<table>
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<th>Title</th>
<th>Semi-operator monotonicity for operator monotone functions (Research on structures of operators via methods in geometry and probability theory)</th>
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</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>SANO, Takashi</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2013), 1839: 40-42</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/194947">http://hdl.handle.net/2433/194947</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Semi-operator monotonicity for operator monotone functions

Takashi SANO
Department of Mathematical Sciences, Faculty of Science, Yamagata University

We review results on operator monotone functions. For details, we refer to [11].

Loewner and Kwong matrices

Let $f(t)$ be a continuously differentiable function from the interval $(0, \infty)$ into itself. For distinct $t_1, \ldots, t_n$ in $(0, \infty)$, we define the $n \times n$ matrix $L_{f(t)}(t_1, \ldots, t_n)$ as

$$L_{f(t)}(t_1, \ldots, t_n) := \left[ \frac{f(t_i) - f(t_j)}{t_i - t_j} \right],$$

where the diagonal entries are understood as the first derivatives $f'(t_i)$. This matrix is called a Loewner matrix. Similarly we define the $n \times n$ matrix $K_{f(t)}(t_1, \ldots, t_n)$ as

$$K_{f(t)}(t_1, \ldots, t_n) := \left[ \frac{f(t_i) + f(t_j)}{t_i + t_j} \right],$$

which we call an Kwong matrix. (In [2, 8] it is called an anti-Loewner matrix.) See [3, 4, 5] on Loewner and Kwong matrices.

We also define the $n \times n$ matrix $L_{f(t)}^{(m)}(t_1, \ldots, t_n)$ and $K_{f(t)}^{(m)}(t_1, \ldots, t_n)$ as

$$L_{f(t)}^{(m)}(t_1, \ldots, t_n) := \left[ \frac{(f(t_i))^m - (f(t_j))^m}{t_i^m - t_j^m} \right],$$

$$K_{f(t)}^{(m)}(t_1, \ldots, t_n) := \left[ \frac{(f(t_i))^m + (f(t_j))^m}{t_i^m + t_j^m} \right]$$

for a positive integer $m$.

It is well-known that $f(t)$ is operator monotone if and only if for all $n$ and $t_1, \ldots, t_n$, the Loewner matrices $L_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite; see [10]. If $f(t)$ is operator monotone, the Kwong matrices $K_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite; see [9]. The latter is recently characterized by
Audenaert [2]. On the other hand, it is known that if $f(t)$ is operator monotone, so is the function $t \mapsto \{f(t^{1/m})\}^m$ for any positive integer $m$. See [1, 7]. Hence, combining them, we see that if $f$ is operator monotone, then the Loewner matrices $L_{\{f(t^{1/m})\}^m}(t_1, \ldots, t_n)$ and the Kwong matrices $K_{\{f(t^{1/m})\}^m}(t_1, \ldots, t_n)$ are positive semidefinite; therefore, so are $L_{f(t)}^{(m)}(t_1, \ldots, t_n)$ and $K_{f(t)}^{(m)}(t_1, \ldots, t_n)$.

We have an alternative proof for operator monotonicity of the function $t \mapsto \{f(t^{1/m})\}^m$ by Theorem 1.6 that if $f$ is operator monotone, then $L_{f(t)}^{(m)}(t_1, \ldots, t_n)$ are positive semidefinite for all $n$ and $t_1, \ldots, t_n$ in $(0, \infty)$. We also have in Theorem 1.5 that if $f$ is operator monotone, then $K_{f(t)}^{(m)}(t_1, \ldots, t_n)$ are positive semidefinite for all $n$ and $t_1, \ldots, t_n$ in $(0, \infty)$.

We recall several facts:

**Theorem 1.1** (Löwner [10]) Let $f$ be a $C^1$ function on $(0, \infty)$. Then $f$ is operator monotone if and only if $L_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers $n$ and $t_1, \ldots, t_n$ in $(0, \infty)$.

**Theorem 1.2** (Kwong [9]) Let $f$ be a positive $C^1$ function on $(0, \infty)$. If $f$ is operator monotone, then $K_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers $n$ and $t_1, \ldots, t_n$ in $(0, \infty)$.

**Theorem 1.3** (Audenaert [2]) Let $f$ be a positive $C^1$ function on $(0, \infty)$. For all positive integers $n$ and $t_1, \ldots, t_n$ in $(0, \infty)$ $K_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite if and only if $f(\sqrt{t})\sqrt{t}$ is operator monotone.

**Theorem 1.4** (Ando [1], Fujii-Fujii [7]) Let $f$ be an operator monotone function from $(0, \infty)$ into itself. Then so is the function $t \mapsto \{f(t^{1/m})\}^m$ for any positive integer $m$.

We have the following theorems in [11]:

**Theorem 1.5** Let $f$ be an operator monotone function from $(0, \infty)$ into itself. Then for any positive integer $m$, $K_{f(t)}^{(m)}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers $n$ and $t_1, \ldots, t_n$ in $(0, \infty)$; or $K_{\{f(t^{1/m})\}^m}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers $n$ and $t_1, \ldots, t_n$ in $(0, \infty)$.

**Theorem 1.6** Let $f$ be an operator monotone function from $(0, \infty)$ into itself. Then for any positive integer $m$, $L_{f(t)}^{(m)}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers $n$ and $t_1, \ldots, t_n$ in $(0, \infty)$; or $L_{\{f(t^{1/m})\}^m}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers $n$ and $t_1, \ldots, t_n$ in $(0, \infty)$.
参考文献


