数理解析研究所講究録
第1839巻 2013年 43-48

フルタ型の作用素不等式から導かれる関数不等式について
On functional inequalities derived from operator inequalities of Furuta type

渡邉　恵一（新潟大学理学部）
Keiichi Watanabe (Niigata University)

Results.

Theorem 1. Let $0 \leq p$, $1 \leq q$ and $0 \leq r$ with $p+r \leq (1+r)q$. If $0 < x$, then

$$x^{\frac{1+r-\frac{p+r}{q}}{2}}\left(x^{p}-1\right)\left(x^{\frac{p+r}{q}}-1\right) \leq \frac{p}{q}\left(x^{p+r}-1\right)(x-1).$$

Remark. degrees :

$$\text{left} : \frac{1+r-\frac{p+r}{q}}{2}+p+\frac{p+r}{q}=\frac{1+r+\frac{p+r}{q}}{2}+p$$

$$\text{right} : p+r+1$$

If $1+r<\dfrac{p+r}{q}$, then the reverse inequality holds for sufficiently large real number x.

Corollary. Let $1 \leq p$ and $0 \leq r$. If $0 < x$, then

$$(p+r)(x^{p}-1)(x^{1+r}-1) \leq p(1+r)(x^{p+r}-1)(x-1).$$

Corollary. Let $0 < p_2 \leq p_1$, $0 < q_2 \leq q_1$, $p_1 + p_2 = q_1 + q_2$ and $p_1 \leq q_1$. If $0 < x$, then

$$\frac{x^{p_1} - 1}{p_1} \cdot \frac{x^{p_2} - 1}{p_2} \leq \frac{x^{q_1} - 1}{q_1} \cdot \frac{x^{q_2} - 1}{q_2}.$$

Definition. $p_1, \cdots, p_n$: real numbers,

$p_{[1]} \geq \cdots \geq p_{[n]}$: decreasing rearrangement.

$(p_1, \cdots, p_n) \prec (q_1, \cdots, q_n)$

$\overset{def}{\Longleftrightarrow}$

$$p_{[1]} \leq q_{[1]}$$

$$p_{[1]} + p_{[2]} \leq q_{[1]} + q_{[2]}$$

$$\vdots$$

$$p_{[1]} + \cdots + p_{[n-1]} \leq q_{[1]} + \cdots + q_{[n-1]}$$

$$p_{[1]} + \cdots + p_{[n-1]} + p_{[n]} = q_{[1]} + \cdots + q_{[n-1]} + q_{[n]}$$

Theorem 2. If positive real numbers satisfy $(p_1, \cdots, p_n) \prec (q_1, \cdots, q_n)$, then

$$\prod_{i=1}^{n} \frac{x^{p_i} - 1}{p_i} \leq \prod_{i=1}^{n} \frac{x^{q_i} - 1}{q_i} \tag{1}$$

for arbitrary $1 < x$.

If n is even, then (1) holds for $0 < x < 1$.

If n is odd, then the reverse inequality of (1) holds for $0 < x < 1$.

Method 1.

This method is to prove Theorem 1 at first, whose proof is making use of the Furuta inequality and an improvement of Tanahashi's argument on the best possibility of it. An ordinary argument of majorization leads to Theorem 2.

Theorem (Furuta '87). Let $0 \le p$, $1 \le q$ and $0 \le r$ with $p + r \le (1 + r)q$. If $0 \le B \le A$, then

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \le A^{\frac{p+r}{q}}.$$

Theorem (Tanahashi '96). Let $0 < p,\ q,\ r$. If $(1+r)q < p+r$ or $0 < q < 1$, then there exist 2×2 matrices $A,\ B$ with $0 < B \le A$ that do not satisfy the inequality

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \le A^{\frac{p+r}{q}}.$$

Outline of Tanahashi's argument.

$$A = \begin{pmatrix} a & \sqrt{\varepsilon(a-b-\delta)} \\ \sqrt{\varepsilon(a-b-\delta)} & b+\varepsilon+\delta \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$$

where

$$0 < b < 1 < a, \quad 0 < \varepsilon, \quad \varepsilon(1-b) \le \delta(a-1+\varepsilon)$$

and $\delta = \dfrac{1-b}{a-1}\varepsilon$. Then $0 < B \le A$.

Assume that the Furuta inequality holds for the combination of the parameters. Then we would have

$$\left(U^*A^{\frac{r}{2}}UU^*B^pUU^*A^{\frac{r}{2}}U\right)^{\frac{1}{q}} \le U^*A^{\frac{p+r}{q}}U,$$

where U is a unitary matrix which diagonalizes A.

(1) Put $0 \le \det(R-L)$ in order as much as possible, where R (resp. L) is the right (resp. left) hand side of the above inequality.

(2) Estimate the first order of each term with respect to $\varepsilon \to +0$.

(3) If $(1+r)q < p+r$, then let $b \to +0$.
If $0 < q < 1$, then let $a \to \infty$.
This yields a contradiction.

Some improvements. We use

$$A = \begin{pmatrix} a & \sqrt{(a-1)y} \\ \sqrt{(a-1)y} & b+y \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$$

where

$$1 < a < b, \quad 0 < y.$$

The benefit of this modification of matrix A is that it considerably simplifies arguments.

Tanahashi's proof has finished with obtaining a contradiction in a refutation. It is naturally concentrated on the purpose which shows the best possibility of the Furuta inequality. In contrast, we obtain a functional inequality in Theorem 1 by applying l'Hopital's rule.

Method 2.

Once we can formulate Theorem 2, it is easily deduced from a classical theorem on majorization and convex functions.

Theorem (Schur, Hardy-Littlewood-Pólya, Karamata). Let $p_1, \cdots, p_n$, $q_1, \cdots, q_n$ be sequences of real numbers from an interval (α, β). If $(p_1, \cdots, p_n) \prec (q_1, \cdots, q_n)$, then

$$\sum_{i=1}^{n} f(p_i) \leq \sum_{i=1}^{n} f(q_i)$$

for every real valued convex function f on (α, β).

Proposition. Let $1 < x$ be a fixed real number. Then

$$f(t) = \log\left(\frac{x^t - 1}{t}\right)$$

is convex on the interval $(0, \infty)$.

References

[1] T. Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc. 101 (1987), no. 1, 85–88.

[2] G.H. Hardy, J.E. Littlewood and G. Pólya, Inequalities. 2nd ed. Cambridge University Press, 1952.

[3] J. Karamata, *Sur une inegalite relative aux fonctions convexes*, Publ. Math. Univ. Belgrade 1 (1932), 145–148.

[4] A.W. Marshall, I.Olkin, B.C. Arnold, Inequalities: Theory of Majorization and Its Applications, 2nd ed. Springer, 2011.

[5] K. Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc. 124 (1996), 141–146.

[6] K. Watanabe, *An application of matrix inequalities to certain functional inequalities involving fractional powers*, J. Inequal. Appl. 2012:221. doi:10.1186/1029-242X-2012-221.

[7] K. Watanabe, *On a relation between a Schur, Hardy-Littlewood-Pólya, Karamata's theorem and an inequality of some products of* $x^p - 1$ *derived from the Furuta inequality*, preprint.