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Kyoto University
ON GENERALIZED POWERS-STØRMER'S INEQUALITY

HIROYUKI OSAKA\textsuperscript{A}

Abstract. A generalization of Powers-Størmer's inequality for operator monotone functions on $[0, +\infty)$ and for positive linear functional on general $C^*$-algebras will be proved. It also will be shown that the generalized Powers-Størmer inequality characterizes the tracial functionals on $C^*$-algebras.

1. INTRODUCTION

Powers-Størmer’s inequality (see, for example, [16, Lemma 2.4], [4, Theorem 11.19]) asserts that for $s \in [0,1]$ the following inequality

(1) \[ 2 \text{Tr}(A^s B^{1-s}) \geq \text{Tr}(A + B - |A - B|) \]

holds for any pair of positive matrices $A, B$. This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory [1]. This inequality was first proven in [1], using an integral representation of the function $t^s$. After that, N. Ozawa gave a much simpler proof for the same inequality, using fact that for $s \in [0,1]$ function $f(t) = t^s$ ($t \in [0, +\infty)$) is an operator monotone ([11, Proposition 1.1]). Recently, Y. Ogata in [13] extended this inequality to standard von Neumann algebras. The motivation of this paper is that if the function $f(t) = t^s$ is replaced by another operator monotone function (this class is intensively studied, see [8][14]), then $\text{Tr}(A + B - |A - B|)$ may get smaller upper bound than what is used in quantum hypothesis testing. Based on N. Ozawa’s proof we formulate Powers-Størmer’s inequality for an arbitrary operator monotone function on $[0, +\infty)$ in the context of general $C^*$-algebras.

2. DOUBLE PILING STRUCTURES FOR MATRIX FUNCTIONS

Throughout this note, $M_n$ stands for the algebra of all $n \times n$ matrices, $M_n^+$ denote the set of positive semi-definite matrices. We call a function

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f matrix convex of order \( n \) or \( n \)-convex in short (resp. matrix concave of order \( n \) or \( n \)-concave) whenever the inequality

\[
f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B), \quad \lambda \in [0, 1]
\]

(resp. \( f(\lambda A + (1 - \lambda)B) \geq \lambda f(A) + (1 - \lambda)f(B), \quad \lambda \in [0, 1] \)) holds for every pair of selfadjoint matrices \( A, B \in M_n \) such that all eigenvalues of \( A \) and \( B \) are contained in \( I \). Matrix monotone functions on \( I \) are similarly defined as the inequality

\[
A \leq B \implies f(A) \leq f(B)
\]

for any pair of selfadjoint matrices \( A, B \in M_n \) such that \( A \leq B \) and all eigenvalues of \( A \) and \( B \) are contained in \( I \). We call a function \( f \) operator convex (resp. operator concave) if for each \( k \in \mathbb{N} \), \( f \) is \( k \)-convex (resp. \( k \)-concave) and operator monotone if for each \( k \in \mathbb{N} \) \( f \) is \( k \)-monotone.

In [15] Tomiyama and the author discussed about the following 3 assertions at each level \( n \) among them in order to see clear insight of the double piling structure of matrix monotone functions and of matrix convex functions:

**Theorem 2.1.** Let \( n \in \mathbb{N} \) and \( f : [0, \alpha) \to \mathbb{R} \) and consider the following assertions.

(i) \( f(0) \leq 0 \) and \( f \) is \( n \)-convex in \([0, \alpha)\),

(ii) For each matrix \( a \) with its spectrum in \([0, \alpha)\) and a contraction \( c \) in the matrix algebra \( M_n \),

\[
f(c^*ac) \leq c^*f(a)c,
\]

(iii) The function \( f(t) = g(t) \) is \( n \)-monotone in \((0, \alpha)\).

Then we have

\[
(i)_{n+1} \prec (ii)_n \sim (iii)_n < (i)_{\lceil\frac{n}{2}\rceil},
\]

where the denotation \( (A)_m \prec (B)_n \) means that "if \( (A) \) holds for the matrix algebra \( M_m \), then \( (B) \) holds for the matrix algebra \( M_n \)."

The following result is proved in [5].

**Lemma 2.1.** Let \( f \) be a strictly positive, continuous function on \([0, \infty)\). If the function \( f \) is \( 2n \)-monotone, then for any positive semi-definite \( A \) and a contraction \( C \) in \( M_n \) we have

\[
C^*f(A)C \leq f(C^*AC).
\]

The following result is essentially proved in [7, Theorem 2.4], but for the reader's convenience we will include a proof.
Proposition 2.1. Let $f$ be a strictly positive, continuous function on $[0, \infty)$. If $f$ is $2n$-monotone, the function $g(t) = \frac{t}{f(t)}$ is $n$-monotone on $[0, \infty)$.

Proof. Let $A, B$ be positive matrices in $M_n$ such that $0 < A \leq B$.

Let $C = B^{-\frac{1}{2}}A^\frac{1}{2}$. Then $\|C\| \leq 1$. Since $f$ is $2n$-monotone, $-f$ satisfies the Jensen type inequality from Lemma 2.1, that is,

$$-f(A) = -f(C^*BC) \leq -C^*f(B)C$$

$$-f(A) \leq -A^{\frac{1}{2}}B^{-\frac{1}{2}}f(B)B^{-\frac{1}{2}}A^{\frac{1}{2}}$$

$$-A^{-\frac{1}{2}}f(A)A^{-\frac{1}{2}} \leq -B^{-\frac{1}{2}}f(B)B^{-\frac{1}{2}}$$

$$-A^{-1}f(A) \leq -B^{-1}f(B)$$

Therefore, since $-1/t$ is operator monotone, $-1/(-f(t)/t) = t/f(t)$ is $n$-monotone. $\square$

Remark 1. The condition of $2n$-monotonicity of $f$ is needed to guarantee the $n$-monotonicity of $g$. Indeed, it is well-known that $t^3$ is monotone, but not 2-monotone. In this case the function $g(t) = \frac{t}{t^3} = \frac{1}{t^2}$ is obviously not 1-monotone.

Corollary 2.1. Let $f$ be a $2n$-monotone, continuous function on $[0, \infty)$ such that $f((0, \infty)) \subset (0, \infty)$, and let $g$ be a Borel function on $[0, \infty)$ defined by $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$. Then for any pair of positive matrices $A, B \in M_n$ with $A \leq B$, $g(A) \leq g(B)$.

Similarly, we can get the concave version of the above observation [10].

Theorem 2.2. For $n \in \mathbb{N}$ and $f : [0, \alpha) \to \mathbb{R}$ we consider the following assertions:

(iv) $f(0) \geq 0$ and $f$ is $n$-concave in $[0, \alpha)$,

(v) For each matrix $a$ with spectrum in $[0, \alpha)$ and a contraction $c$ in the matrix algebra $M_n$,

$$f(c^*ac) \geq c^*f(a)c,$$

(vi) The function $\frac{t}{f(0)}$ is $n$-monotone in $(0, \alpha)$.

We can show that

$$(iv)_{n+1} \prec (v)_{n} \sim (vi)_{n} \prec (iv)_{\lfloor \frac{n}{2} \rfloor}.$$
3. Generalized Powers-Størmer's Inequality

In this section we investigate the generalized Powers-Størmer inequality from the point of matrix functions. Note that the $2n$-monotonicity of a function $f$ on $[0, \infty)$ implies the $n$-concavity of $f$ by [2, Theorem V.2.5].

**Theorem 3.1.** Let $\text{Tr}$ be the canonical trace on $M_n$ and $f$ be a $(n+1)$-concave (or $2n$-monotone) function on $[0, \infty)$ such that $f((0, \infty)) \subset (0, \infty)$. Then for any pair of positive matrices $A, B \in M_n$

\begin{equation}
\text{Tr}(A) + \text{Tr}(B) - \text{Tr}(|A - B|) \leq 2 \text{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}),
\end{equation}

where $g(t) = \begin{cases} 
\frac{t}{f(t)} & (t \in (0, \infty)) \\
0 & (t = 0)
\end{cases}$.

**Proof.** Note that we know that a function $g$ is $n$-monotone from Corollary 2.1 and Theorem 2.2.

Let $A, B$ be any positive matrices in $M_n$.

For operator $(A - B)$ let us denote by $P = (A - B)^+$ and $Q = (A - B)^-$ its positive and negative part, respectively. Then we have

\begin{equation}
A - B = P - Q \quad \text{and} \quad |A - B| = P + Q,
\end{equation}

from that it follows that

\begin{equation}
A + Q = B + P.
\end{equation}

On account of (4) the inequality (2) is equivalent to the following

\begin{equation}
\text{Tr}(A) - \text{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) \leq \text{Tr}(P).
\end{equation}

Since $B + P \geq B \geq 0$ and $B + P = A + Q \geq A \geq 0$ and $g$ is $n$-monotone, we have $g(A) \leq g(B + P)$ and

\begin{align*}
\text{Tr}(A) - \text{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) & = \text{Tr}(f(A)^{\frac{1}{2}}g(A)f(A)^{\frac{1}{2}}) - \text{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) \\
& \leq \text{Tr}(f(A)^{\frac{1}{2}}g(B+P)f(A)^{\frac{1}{2}}) - \text{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) \\
& = \text{Tr}(f(A)^{\frac{1}{2}}(g(B+P) - g(B))f(A)^{\frac{1}{2}}) \\
& \leq \text{Tr}(f(B+P)^{\frac{1}{2}}(g(B+P) - g(B))f(B+P)^{\frac{1}{2}}) \\
& = \text{Tr}(f(B+P)^{\frac{1}{2}}g(B+P)f(B+P)^{\frac{1}{2}}) \\
& - \text{Tr}(f(B+P)^{\frac{1}{2}}g(B)f(B+P)^{\frac{1}{2}}) \\
& \leq \text{Tr}(B+P) - \text{Tr}(f(B)^{\frac{1}{2}}g(B)f(B)^{\frac{1}{2}}) \\
& = \text{Tr}(B+P) - \text{Tr}(B) \\
& = \text{Tr}(P).
\end{align*}
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Hence, we have the conclusion.

\[ \square \]

**Corollary 3.1.** Let \( f \) be an operator monotone function on \([0, \infty)\) such that \( f((0, \infty)) \subset (0, \infty) \). Then for any pair of positive matrices \( A, B \in M_n \)

\[
(5) \quad \text{Tr}(A) + \text{Tr}(B) - \text{Tr}(|A - B|) \leq 2 \text{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}),
\]

where \( g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases} \).

Note that the operator monotonicity is equivalent to the operator concavity in the case that \( f([0, \infty)) \subset [0, \infty) \) [2, Theorem V. 2.5].

**Corollary 3.2.** [1, Theorem 1] Let \( A \) and \( B \) be positive matrices, then for all \( s \in [0, 1] \)

\[
\text{Tr}(A + B - |A - B|) \leq \text{Tr}(A^s B^{1-s}).
\]

**Remark 2.** As pointed in Proposition 2.1, 2-monotonicity of \( f \) is needed to guarantee the inequality (2). Indeed, let \( f(t) = t^3 \) and \( n = 1 \). Then, for any \( a, b \in (0, \infty) \), the inequality (2) would imply

\[
a \leq f(a)^{\frac{1}{2}}g(b)f(a)^{\frac{1}{2}},
\]

that is,

\[
\frac{a}{f(a)} \leq \frac{b}{f(b)}.
\]

Since \( \frac{t}{f(t)} \) is, however, not 1-monotone, the latter inequality is impossible.

**Remark 3.** For matrices \( A, B \in M_n^+ \) let us denote

\[
(6) \quad Q(A, B) = \min_{s \in [0,1]} \text{Tr}(A^{(1-s)/2}B^s A^{(1-s)/2})
\]

and

\[
(7) \quad Q_{\mathcal{F}_{2n}}(A, B) = \inf_{f \in \mathcal{F}_{2n}} \text{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}),
\]

where \( \mathcal{F}_{2n} \) is the set of all \( 2n \)-monotone functions on \([0, +\infty)\) satisfy condition of the Theorem 3.1 and \( g(t) = tf(t)^{-1} (t \in [0, +\infty)) \).

Since the class of \( 2n \)-monotone functions is large enough [14], we know that \( Q_{\mathcal{F}_{2n}}(A, B) \leq Q(A, B) \). Hence, we hope on finding another \( 2n \)-monotone function \( f \) on \([0, +\infty)\) such that

\[
(8) \quad \text{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) < Q(A, B).
\]

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If we can find such a function, then we may get smaller upper bound than what is used in quantum hypothesis testing [1]. For example, considering the trace distance \( T(A, B) = \frac{\text{Tr}(|A - B|)}{2} \), we might have the following better estimate

\[
\frac{1}{2} \text{Tr}(A + B) - Q_{\mathcal{F}_{2n}}(A, B) \leq T(A, B) \leq \sqrt{\left(\frac{1}{2} \text{Tr}(A + B)\right)^2 - Q_{\mathcal{F}_{2n}}(A, B)^2}.
\]

(See the estimate (6) in [1].)

### 4. CHARACTERIZATIONS OF THE TRACE PROPERTY

In this section the generalized Powers-Størmer inequality in the previous section implies the trace property for a positive linear functional on operator algebras.

**Lemma 4.1.** Let \( \varphi \) be a positive linear functional on \( M_n \) and \( f \) be a continuous function on \([0, \infty)\) such that \( f(0) = 0 \) and \( f((0, \infty)) \subset (0, \infty) \). If the following inequality

\[
\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}})
\]

holds true for all \( A, B \in M_n^+ \), then \( \varphi \) should be a positive scalar multiple of the canonical trace \( \text{Tr} \) on \( M_n \), where \( g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases} \).

**Theorem 4.1.** Let \( \varphi \) be a positive normal linear functional on a von Neumann algebra \( \mathcal{M} \) and \( f \) be a continuous function on \([0, \infty)\) such that \( f(0) = 0 \) and \( f((0, \infty)) \subset (0, \infty) \). If the following inequality

\[
\varphi(A) + \varphi(B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}})
\]

holds true for any pair \( A, B \in \mathcal{M}^+ \), then \( \varphi \) is a trace, where \( g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases} \).

**Corollary 4.1.** Let \( \varphi \) be a positive linear functional on a \( C^* \)-algebra \( \mathcal{A} \) and \( f \) be a continuous function on \([0, \infty)\) such that \( f(0) = 0 \) and \( f((0, \infty)) \subset (0, \infty) \). If the following inequality

\[
\varphi(A) + \varphi(B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}})
\]

holds true for any pair \( A, B \in \mathcal{A}^+ \), then \( \varphi \) is a tracial functional, where \( g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases} \).

**Remark 4.** Let \( \mathcal{A} \) be a von Neumann algebra and \( \varphi \) be a positive normal linear functional on \( \mathcal{A} \). The set \( P(\mathcal{A}) \) of all orthogonal projections from \( \mathcal{A} \) is enough as a testing space for some inequality to characterize
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the trace property of \( \varphi \) (see [3]). But, in the case of the inequality (10) the set \( P(\mathcal{A}) \) is not enough as a testing set.

Indeed, let \( P, Q \) be arbitrary orthogonal projections from a von Neumann algebra \( \mathcal{M} \). Since \( Q \geq P \wedge Q \) it follows that \( PQP \geq P(P \wedge Q)P = P \wedge Q \). So \( PQP \geq P \wedge Q \) holds for any pair of projections. From that it follows

\[
\varphi(P + Q - |P - Q|) = 2\varphi(P \wedge Q) \leq 2\varphi(PQP) = 2\varphi(f(P)^{1/2}g(Q)f(P)^{1/2})
\]

whenever \( \varphi \) is an arbitrary positive linear functional on \( \mathcal{M} \).

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H. OSAKA

DEPARTMENT OF MATHEMATICAL SCIENCES, RITSUMEIKAN UNIVERSITY, KUSATSU, SHIGA 525-8577, JAPAN.

E-mail address: osaka@se.ritsumei.ac.jp