $\log_I(x)$ is an operator monotone function which fulfills the functional equation

$$\log_I(f(x)) = f'(1) \log_I(x) \tag{18}$$

on the interval $(0, \infty)$.

Remark 3.1. The functional equation (18) was first studied by Schröder for holomorphic functions on the unit disk in [35] long ago. Later Koenigs suggested in [21] the iterative construction given above in Theorem 3.1 to provide a solution to the functional equation on the unit disk. Usually in this setting the function $\log_I$ is said to be a Koenigs eigenfunction for function composition as an operator acting on a certain Hardy space of holomorphic functions on the complex unit disk. He proved also that the rate of convergence of the iteration to $\log_I$ is geometric, moreover that Koenigs function is the unique solution of the functional equation in the class of analytic functions. See also [11, 34, 37] for other results in this setting.

Example 3.1. Consider the one parameter family of functions

$$f_q(x) = [(1-t) + tx^q]^{1/q}$$

for $t \in (0,1)$. These are in $\mathfrak{P}(t)$ if and only if $q \in [-1,1]$, because for other values of q the function is not operator monotone, see exercise 4.5.11 in [6]. It is easy to see that

$$f_q(x)^{\circ n} = \left[t^n x^q + \sum_{k=0}^{n-1} t^k(1-t)\right]^{1/q} = \left[t^n x^q - t^n + 1\right]^{1/q}.$$

In this case we can easily calculate the limit function $\log_{I,f_q}(x)$ by turning the limit into a derivative:

$$\begin{aligned}\log_{I,f_q}(x) &= \lim_{n\to\infty} \frac{(t^n x^q - t^n + 1)^{1/q} - 1}{t^n} = \lim_{s\to 0} \frac{(sx^q - s + 1)^{1/q} - 1}{s} \\ &= \frac{\partial}{\partial s}(sx^q - s + 1)^{1/q}\bigg|_{s=0} = \frac{x^q - 1}{q}.\end{aligned}$$

The limit functions indeed are operator monotone again if and only if $q \in [-1,1]$. This family has a singularity at $q = 0$ but it is easy to verify that it is a removable singularity, so in fact we have

$$\begin{aligned} f_0(x) &= x^t \\ \log_{I,f_0}(x) &= \log(x), \end{aligned}$$

where $\log(x)$ and x^t are also well known to be operator monotone. Particularly x^t as a representing function corresponds to the weighted geometric mean.

Proposition 3.2. *The limit function* $\log_I(x)$ *in Theorem 3.1 satisfies the following:*

(i) $\log_I(x)$ *maps* $\mathbb{P}$ *to* $S(E)$ *injectively,*
(ii) $1 - x^{-1} \le \log_I(x) \le x - 1$ *for all* $x > 0$,
(iii) *If* $\log_{I,f}(x)$ *and* $\log_{I,g}(x)$ *are the corresponding limit functions for* $f, g \in \mathfrak{P}(t)$ *such that* $f(x) \le g(x)$ *for all* $x > 0$, *then* $\log_{I,f}(x) \le \log_{I,g}(x)$ *for all* $x > 0$,
(iv) $\log_I(1) = 0$ *and* $\log_I'(1) = 1$.

We will use the notation $\mathfrak{L}$ to denote the set of operator monotone functions $g(x)$ on $(0,\infty)$ such that $g(1) = 0$ and $g'(1) = 1$.

Proposition 3.3. *Let* $f \in \mathfrak{P}(t)$. *Then*

$$f(x) = \exp_I\left(f'(1)\log_I(x)\right), \tag{19}$$

where $\log_I \in \mathfrak{L}$ *is the unique solution of the functional equation* (18) *in the wider class of continuously differentiable and invertible functions on* $(0,\infty)$ *which vanish at* 1 *and have derivative* 1 *at* 1 *and* $\exp_I$ *is its inverse.*

By Proposition 3.3 it is clear, that for each $f \in \mathfrak{P}(t)$ we have a unique corresponding $\log_I(x)$ in $\mathfrak{L}$. Since this $\log_I$ is operator monotone, it has an (analytic) inverse $\exp_I$.

Definition 3.1 (Exponential and logarithm maps). We say that for an $f \in \mathfrak{P}(t)$ the corresponding unique solution $\log_I(x)$ in $\mathfrak{L}$ of the functional equation (18) is the logarithm map corresponding to $f(x)$, while its inverse $\exp_I(x)$ is the exponential map corresponding to $f(x)$.

4. Semigroups of representing functions

In the following we will go the other way around and see whether the function

$$f_t(x) = \exp_I(t \log_I(x))$$

is in $\mathfrak{P}(t)$ for all $\log_I \in \mathfrak{L}$ and $t \in (0,1)$. We say that a holomorphic function g on the complex plain has a ramification point at z if $g'(z) = 0$.

Theorem 4.1. *Let $\log_I \in \mathfrak{L}$. Then $f_t \in \mathfrak{P}(t)$ for all $t \in (0,1)$ if and only if $log_I(z)$ has no ramification point in $\mathbb{H}^+$.*

What happens if $\log_I$ has a ramification point in $\mathbb{H}^+$? What can then be said about $f_t(z)$?

Proposition 4.2. *Let $\log_I \in \mathfrak{L}$ be induced by an $f_{t_0} \in \mathfrak{P}(t_0)$ using Proposition 3.3. Then $f_t \in \mathfrak{P}(t)$ for all $0 < t \le t_0$.*

Remark 4.1. In general one can assure that if for a given $\log_I \in \mathfrak{L}$ with ramification points $t \log_I(\mathbb{H}^+)$ avoids the image of the ramification points (of $\log_I$) under $\log_I$ in $\mathbb{H}^+$, then $f_t \in \mathfrak{P}(t)$.

Remark 4.2 (Semigroup property)**.** By Proposition 4.2 if $f_{t_0} \in \mathfrak{P}(t_0)$, then for all $0 < s, t \le t_0$ we have

$$f_{st} = f_s \circ f_t = f_t \circ f_s$$

and $f_{st} \in \mathfrak{P}(st)$. I.e. f_t is a semigroup of holomorphic functions with respect to function composition, see [11, 23].

Proposition 4.3. *Let $f_t \in \mathfrak{P}(t)$ and $\log_I \in \mathfrak{L}$ its corresponding logarithm map such that it fulfills the functional equation* (18)*. Then z_0 is a ramification point of f_t if and only if it is a ramification point of* $\log_I$*.*

5. Contraction property of 2-variable operator means for Thompson's part metric

On $\mathbb{P}$ the partial ordering induces a complete metric space structure [38]. Thompson's part metric is defined as

$$d_\infty(A,B) = \max\{\log M(A/B), M(B/A)\} \tag{20}$$

for any $A, B \in \mathbb{P}$, where $M(A/B) = \inf\{\alpha : A \le \alpha B\}$. The metric space $(\mathbb{P}, d_\infty)$ is complete and has some several other nice properties [24].

The new result for d_∞ here is the following.

Proposition 5.1. *Let $A_i, B_i \in \mathbb{P}$, $1 \le i \le k$ and suppose that $d_\infty(A_m, B_m) \ge d_\infty(A_i, B_i)$. Then we have*

$$e^{d_\infty(\sum_{i=1}^k A_i, \sum_{i=1}^k B_i)} \le \max\left\{ \frac{\sum_{i=1}^k e^{d_\infty(A_i,B_i)} e^{-d_\infty(A_m,A_i)}}{\sum_{i=1}^k e^{-d_\infty(A_m,A_i)}}, \frac{\sum_{i=1}^k e^{d_\infty(A_i,B_i)} e^{-d_\infty(B_m,B_i)}}{\sum_{i=1}^k e^{-d_\infty(B_m,B_i)}} \right\}.$$

Let $\overline{B}_A(r) = \{X \in \mathbb{P} : d_\infty(A, X) \le r\}$. The important result here is the following theorem for 2-variable operator means.

Theorem 5.2. *Let $M \in \mathfrak{M}$ and $f(X) = M(A, X)$. If M is not the right trivial mean (i.e. $M(A,B) \neq B$) then the mapping $f(X)$ is a strict contraction on $\overline{B}_A(r)$ for all $r < \infty$, i.e. there exists $0 < \rho_r < 1$ such that*

$$d_\infty(f(X), f(Y)) \leq \rho_r d_\infty(X, Y)$$

for all $X, Y \in \overline{B}_A(r)$.

If M is the right trivial mean (i.e. $M(A,B) = B$) then $f(X)$ is non-expansive on $\mathbb{P}$, that is

$$d_\infty(f(X), f(Y)) \leq d_\infty(X, Y)$$

for all $A, X, Y \in \mathbb{P}$.

Remark 5.1. In [24] Lawson and Lim provided an extension of the geometric, logarithmic and some other iterated means to several variables over $\mathbb{P}$ relying on the Ando-Li-Mathias construction provided in [1]. They established the above contraction property for these means. Our Theorem 5.2 shows that in fact the construction is applicable to all operator means, hence providing multivariable extensions which work in the possibly infinite dimensional setting of $\mathbb{P}$. This were only known in the finite dimensional setting so far for symmetric means which case was proved in [32].

In the next section we will utilize Theorem 5.2 to define new multivariable operator means.

6. Induced operator means

Let Δ_n denote the convex set of positive probability vectors, i.e. if $\omega = (w_1, \ldots, w_n) \in \Delta_n$, then $w_i > 0$ and $\sum_{i=1}^n w_i = 1$. We will use the following notations:

For $\mathbb{A} = (A_1, \ldots, A_k) \in \mathbb{P}^k$, $M \in \mathrm{GL}(E), \mathbf{a} = (a_1, \ldots, a_k) \in (0,\infty)^k, \omega = (w_1, \ldots, w_k) \in \Delta_k$, $f : (0,\infty) \to (0,\infty)$, and for a permutation σ on k-letters let

$$\begin{aligned}
M\mathbb{A}M^* &= (MA_1M^*, \ldots, MA_kM^*), \quad \mathbb{A}_\sigma = (A_{\sigma(1)}, \ldots, A_{\sigma(k)}), \\
\mathbb{A}^{(n)} &= \underbrace{(\mathbb{A}, \ldots, \mathbb{A})}_{n} \in \mathbb{P}^{nk}, \quad \omega^{(n)} = \frac{1}{n}\underbrace{(\omega, \ldots, \omega)}_{n} \in \Delta_{nk}, \\
f(\mathbf{a}) &= (f(a_1), \ldots, f(a_k)), \quad \omega \odot \mathbf{a} = \frac{1}{\sum_{i=1}^k w_i a_i}(w_1a_1, \ldots, w_ka_k) \in \Delta_k, \\
\hat{\omega} &= \frac{1}{1 - w_k}(w_1, \ldots, w_{k-1}) \in \Delta_{k-1}, \qquad \mathbf{a} \cdot \mathbb{A} = (a_1A_1, \ldots, a_kA_k).
\end{aligned}$$

The next lemma is a direct consequence of the contraction result Theorem 5.2.

Lemma 6.1. *Let $\omega \in \Delta_k$ and $A_i \in \mathbb{P}$, $1 \leq i \leq k$ and $M \in \mathfrak{M}$ which is not the right trivial mean. Then the function*

$$f_M(X) = \sum_{i=1}^k w_i M(X, A_i) \tag{21}$$

is a strict contraction with respect to the Thompson metric $d_\infty(\cdot,\cdot)$ on every bounded $S \subseteq \mathbb{P}$ such that $A_i \in S$ for all $1 \leq i \leq k$.

The above lemma yields the following.

Proposition 6.2. *Let $\omega \in \Delta_k$ and $A_i \in \mathbb{P}$, $1 \leq i \leq k$ and $M \in \mathfrak{M}$. Then the equation*

$$X = \sum_{i=1}^{k} w_i M(X, A_i) \tag{22}$$

has a unique positive definite solution in $\mathbb{P}$.

Definition 6.1 (Induced Operator Mean). Let $M(\cdot,\cdot) \in \mathfrak{M}$, $\mathbb{A} = (A_1, \ldots, A_k) \in \mathbb{P}^k$ and $\omega \in \Delta_k$. We denote by $M(\omega; \mathbb{A})$ the unique solution of the equation

$$X = \sum_{i=1}^{k} w_i M(X, A_i). \tag{23}$$

We call $M(\omega; \mathbb{A})$ the ω-weighted induced operator mean of $M \in \mathfrak{M}$ of $A_1, \ldots, A_n$.

Proposition 6.3. *Let $\mathbb{A} = (A_1, \ldots, A_k), \mathbb{B} = (B_1, \ldots, B_k) \in \mathbb{P}^k, \omega \in \Delta_k$ and $M, N \in \mathfrak{M}$ and $M(\omega; \mathbb{A})$, $N(\omega; \mathbb{A})$ the corresponding induced operator means. Then*

1. $M(\omega; \mathbb{A}) = A$ *if* $A_i = A$ *for all* $1 \leq i \leq k$;
2. $M(\omega_\sigma; \mathbb{A}_\sigma) = M(\omega; \mathbb{A})$ *for any permutation* σ;
3. $M(\omega; \mathbb{A}) \leq M(\omega; \mathbb{B})$ *if* $A_i \leq B_i$ *for all* $i = 1, 2, \ldots, k$;
4. *if* $M(A, B) \leq N(A, B)$ *for all* $A, B \in \mathbb{P}$ *then* $M(\omega; \mathbb{A}) \leq N(\omega; \mathbb{A})$;
5. $M(\omega; X\mathbb{A}X^*) = XM(\omega; \mathbb{A})X^*$ *for any* $X \in \mathrm{GL}(E)$;
6. $(1-u)M(\omega; \mathbb{A}) + uM(\omega; \mathbb{B}) \leq M(\omega; (1-u)\mathbb{A} + u\mathbb{B})$ *for any* $u \in [0, 1]$;
7. $d_\infty(M(\omega; \mathbb{A}), M(\omega; \mathbb{B})) \leq \max_{1 \leq i \leq k}\{d_\infty(A_i, B_i)\}$;
8. $M(\omega^{(n)}; \mathbb{A}^{(n)}) = M(\omega; \mathbb{A})$ *for any* $n \in \mathbb{N}$;
9. $M(\omega; A_1, \ldots, A_{k-1}, X) = X$ *if and only if* $X = M(\hat{\omega}; A_1, \ldots, A_{k-1})$. *In particular,* $M(A_1, \ldots, A_k, X) = X$ *if and only if* $X = M(A_1, \ldots, A_k)$;
10. $\Phi(M(\omega; \mathbb{A})) \leq M(\omega; \Phi(\mathbb{A}))$ *for any positive unital linear map* Φ, *where* $\Phi(\mathbb{A}) = (\Phi(A_1), \ldots, \Phi(A_k))$.

Corollary 6.4. *If $k = 2$, $M(w_1, w_2; A, B) \in \mathfrak{M}$ is an operator mean (induced by another mean $M(A, B) \in \mathfrak{M}$).*

Proposition 6.5. *Let $\omega \in \Delta_2$, $A, B \in \mathbb{P}$ and $M \in \mathfrak{M}$ with representing function f. Then*

$$M(w_1, w_2; A, B) = A^{1/2} g\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}$$

where

$$g^{-1}(x) = x f^{-1}\left(\frac{1 - w_1 f(x^{-1})}{w_2}\right). \tag{24}$$

7. Generalized Karcher equations and one parameter families of induced operator means

In this section we generalize the results of [27, 26] which were given for the one parameter family of power means. We will provide solutions of nonlinear operator equations that are given in the following

Definition 7.1 (Generalized Karcher equation). Let $\log_I \in \mathfrak{L}$ and $\log_X(A) = X^{1/2} \log_I(X^{-1/2} A X^{-1/2}) X^{1/2}$. The generalized Karcher equation induced by $\log_I$

is the operator equation

$$\sum_{i=1}^{k} w_i \log_X(A_i) = 0$$

where $X, A_i \in \mathbb{P}$.

By Proposition 2.5 we have that if $M \in \mathfrak{M}$ is not the left or right trivial mean, then $f \in \mathfrak{P}(t)$, $t = f'(1)$. The previous results, in particular Theorem 3.1 ensures us, that all $f \in \mathfrak{P}(t_0)$ can be uniquely written as

$$f(x) = \exp_I(t_0 \log_I(x)) \tag{25}$$

where $\log_I \in \mathfrak{L}$ is the unique logarithm map corresponding to f and $\exp_I$ is the inverse of $\log_I$. We also have by Theorem 4.1 that if $\log_I$ has no ramification points in the upper half-plane $\mathbb{H}^+$, then the one parameter family $f_t(x) = \exp_I(t \log_I(x))$ is in $\mathfrak{P}(t)$ for all $t \in (0,1)$. In the general situation of ramification points if a given $\log_I \in \mathfrak{L}$ is induced by an $f \in \mathfrak{P}(t_0)$, then by Proposition 4.2 $f_t \in \mathfrak{P}(t)$ for all $0 < t \leq t_0$. This makes it possible to consider one parameter families of induced operator means.

Throughout this section we suppose that $M \in \mathfrak{M}$ with representing function $f(x)$ given as (25) and $f(x)$ is not the left or right trivial mean. This means that $f \in \mathfrak{P}(t_0)$ and also then

$$f_t(x) = \exp_I(t \log_I(x))$$

is well defined for $0 < t \leq t_0$, i.e. $f_t \in \mathfrak{P}(t)$ and $M_t(A,B)$ denotes its corresponding mean in $\mathfrak{M}$. Also we assume that $A_i \in \mathbb{P}$ for all $1 \leq i \leq k$ and that $\omega \in \Delta_k$.

Proposition 7.1. *The one parameter family of induced operator means $M_t(\omega; \mathbb{A})$ induced by the $M_t(A,B) \in \mathfrak{M}$ with representing function $f_t(x)$ is continuous for $t \in (0, t_0]$ on any bounded set $S \subseteq \mathbb{P}$.*

Theorem 7.2. *There exists $X_0 \in \mathbb{P}$ such that*

$$\lim_{t \to 0+} M_t(\omega; \mathbb{A}) = X_0.$$

Furthermore for $0 < t \leq s \leq t_0$ we have

$$X_0 \leq M_t(\omega; \mathbb{A}) \leq M_s(\omega; \mathbb{A}) \leq M_{t_0}(\omega; \mathbb{A}).$$

Definition 7.2. Let $\Lambda_M(\omega; \mathbb{A}) = \lim_{t \to 0+} M_t(\omega; \mathbb{A})$ and call it the ω-weighted lambda extension of $M \in \mathfrak{M}$.

Let us summarize our results for $\Lambda_M(\omega; \mathbb{A})$.

Theorem 7.3. *Let $\mathbb{A} = (A_1, \dots, A_k), \mathbb{B} = (B_1, \dots, B_k) \in \mathbb{P}^k, \omega \in \Delta_k$ and $M, N \in \mathfrak{M}$ and $\Lambda_M(\omega; \mathbb{A})$, $\Lambda_N(\omega; \mathbb{A})$ the corresponding lambda extensions. Then*

(1) $\Lambda_M(\omega; \mathbb{A}) = A$ *if* $A_i = A$ *for all* $1 \leq i \leq k$;
(2) $\Lambda_M(\omega_\sigma; \mathbb{A}_\sigma) = \Lambda_M(\omega; \mathbb{A})$ *for any permutation* σ;
(3) $\Lambda_M(\omega; \mathbb{A}) \leq \Lambda_M(\omega; \mathbb{B})$ *if* $A_i \leq B_i$ *for all* $i = 1, 2, \dots, k$;
(4) *if* $M(A,B) \leq N(A,B)$ *for all* $A, B \in \mathbb{P}$ *then* $\Lambda_M(\omega; \mathbb{A}) \leq \Lambda_N(\omega; \mathbb{A})$;
(5) $\Lambda_M(\omega; X\mathbb{A}X^*) = X\Lambda_M(\omega; \mathbb{A})X^*$ *for any* $X \in \mathrm{GL}(E)$;
(6) $(1-u)\Lambda_M(\omega; \mathbb{A}) + u\Lambda_M(\omega; \mathbb{B}) \leq \Lambda_M(\omega; (1-u)\mathbb{A} + u\mathbb{B})$ *for any* $u \in [0,1]$;
(7) $d_\infty(\Lambda_M(\omega; \mathbb{A}), \Lambda_M(\omega; \mathbb{B})) \leq \max_{1 \leq i \leq k}\{d_\infty(A_i, B_i)\}$;
(8) $\Lambda_M(\omega^{(n)}; \mathbb{A}^{(n)}) = \Lambda_M(\omega; \mathbb{A})$ *for any* $n \in \mathbb{N}$;
(9) $\Phi(\Lambda_M(\omega; \mathbb{A})) \leq \Lambda_M(\omega; \Phi(\mathbb{A}))$ *for any positive unital linear map* Φ*, where* $\Phi(\mathbb{A}) = (\Phi(A_1), \dots, \Phi(A_k))$;

(10) $\left(\sum_{i=1}^{k} A_i^{-1}\right)^{-1} \leq \Lambda_M(\omega; \mathbb{A}) \leq \sum_{i=1}^{k} A_i$;

(11) $\Lambda_M(\omega; A_1, \dots, A_{k-1}, X) = X$ *if and only if* $X = \Lambda_M(\hat{\omega}; A_1, \dots, A_{k-1})$. *In particular,* $\Lambda_M(A_1, \dots, A_k, X) = X$ *if and only if* $X = \Lambda_M(A_1, \dots, A_k)$;

(12) $\Lambda_M(\omega; \mathbb{A})$ *is the unique solution of the operator equation* $\sum_{i=1}^{k} w_i \log_X(A_i) = 0$ *where* $\log_X(A) = X^{1/2} \log_I \left(X^{-1/2} A X^{-1/2}\right) X^{1/2}$.

Corollary 7.4. *If* $k = 2$, $\Lambda_M(w_1, w_2; A, B) \in \mathfrak{M}$ *is an operator mean.*

Proposition 7.5. *Let* $\omega \in \Delta_2$, $A, B \in \mathbb{P}$ *and* $M \in \mathfrak{M}$ *with representing function* $f(x) = \exp_I(t \log_I(x))$. *Then*

$$\Lambda_M(w_1, w_2; A, B) = A^{1/2} g\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}$$

where

$$g^{-1}(x) = x \exp_I\left(-\frac{w_1}{w_2} \log_I(x^{-1})\right). \tag{26}$$

8. Determining the set of lambda extensions in 2-variables

How big is the set of lambda extensions $\Lambda_M(w_1, w_2; A, B)$ in $\mathfrak{M}$? Is any element of $\mathfrak{M}$ a lambda extension of some $M \in \mathfrak{M}$? If that is the case then every operator mean in the sense of Kubo-Ando occurs as a lambda extension, hence has a multivariable extension $\Lambda_M(\omega; \mathbb{A})$ with the same $M \in \mathfrak{M}$. We investigate this question now.

Proposition 8.1. *Let* $M \in \mathfrak{M}$ *with representing function* $f(x) = \exp_I(t \log_I(x))$. *Then the representing function* $g(x)$ *of the corresponding lambda extension* $\Lambda_M(w_1, w_2; A, B)$ *for all* $\omega \in \Delta_2$, $A, B \in \mathbb{P}$ *is in* $\mathfrak{P}(w_2)$.

By the previous Proposition 8.1 it is clear that if an operator mean is a lambda extension, then the derivative $g'(1) = w_2$ of its representing function, where $\omega = (w_1, w_2)$ is the weight of the lambda extension. In the next result we will use the following fact. A real function $g(x)$ is a representing function of a mean in $\mathfrak{M}$ if and only if the function $g^*(x) = \frac{x}{g(x)}$ is a representing function of a mean in $\mathfrak{M}$, i.e. it is positive operator monotone on $(0, \infty)$ (cf. Proposition 7.1 [36]). In this setting we say that $g^*(x)$ is the conjugate pair of $g(x)$ and vice versa.

Theorem 8.2. *Let* $M \in \mathfrak{M}$ *be an operator mean with representing function* $g(x)$ *such that* $g'(1) \neq 0, 1/2, 1$. *Let* $g^*(x) = \frac{x}{g(x)}$ *denote the conjugate pair. Define the function* $h(x)$ *s.t.*

(1) *if* $g'(1) < 1/2$ *then* $h(x) := \frac{x}{g^{*-1}(x)}$

(2) *if* $g'(1) > 1/2$ *then* $h(x) := x g^{-1}(x^{-1})$.

Then M *is a lambda extension if and only if there exists a positive integer* n, *such that the function* $h^{\circ 2n}(x)$ *is in* $\mathfrak{m}$, *i.e. it is a representing function of an operator mean in* $\mathfrak{M}$. *Moreover in this case the function* $\log_I \in \mathfrak{L}$ *in* (26) *in Proposition 7.5 is unique.*

We were unable to derive similar characterizations in the case when $g'(1) = 1/2$. There is one clue however.

Proposition 8.3. *Let* $M \in \mathfrak{M}$ *be an operator mean with representing function* $g(x)$ *such that* $g'(1) = 1/2$. *Define the function as* $h(x) := x g^{-1}(x^{-1})$. *Then* M

is a lambda extension if and only if $g(x) = xg(1/x)$ *and there exists a holomorphic function* $k(z)$ *with* $k(1) = 0$, $k'(1) = 1/2$ *s.t.*

$$\log_I(z) = -k(h(z)) + k(z) \tag{27}$$

where $\log_I \in \mathfrak{L}$ *and there exists a* $t \in (0, 1]$ *such that the function*

$$f_t(x) = \exp_I(t \log_I(x))$$

is in $\mathfrak{P}(t)$ *(where* $\exp_I$ *denotes the inverse of* $\log_I$ *as usual).*

9. Problems

We finish with some open problems.

Problem 9.1. In view of Proposition 3.3 for various $f \in \mathfrak{P}(t)$ find the corresponding logarithm map $\log_I \in \mathfrak{L}$. In some well known cases we know how $\log_I$ looks like, refer to Example 3.1. For example one can choose $f(x) = \frac{x-1}{\log x}$. Then what is the corresponding $\log_I$?

Problem 9.2. Refine Theorem 8.2 and Proposition 8.3.

Problem 9.3. Derive similar results to Theorem 8.2 and Proposition 8.3 in the case of induced operator means.

Problem 9.4. In view of Proposition 8.3 decide whether the logarithmic mean with representing function $f(x) = \frac{x-1}{\log x}$ is a lambda extension (or induced operator mean) or not.

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