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Beckner's inequality and its application to Banach spaces

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1 Introduction

The study of Banach space geometry provides basic concepts and tools in various fields of functional analysis. The origin of geometric properties defined for Banach spaces is probably the uniform convexity introduced by Clarkson. As the uniform convexity of the space L_p was shown by Clarkson's inequality, most of such geometric properties are closely related to various norm inequalities.

Theorem (Clarkson's inequality). *Let* $1<p\leq 2$ *and* $1/p+1/p'=1$. *Then,*

$$(\|f+g\|^{p'}+\|f-g\|^{p'})^{1/p'}\leq 2^{1/p'}(\|f\|^{p}+\|g\|^{p})^{1/p}$$

for all $f, g\in L^{p}$.

Theorem (Hanner's inequality). *If* $1<p\leq 2$ *then*

$$\|f+g\|^{p}+\|f-g\|^{p}\geq\big|\|f\|+\|g\|\big|^{p}+\big|\|f\|-\|g\|\big|^{p}$$

for all $f, g\in L^{p}$, *and if* $2\leq p<\infty$ *then*

$$\|f+g\|^{p}+\|f-g\|^{p}\leq\big|\|f\|+\|g\|\big|^{p}+\big|\|f\|-\|g\|\big|^{p}$$

for all $f, g\in L^{p}$.

In this note, we consider the following classical inequality which was proved by Beckner [2] (cf. [4, Lemma 1.e.14]).

Theorem 1.1. *Let* $1<p\leq q<\infty$, *and let* $\gamma_{p,q}=\sqrt{(p-1)/(q-1)}$. *Then,*

$$\left(\frac{|u+\gamma_{p,q}v|^{q}+|u-\gamma_{p,q}v|^{q}}{2}\right)^{\frac{1}{q}}\leq\left(\frac{|u+v|^{p}+|u-v|^{p}}{2}\right)^{\frac{1}{p}}$$

for all $u, v\in \mathbb{R}$.

It is also known that $\gamma_{p,q}$ in Theorem 1.1 is the best constant, that is, if $a \geq 0$ and

$$\left(\frac{|u+av|^q+|u-av|^q}{2}\right)^{\frac{1}{q}} \leq \left(\frac{|u+v|^p+|u-v|^p}{2}\right)^{\frac{1}{p}}$$

for all $u, v \in \mathbb{R}$, then we have $a \leq \gamma_{p,q}$. We note that the case $0 \leq a \leq 1$ is essential in this direction. Indeed, letting $u = 0$ and $v = 1$ in the above inequality, we obtain $a \leq 1$. The proof of this fact can be found in the proof of [11, Theorem 6].

Our aim is to present an elementary proof of Theorem 1.1 and the above fact (cf. [3, 5, 6, 7]). It is needless to say that Theorem 1.1 is trivial if $p = q$. So we only consider the case $p \neq q$. Suppose that $1 < p < q < \infty$ and that $b \in [0, 1]$. Let A_b be the linear operator from $(\mathbb{R}^2, \|\cdot\|_p)$ into $(\mathbb{R}^2, \|\cdot\|_q)$ defined by

$$A_b = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$$

and let $\|A_b\|_{p,q}$ denote the operator norm of A_b. Put $f_{p,q,b}$ be the real-valued function on $[0, 1]$ defined by

$$f_{p,q,b}(t) = \left(\left(t^{\frac{1}{p}} + b(1-t)^{\frac{1}{p}}\right)^q + \left(bt^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)^q\right)^{\frac{1}{q}}.$$

First, we prove the following two lemmas.

Lemma 1.2. $\|A_b\|_{p,q} = \max_{0 \leq t \leq 1/2} f_{p,q,b}(t)$.

Lemma 1.3. *Let $a \in [0, 1]$ and let $b = (1-a)/(1+a)$. Then, the following are equivalent:*

(i) *The inequality*

$$\left(\frac{|u+av|^q+|u-av|^q}{2}\right)^{\frac{1}{q}} \leq \left(\frac{|u+v|^p+|u-v|^p}{2}\right)^{\frac{1}{p}}$$

hold for all $u, v \in \mathbb{R}$.

(ii) $f_{p,q,b}(1/2) = \max_{0 \leq t \leq 1/2} f_{p,q,b}(t)$.

Now, let

$$\delta_{p,q} = \frac{1-\gamma_{p,q}}{1+\gamma_{p,q}} = \frac{\sqrt{q-1}-\sqrt{p-1}}{\sqrt{q-1}+\sqrt{p-1}} = \frac{p+q-2-2\sqrt{(p-1)(q-1)}}{q-p},$$

and let $\alpha = 1/p$ and $\beta = q-1$, respectively. We note that $0 < \alpha < 1$ and $\beta+1 > \alpha\beta-\alpha+1$. Henceforth, $\delta_{p,q}$ is simply denoted by δ.

Lemma 1.4. *Let $b \in [0, \delta]$ and let $g_{1,b}$ be the real-valued function on $[0, 1]$ defined by*

$$g_{1,b}(u) = -\beta bu^2 + (\alpha\beta+\alpha-1)(1+b^2)u - (2\alpha-1)\beta b - 2(1-\alpha)bu^{\frac{1}{1-\alpha}}.$$

(i) *If $1 < p < 2$, then there exists a real number $u_0 \in (0, 1)$ such that $g_{1,b}(u_0) = 0$, $g_{1,b}(u) < 0$ for all $u \in [0, u_0)$, and $g_{1,b}(u) > 0$ for all $u \in (u_0, 1)$.*

(ii) *If* $2 \le p < \infty$, *then* $g_{1,b}(u) > 0$ *for all* $u \in (0,1)$.

Lemma 1.5. *Let* $b \in [0,\delta]$ *and let* g_2 *be the real-valued function on* $[0,1]$ *defined by*

$$g_{2,b}(s) = (\alpha\beta + \alpha - 1)(1+b^2)s^\alpha - \alpha\beta b(s^{2\alpha-1} + s) - (1-\alpha)b(s^{2\alpha}+1).$$

(i) $g_{2,\delta}(s) \le 0$ *for all* $s \in [0,1]$.

(ii) *If* $0 \le b < \delta$, *then there exists a real number* $s_0 \in (0,1)$ *such that* $g_{2,b}(s_0) = 0$, $g_{2,b}(s) < 0$ *for all* $s \in [0, s_0)$, *and* $g_{2,b}(s) > 0$ *for all* $s \in (s_0, 1)$.

Lemma 1.6. *Let* $g_{3,b}$ *be a real-valued function on* $[0,1]$ *defined by*

$$g_{3,b}(s) = (s^\alpha + b)^\beta(s^{\alpha-1} - b) + (bs^\alpha + 1)^\beta(bs^{\alpha-1} - 1).$$

(i) $g_{3,\delta}(s) \ge 0$ *for all* $s \in [0,1]$.

(ii) *If* $0 \le b < \delta$, *then there exists a real number* $s_1 \in (0,1)$ *such that* $g_{3,b}(s_1) = 0$, $g_{3,b}(s) > 0$ *for all* $s \in [0, s_1)$, *and* $g_{3,b}(s) < 0$ *for all* $s \in (s_1, 1)$.

Proof of Theorem 1.1. Suppose that $b \in [0,\delta]$. Let g_b be the real-valued function on $[0,1/2]$ defined by

$$g_b(t) = (f_{p,q,b}(t))^q = (t^{\frac{1}{p}} + b(1-t)^{\frac{1}{p}})^q + (bt^{\frac{1}{p}} + (1-t)^{\frac{1}{p}})^q.$$

The derivative of g_b is

$$g_b'(t) = \frac{q}{p}(1-t)^{\frac{q}{p}-1} g_{3,b}\left(\frac{t}{1-t}\right).$$

By Lemma 1.6 (i), we have $g_\delta'(t) \ge 0$ for all $t \in [0,1/2]$. Thus the function g_δ is nondecreasing on $[0,1/2]$, and hence we obtain $g_\delta(1/2) = \max_{0\le t\le 1/2} g(t)$. This means that $f_{p,q,\delta}(1/2) = \max_{0\le t\le 1/2} f_{p,q,\delta}(t)$. Thus, by Lemma 1.3, we have

$$\left(\frac{|u+\gamma_{p,q}v|^q + |u-\gamma_{p,q}v|^q}{2}\right)^{\frac{1}{q}} \le \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{\frac{1}{p}}$$

for all $u, v \in \mathbb{R}$. This proves Theorem 1.1.

Finally, we show that $\gamma_{p,q}$ is the best constant for Beckner's inequality. Suppose that $\gamma_{p,q} < a \le 1$. Let $b = (1-a)/(1+a)$. By Lemma 1.3, it is enough to prove that $f_{p,q,b}(1/2) < \max_{0\le t\le 1/2} f_{p,q,b}(t)$. To this end, we remark that $0 \le b < \delta$. By Lemma 1.6 (ii), g_b is strictly increasing on $[0, s_2]$ and strictly decreasing on $[s_2, 1/2]$, where $s_2 = s_1/(1+s_1)$.

t	0	$\cdots$	s_2	$\cdots$	$1/2$
g_b'		$+$	0	$-$	
g_b		$\nearrow$		$\searrow$	

From this fact, we have $f_{p,q,b}(1/2) < f_{p,q,b}(s_2) = \max_{0\le t\le 1/2} f_{p,q,b}(t)$. The proof is complete. $\square$

2 Application to Banach spaces

In this section, we consider an application of Beckner's inequality. First, we extend Beckner's inequality to normed linear spaces.

Theorem 2.1 (Lindenstrauss and Tzafriri [4]). *Let X be a normed linear space. Suppose that $1 < p \leq q < \infty$ and $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then,*

$$\left(\frac{\|x+\gamma_{p,q}y\|^q + \|x-\gamma_{p,q}y\|^q}{2}\right)^{1/q} \leq \left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p}$$

for all $x, y \in X$.

Proof. Take arbitrary $x, y \in X$. Put

$$z = x + y \quad \text{and} \quad w = x - y,$$

respectively. Putting

$$u = \frac{\|z\| + \|w\|}{2} \quad \text{and} \quad v = \frac{\|z\| - \|w\|}{2},$$

then we have

$$\begin{aligned}
&\left(\frac{\|x+\gamma_{p,q}y\|^q + \|x-\gamma_{p,q}y\|^q}{2}\right)^{1/q} \\
&\leq \left(\frac{1}{2}\left(\frac{1+\gamma_{p,q}}{2}\|z\| + \frac{1-\gamma_{p,q}}{2}\|w\|\right)^q + \frac{1}{2}\left(\frac{1-\gamma_{p,q}}{2}\|z\| + \frac{1+\gamma_{p,q}}{2}\|w\|\right)^q\right)^{1/q} \\
&= \left(\frac{|u+\gamma_{p,q}v|^q + |u-\gamma_{p,q}v|^q}{2}\right)^{\frac{1}{q}} \leq \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{\frac{1}{p}} \\
&= \left(\frac{\|z\|^p + \|w\|^p}{2}\right)^{\frac{1}{p}} = \left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{\frac{1}{p}},
\end{aligned}$$

by Beckner's inequality. □

From this result, we remark that in any normed linear space, $\gamma_{p,q}$ is the best constant for Beckner's inequality.

Finally, we see an application of Beckner's inequality to Banach spaces. We recall some notions about q-uniform convexity and p-uniform smoothness.

Definition 2.2. *A Banach space X is said to be uniformly convex if*

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X,\ \|x-y\| = \varepsilon\right\} > 0$$

for all $\varepsilon \in (0, 2]$. The value $\delta_X(\varepsilon)$ is called the modulus of convexity of X.

Definition 2.3. *Let $2 \leq q < \infty$. Then, a Banach space X is said to be q-uniformly convex if there exists a positive number C such that $\delta_X(\varepsilon) \geq C\varepsilon^q$ for all $\varepsilon \in [0, 2]$.*

Clearly, q-uniform convexity implies uniform convexity.

Definition 2.4. *A Banach space X is said to be uniformly smooth if*

$$\lim_{\tau \to 0^+} \frac{\rho_X(\tau)}{\tau} = 0,$$

where $\rho_X(\tau)$ is the modulus of smoothness of X defined by

$$\rho_X(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S_X \right\}$$

for all $\tau \geq 0$.

Definition 2.5. *Let $1 < p \leq 2$. Then, a Banach space X is said to be p-uniformly smooth if there exists a positive number K such that $\rho_X(\tau) \leq K\tau^p$ for all $\tau \geq 0$.*

It is obvious that p-uniform smoothness implies uniform smoothness. The following is an well known characterization of p-uniform smoothness; see, for example, [1].

Theorem 2.6. *Let X be a Banach space and let $1 < p \leq 2$. Then, the following are equivalent:*

(i) *X is p-uniformly smooth.*

(ii) *There exists a positive number K such that*

$$\frac{\|x + y\|^p + \|x - y\|^p}{2} \leq \|x\|^p + \|Ky\|^p$$

for all $x, y \in X$.

(iii) *For any positive number $s \in [1, \infty)$, there exists a positive number K_s such that*

$$\left(\frac{\|x + y\|^s + \|x - y\|^s}{2} \right)^{1/s} \leq (\|x\|^p + \|K_s y\|^p)^{1/p}$$

for all $x, y \in X$.

(iv) *There exist positive numbers $s \in [1, \infty)$ and K_s such that*

$$\left(\frac{\|x + y\|^s + \|x - y\|^s}{2} \right)^{1/s} \leq (\|x\|^p + \|K_s y\|^p)^{1/p}$$

for all $x, y \in X$.

Proof. (i) $\Rightarrow$ (ii): Suppose that X is p-uniformly smooth. Then, there exists a positive number K_1 such that $\rho_X(\tau) \leq K_1\tau^p$ for all $\tau \geq 0$. For each $x \in S_X$ and each $y \in B_X \setminus \{0\}$, we have

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 \leq \rho_X(\|y\|) \leq K_1 \|y\|^p,$$

Put

$$\alpha = \frac{\|x+y\| + \|x-y\|}{2} \quad \text{and} \quad \alpha\beta = \frac{\|x+y\| - \|x-y\|}{2},$$

respectively. Then we have

$$\|x+y\| = \alpha + \alpha\beta \quad \text{and} \quad \|x-y\| = \alpha - \alpha\beta$$

and

$$\begin{aligned}
&\left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p} - (1 + K_1\|y\|^p) \\
&\le \left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p} - \frac{\|x+y\| + \|x-y\|}{2} \\
&\le \alpha\left(\left(\frac{(1+\beta)^p + (1-\beta)^p}{2}\right)^{1/p} - 1\right) \\
&\le \alpha\left(\left(\frac{(1+\beta)^2 + (1-\beta)^2}{2}\right)^{1/2} - 1\right) \\
&\le \alpha((1+\beta^2) - 1) = \alpha\beta^2.
\end{aligned}$$

On the other hand, we have

$$(\alpha\beta)^2 = \left(\frac{\|x+y\| - \|x-y\|}{2}\right)^2 \le \left(\frac{\|x\| + \|y\| - (\|x\| - \|y\|)}{2}\right)^2 = \|y\|^2$$

and

$$\alpha = \frac{\|x+y\| + \|x-y\|}{2} \ge \frac{\|x+y+(x-y)\|}{2} = 1.$$

Thus we obtain

$$\begin{aligned}
\frac{\|x+y\|^p + \|x-y\|^p}{2} &\le ((1 + K_1\|y\|^p) + \alpha\beta^2)^p \\
&\le (1 + K_1\|y\|^p + \|y\|^2)^p \\
&\le \|x\|^p + \|Ky\|^p
\end{aligned}$$

for some $K > 0$. From this, one can show that (ii) holds.

(ii) $\Rightarrow$ (iii): Let $1 \le s < \infty$. Then, by Beckner's inequality, we have

$$\begin{aligned}
\left(\frac{\|x+My\|^s + \|x-My\|^q}{2}\right)^{1/s} &\le \left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p} \\
&\le (\|x\|^p + \|Ky\|^p)^{1/p}
\end{aligned}$$

for all $x, y \in X$, where $M = \min\{1, \gamma_{p,s}\}$. Thus, putting $K_s = KM^{-1}$ and replacing y with $M^{-1}y$, we obtain

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \le (\|x\|^p + \|K_s y\|^p)^{1/p}$$

for all $x, y \in X$.

(iii) $\Rightarrow$ (iv): Obviously holds.

(iv) $\Rightarrow$ (i): Suppose that (iv) holds. Then, there exist positive numbers $s \in [1, \infty)$ and K_s such that

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \leq (\|x\|^p + \|K_s y\|^p)^{1/p}$$

for all $x, y \in X$. Let $x, y \in S_X$ and $\tau \geq 0$. It follows that

$$\begin{aligned}\frac{\|x+y\| + \|x-y\|}{2} \leq \left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} &\leq (\|x\|^p + \|K_s y\|^p)^{1/p} \\ &\leq 1 + \frac{(K_s \tau)^p}{p},\end{aligned}$$

and hence $\rho_X(\tau) \leq (K_s^p/p)\tau^p$. This proves (iv) $\Rightarrow$ (i). □

It is well known that uniform convexity and uniform smoothness are dual properties of each other. A similar fact is true for q-uniform convexity and p-uniform smoothness. To see this, we need the following lemma.

Lemma 2.7 (Takahashi-Hashimoto-Kato [9]). *Let X be a Banach space. Suppose that $1 < p \leq 2$, $1 \leq s < \infty$, $1/p + 1/q = 1$, $1/s + 1/t = 1$ and $K > 0$. Then, the following are equivalent:*

(i) *The inequality*

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \leq (\|x\|^p + \|Ky\|^p)^{1/p}$$

hold for all $x, y \in X$.

(ii) *The inequality*

$$\left(\frac{\|f+g\|^t + \|f-g\|^t}{2}\right)^{1/t} \geq (\|f\|^q + \|K^{-1}g\|^q)^{1/q}$$

hold for all $f, g \in X^*$.

The same is true if X is replaced with X^*.

Theorem 2.8. *Let X be a Banach space and let $2 \leq q < \infty$. Then, the following are equivalent:*

(i) *X is q-uniformly convex.*

(ii) *There exists a positive number C such that*

$$\frac{\|x+y\|^q + \|x-y\|^q}{2} \geq \|x\|^q + \|Cy\|^q$$

for all $x, y \in X$.

(iii) *For any* $t \in (1, \infty]$, *there exists a positive number* C_t *such that*

$$\left(\frac{\|x+y\|^t + \|x-y\|^t}{2} \right)^{1/t} \geq (\|x\|^q + \|C_t y\|^q)^{1/q}$$

for all $x, y \in X$.

(iv) *There exist positive numbers* $t \in (1, \infty]$ *and* C_t *such that*

$$\left(\frac{\|x+y\|^t + \|x-y\|^t}{2} \right)^{1/t} \geq (\|x\|^q + \|C_t y\|^q)^{1/q}$$

for all $x, y \in X$.

Finally, we have the following result.

Corollary 2.9. *Let* X *be a Banach space.*

(i) X *is p-uniformly smooth if and only if* X^* *is q-uniformly convex.*

(ii) X *is q-uniformly convex if and only if* X^* *is p-uniformly smooth.*

The results in this section are summarized as follows:

p-uniformly smooth	$\Rightarrow$	uniformly smooth	$\Rightarrow$	smooth
$\updownarrow$		$\updownarrow$		$(\updownarrow)$
q-uniformly convex	$\Rightarrow$	uniformly convex	$\Rightarrow$	strictly convex

The smoothness and strict convexity are dual properties of each other if the space is reflexive.

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