Some reduced expressions of the classical Weyl groups and the Weyl groupoids of the Lie superalgebras osp$(2m|2n)$ (Hopf algebras and quantum groups: their possible applications)

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Some reduced expressions of the classical Weyl groups and the Weyl groupoids of the Lie superalgebras $\text{osp}(2m|2n)$

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Abstract

We give some reduced expressions of the classical Weyl groups $W(A_{N-1}), W(B_N) = W(C_N), W(D_N)$ and the Weyl groupoid of the Lie superalgebra $\text{osp}(2m|2(N - m))$.

1 Some reduced expressions of the classical Weyl groups

For $m, n \in \mathbb{Z}$, let $J_{n,m} := \{k \in \mathbb{Z} | m \leq k \leq n\}$.

Let $N \in \mathbb{N}$. Let $M_N(\mathbb{R})$ be the $\mathbb{R}$-algebra of $N \times N$-matrices. For $k, r \in J_{1,N}$, let $E_{k,r} := [\delta_{k,k'}\delta_{r,r'}]_{k',r' \in J_{1,N}} \in M_N(\mathbb{R})$, that is $E_{k,r}$ is the matrix unite such that its $(k, r)$-component is 1 and the other components is 0. Then $M_N(\mathbb{R}) = \bigoplus_{k,r \in J_{1,N}} \mathbb{R}E_{k,r}$. Let $\mathbb{R}^N$ denote the $\mathbb{R}$-linear space of $N \times 1$-matrices. For $k \in J_{1,N}$, let $e_k$ is the element of $\mathbb{R}^N$ such that its $(k,1)$-component is 1 and the other components is 0. That is $\{e_k | k \in J_{1,N}\}$ is the standard basis of $\mathbb{R}^N$. The $\mathbb{R}$-algebra $M_N(\mathbb{R})$ acts on $\mathbb{R}^N$ in the ordinal way, that is $E_{k,r}e_p = \delta_{r,p}e_r$. Let $GL_N(\mathbb{R})$ be the group of invertible $N \times N$-matrices, that is $GL_N(\mathbb{R}) = \{X \in M_N(\mathbb{R}) | \det X \neq 0\}$. Let $(, ) : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ be the $\mathbb{R}$-bilinear map defined by $(e_k, e_r) := \delta_{kr}$.

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Definition 1.1. For \( v \in \mathbb{R}^N \setminus \{0\} \), define \( s_v \in \text{GL}_N(\mathbb{R}) \) by
\[
 s_v(u) := u - \frac{2(u,v)}{(v,v)}v \quad (u \in \mathbb{R}^N),
\]
that is, \( s_v \) is the reflection with respect to \( v \).

Note that
\[
 s_v^2 = 1. \tag{1.1}
\]

We say that a subset \( R \) of \( \mathbb{R}^N \setminus \{0\} \) is a root system (in \( \mathbb{R}^N \)) if \(|R|<\infty\), \( s_v(R) = R \) and \( \mathbb{R}v \cap R = \{v, -v\} \) for all \( v \in R \), see [Hum, 1.1].

Let \( R \) be a root system in \( \mathbb{R}^N \). We say that a subset \( \Pi \) of \( R \) is a root basis of \( R \) if \( \Pi \) is a (set) basis of \( \text{Span}_\mathbb{R}(\Pi) \) as an \( \mathbb{R} \)-linear space and \( R \subset \text{Span}_{\mathbb{R}_{\geq 0}}(\Pi) \cup -\text{Span}_{\mathbb{R}_{\geq 0}}(\Pi) \) (this is called a simple system in [Hum, 1.3]).

Let \( R \) be a root system in \( \mathbb{R}^N \). Let \( \Pi \) be a root basis of \( R \). Let \( R^+(\Pi) := R \cap \text{Span}_{\mathbb{R}_{\geq 0}}(\Pi) \). We call \( R^+(\Pi) \) a positive root system of \( R \) associated with \( \Pi \) (this is called a positive system in [Hum, 1.3]).

Definition 1.2. (See [Hum, 2.10].) Let \( R \) be a root system in \( \mathbb{R}^N \). Let \( \Pi \) be a root basis of \( R \).

(1) Assume \( N \geq 2 \). We call \( R \) the \( A_{N-1} \)-type root system if
\[
 R = \{ e_x - e_y \mid x, y \in J_{1,N}, x \neq y \}. 
\]
We call \( \Pi \) the \( A_{N-1} \)-type standard root basis if
\[
 \Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \}. 
\]

(2) Assume \( N \geq 2 \). We call \( R \) the \( B_N \)-type standard root system if
\[
 R = \{ ce_x + c'e_y \mid x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\} \} \cup \{ c''e_x \mid c'' \in \{1, -1\} \}. 
\]
We call \( \Pi \) the \( B_N \)-type standard root basis if
\[
 \Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \} \cup \{ e_N \}. 
\]

(3) Assume \( N \geq 2 \). We call \( R \) the \( C_N \)-type root system if
\[
 R = \{ ce_x + c'e_y \mid x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\} \} \cup \{ 2c''e_x \mid c'' \in \{1, -1\} \}. 
\]
We call \( \Pi \) the \( C_N \)-type standard root basis if
\[
 \Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \} \cup \{ 2e_N \}. 
\]
(4) Assume $N \geq 4$. We call $R$ the $D_N$-type root system if
$$R = \{ ce_x + c'e_y | x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\} \}.$$ 
We call $\Pi$ the $D_N$-type standard root basis if
$$\Pi = \{ e_x - e_{x+1} | x \in J_{1,N-1} \} \cup \{ e_{N-1} + e_N \}.$$ 
Let $R$ be a root system in $\mathbb{R}^N$. Let $\Pi$ be a root basis of $R$. We call $W(\Pi)$ the Coxeter group associated with $(R, \Pi)$. Let $S(\Pi) := \{ s_v \in W(\Pi) | v \in \Pi \}$. We call $(W(\Pi), S(\Pi))$ the Coxeter system associated with $(R, \Pi)$, see [Hum, 1.9 and Theorem 1.5]. Define the map $\ell : W(\Pi) \rightarrow \mathbb{Z}_{\geq 0}$ in the following way, see [Hum, 1.6]. Let $\ell(1) := 0$, where 1 is a unit of $W(\Pi)$. Note that an arbitrary $w \in W(\Pi)$ can be written as a product of finite $s_v$'s with some $v \in \Pi$, say $w = s_{v_1} \cdots s_{v_r}$ for some $r \in \mathbb{N}$ and some $v_x \in \Pi (x \in J_{1,r})$. If $w \neq 1$, let $\ell(w)$ be the smallest $r$ for which such an expression exists, and call the expression reduced. For $w \in W(\Pi)$, we call $\ell(w)$ the length of $w$. Let
$$\mathfrak{L}(w) := \{ v \in R^+(\Pi) | w(v) \in -R^+(\Pi) \}.$$ 
It is well-known that
$$\ell(w) = |\mathfrak{L}(w)|$$
(see [Hum, Corollary 1.7]). It is also well-known that for $v \in \Pi$,
$$s_v(R^+(\Pi) \setminus \{v\}) = R^+(\Pi) \setminus \{v\}$$
(see [Hum, Proposition 1.4]), and
$$\ell(ws_v) = \begin{cases} \ell(w) + 1 & \text{if } w(v) \in R^+(\Pi), \\ \ell(w) - 1 & \text{if } w(v) \in -R^+(\Pi) \end{cases}$$
(see [Hum, Lemma 1.6 and Corollary 1.7]). Assume that $|R| < \infty$. By the above properties, we can see that there exists a unique $w_o \in W(\Pi)$ such that $w_o(\Pi) = -\Pi$, see [Hum, 1.8]. It is well-known that
$$\ell(w_o) = |R^+(\Pi)|,$$
which can easily be proved by (1.2), (1.3) and (1.4). Note that \( w_\circ \) is the only element \( W(\Pi) \) that \( \ell(w) \leq \ell(w_\circ) \) for all \( w \in W(\Pi) \), and \( \ell(w) = \ell(w_\circ) - \ell(w_\circ w^{-1}) \) for all \( w \in W(\Pi) \). We call \( w_\circ \) the longest element of the Coxeter system of \( (W(\Pi), S(\Pi)) \).

Let \( k, r \in J_{1,N} \) be such that \( k \leq r \). For \( z_p \in J_{k,r} \cup (-J_{k,r}) \) \((p \in J_{k,r})\) with \(|u_p| \neq |u_t| \) \((p \neq t)\), let

\[
\begin{align*}
\left\{ k \quad k+1 \quad \ldots \quad r \right\} &= \sum_{p \in J_{k,r}} \frac{z_p}{|z_p|} E_{|z_p|,p} + \sum_{t \in J_{1,N} \setminus J_{k,r}} E_{t,t} \in \GL_N(\mathbb{R}).
\end{align*}
\]

We have

\[
(1.6) \quad s_{ek} = \left\{ \begin{array}{c} k \\ -k \end{array} \right\} \quad (k \in J_{1,N}),
\]

\[
(1.7) \quad s_{e_k-e_{k+1}} = \left\{ \begin{array}{c} k \\ k+1 \\ k \end{array} \right\} \quad (k \in J_{1,N-1}),
\]

and

\[
(1.8) \quad s_{e_k+e_{k+1}} = \left\{ \begin{array}{c} k \\ -(k+1) \\ -k \end{array} \right\} \quad (k \in J_{1,N-1}).
\]

Let \( k, p, r \in J_{k,r} \) with \( k \leq p \leq r \), let

\[
\left\{ k \quad \ldots \quad z_k \quad \ldots \quad p \quad \ldots \quad p+1 \quad \ldots \quad r \right\} := \left\{ \begin{array}{c} k \\ \ldots \end{array} \right\} \left\{ \begin{array}{c} z_k \\ \ldots \end{array} \right\} \left\{ \begin{array}{c} p \\ \ldots \end{array} \right\} \left\{ \begin{array}{c} p+1 \\ \ldots \end{array} \right\} \left\{ \begin{array}{c} r \\ \ldots \end{array} \right\}.
\]

Let \( k, r \in J_{1,N-1} \) with \( k \leq r \). Define \( s_{(k,r)} \) inductively by

\[
(1.9) \quad s_{(k,r)} := \left\{ \begin{array}{c} 1 \quad \text{if} \quad k = r \\ s_{(k,r-1)} s_{e_{r-1}-e_r} \quad \text{if} \quad k < r. \end{array} \right.
\]

Then, if \( r > k \), we have

\[
(1.10) \quad s_{(k,r)} = \left\{ \begin{array}{c} k \\ \ldots \\ p \\ \ldots \\ r-1 \\ r \end{array} \right\},
\]

since (if \( r \geq k+2 \))

\[
(1.11) \quad s_{(k,r)} = s_{(k,r-1)} s_{e_{r-1}-e_r} - \left\{ \begin{array}{c} k \\ \ldots \\ p \\ \ldots \\ r-2 \\ r-1 \end{array} \right\} \left\{ \begin{array}{c} r-1 \\ r \end{array} \right\}
\]

(by (1.7) and an induction)

\[
= \left\{ \begin{array}{c} k \\ \ldots \\ p \\ \ldots \\ r-1 \\ r \end{array} \right\}.
\]
Define $s_{(r,k)}$ inductively by $s_{(r,k)} := s_{e_{r-1}-e_r} s_{(r-1,k)}$ if $r \geq k + 1$. Clearly (if $r > k$) we have

(1.12) \[ s_{(r,k)} = s_{(k,r)}^{-1} = \begin{array}{llllllllll} k & k+1 & \ldots & p & \ldots & r \\ r & k & \ldots & p-1 & \ldots & r-1 \end{array} \]

**Lemma 1.3.** Let $\Pi$ be the $A_{N-1}$-type standard root basis. Let $w_\circ$ be the longest element of $(W(\Pi), S(\Pi))$. Let $s_k := s_{e_k-e_{k+1}} \in S(\Pi)$ for $k \in J_{1,N-1}$.

(1) We have

(1.13) \[ w_\circ = \begin{array}{llllll} 1 & \ldots & p & \ldots & N \\ N & \ldots & N-p+1 & \ldots & 1 \end{array} \]

Moreover

(1.14) \[ w_\circ = (s_1 s_2 \cdots s_{N-1}) (s_1 s_2 \cdots s_{N-2}) \cdots (s_1 s_2) \frac{s_1}{2} \frac{s_1}{1} \]

Furthermore RHS of (1.14) is the reduced expression of $w_\circ$.

(2) Let $m \in J_{2,N-1}$. Then

(1.15) \[ w_\circ = \left( s_{1} s_{2} \cdots s_{m-1} \right) \left( s_{1} s_{2} \cdots s_{m-2} \right) \cdots \left( s_{1} s_{2} \right) \frac{s_{1}}{1} \]

and RHS of (1.15) is a reduced expression of $w_\circ$.

**Proof.** By (1.5), we have

(1.16) \[ \ell(w) = \frac{N(N-1)}{2} \]

Let $k, r \in J_{1,n}$ with $k < r$. Let

\[ x_{(k,r)} := \begin{array}{llllll} k & \ldots & p & \ldots & r \\ r & \ldots & r-p+k & \ldots & k \end{array} \]
Then

\[(1.17) \quad s_{(k,r)}^{(k,r-1)} \cdots s_{(k,k+1)} = x_{(k,r)},\]

since, if \( r \geq k + 2 \), we have

\[
s_{(k,r)}(s_{(k,r-1)} \cdots s_{(k,k+1)}) = \begin{cases} 
k \cdots p \cdots r-1 \; \; r \\
k+1 \cdots p+1 \cdots r \; \; k \\
\end{cases} \cdot x_{(k,r-1)}
\]

(by (1.11) and an induction)

\[= x_{(k,r)}.
\]

We have

\[(1.18) \quad x_{(k,r)} \in W(\Pi) \quad \text{and} \quad \ell(x_{(k,r)}) = \frac{(k-r+1)(k-r)}{2},\]

where the first claim follows from (1.17) and the second claim follows from

by (1.2), since \( \mathfrak{L}(x_{(k,r)}) = \{e_x - e_y | k \leq x < y \leq r\} \).

We obtain the claim (1) from (1.16). (1.17) and (1.18) for \( k = 1 \) and \( r = N \).

For \( k, \; r, \; t \in J_{1,N-1} \) with \( k < r \leq t \), let

\[(1.19) \quad y_{(k,r-1;r,t)} := \begin{cases} 
k \cdots p \quad r-1 \\
k+1 \cdots p+1 \; \; r \\
\end{cases} \cdot x_{(k,r-1)}
\]

We have

\[(1.20) \quad s_{(k+t-r,t)}s_{(k+t-r-1,t-1)} \cdots s_{(k+1,r+1)}s_{(k,r)} = y_{(k,r-1;r,t)}\]

since, if \( t > r \),

\[
(s_{(k+t-r,t)}s_{(k+t-r-1,t-1)} \cdots s_{(k+1,r+1)})s_{(k,r)}
\]

\[= y_{(k+1,r+1;r,t)} \cdot \begin{cases} 
k \cdots p \quad r-1 \\
k+1 \cdots p+1 \; \; r \\
\end{cases} \cdot k
\]

(by (1.11) and an induction)

\[= y_{(k,r-1;r,t)}.
\]
We have
\begin{align}
(1.21) \quad & y_{(k,r-1;r,t)} \in W(\Pi) \quad \text{and} \quad \ell(y_{(k,r-1;r,t)}) = (t - r + 1)(r - k),
\end{align}
where the first claim follows from (1.20) and the second claim follows from
by (1.2), since \( \mathcal{L}(x_{(k,r)}) = \{ e_x - e_y | x \in J_{k,r-1}, \ x \in J_{r,t} \} \).

Let \( m \in J_{2,N-1} \). By (1.13), we have
\begin{align}
(1.22) \quad & w_o = x_{(1,m)}x_{(m+1,N)}y_{(1,N-m;N-m+1,N)}. \nonumber
\end{align}
Then we obtain the claim (2) from (1.16), (1.18), (1.21) and (1.22), since
\[
\frac{m(m-1)}{2} + \frac{(N-m)(N-m-1)}{2} + (N - m)m = \frac{N(N-1)}{2}. \nonumber
\]

Let \( k, r \in J_{1,N} \) with \( k \leq r \). Let
\begin{align}
(1.23) \quad & b_{(k,r)} := s_{e_k} \cdots s_{e_r} = \begin{pmatrix} k & \ldots & p & \ldots & r \\ -k & \ldots & -p & \ldots & -r \end{pmatrix}, \nonumber
\end{align}
see also (1.6). By (1.10), we have
\begin{align}
(1.24) \quad & (s_{(k,r)})^{r-k+1} = 1. \nonumber
\end{align}

By (1.6) and (1.10), we have
\begin{align}
(1.25) \quad & s_{e_t}s_{(k,r)} = s_{(k,r)}s_{e_{t-1}} \nonumber
\end{align}

By (1.23), (1.24) and (1.25), for \( t \in J_{k+1,r} \), we have
\begin{align}
(1.26) \quad & (s_{(k,r)}s_{e_r})^{r-k+1} = (s_{(k,r)})^{r-k+1}s_{e_k} \cdots s_{e_r} = b_{(k,r)}. \nonumber
\end{align}

By (1.6), (1.10) and (1.12), we have
\begin{align}
(1.27) \quad & s_{e_k-e_{k+1}} \cdots s_{e_{r-1}-e_r} s_{e_r} s_{e_{r-1}-e_r} \cdots s_{e_k-e_{k+1}} = s_{(k,r)}s_{e_r}s_{(r,k)} = s_{e_k}. \nonumber
\end{align}

\textbf{Lemma 1.4.} Let \( \Pi \) be the \( B_N \)-type standard root basis. Let \( w_o \) be the longest
element of \( (W(\Pi), S(\Pi)) \). Let \( s_k := s_{e_k-e_{k+1}} \in S(\Pi) \) for \( k \in J_{1,N-1} \) and let
\( s_N := s_{e_N} \in S(\Pi) \).

(1) We have
\begin{align}
(1.28) \quad & w_o = b_{(1,N)} = (s_1s_2 \cdots s_N)^N. \nonumber
\end{align}
Moreover the rightmost hand side of (1.28) is a reduced expression of $w_o$.

(2) Let $k, r \in J_{1,N}$ with $k \leq r$. Then

\begin{equation}
(1.29) \quad b_{(k,r)} = \left(\frac{s_ks_{k+1}\cdots s_{N-1}sNs_{N-1}\cdots s_{r+1}s_r}{2N-k-r+1}\right)^{r-k+1}.
\end{equation}

Moreover RHS of (1.29) is a reduced expression of $b_{(k,r)}$.

(3) Let $k_1, k_2, \ldots, k_{r-1} \in J_{1,N}$ with $k_1 < k_2 < \ldots < k_{r-1}$. Let $b'_y := b_{(k_{y-1},k_y-1)}(y \in J_{1,r})$, where let $k_0 := 1$ and $k_r := N + 1$. Then we have $w_o = b'_1b'_2\cdots b'_r$ and $\ell(w_o) = \sum_{y=1}^{r} \ell(b'_y)$. Moreover $b'_yb'_z = b'_z b'_y$ for $y, z \in J_{1,r}$.

(4) Let $m \in J_{1,N-1}$. Then

\begin{equation}
(1.30) \quad w_o = \left(\frac{s_{N-m+1}s_{N-m+2}\cdots s_N}{m}\right)^{m} \cdot \left(\frac{s_1s_2\cdots s_{N-1}s_{N-1}\cdots s_{N-m+1}s_{N-m}}{N+m}\right)^{N-m}.
\end{equation}

Moreover RHS of (1.30) is a reduced expression of $w_o$.

**Proof.** We can easily show (1.29) by (1.26) and (1.27).

Let $k, r \in J_{1,N}$ be such that $k \leq r$. Note that

\[ \mathcal{L}(b_{(k,r)}) = \{ e_t \mid t \in J_{k,r} \} \cup \{ e_t + ce_{t'} \mid c \in \{-1, 1\}, t \in J_{k,r}, t' \in J_{t',N} \}. \]

Hence by (1.2), we have

\begin{equation}
(1.31) \quad \ell(b_{(k,r)}) = (r - k + 1) + 2\sum_{t=k}^{r} (N - t)
\quad = (r - k + 1) + 2N(r - k + 1) - 2\left( \frac{r(r+1)}{2} - \frac{k(k-1)}{2} \right)
\quad = (r - k + 1)(1 + 2N - (r + k))
\quad = (2N - k - r + 1)(r - k + 1).
\end{equation}

Hence we obtain the second claim of the claim (2). We also obtain the claim (1) since $|R^+(\Pi)| = N^2$.

Let $k, t, r \in J_{1,N}$ be such that $k \leq t < r$. By (1.23), we have

\begin{equation}
(1.32) \quad b_{(k,t)}b_{(t+1,r)} = b_{(k,r)}.
\end{equation}
By (1.31), we have

\[
\ell(b_{(k,t)}) + \ell(b_{(t+1,r)}) \\
= (2N - k - t + 1)(t - k + 1) + (2N - t - r)(r - t) \\
= 2N(r - k + 1) - (k + t - 1)(t - k + 1) - (t + r)(r - t) \\
= 2N(r - k + 1) - (-k^2 + t^2 + 2k - 1) - (r^2 - t^2) \\
= 2N(r - k + 1) + (k^2 - r^2 - 2k + 1) \\
= 2N(r - k + 1) + (k - 1 + r)(k - 1 - r) \\
= (2N - r - k - 1)(r - k + 1) \\
= \ell(b_{(k,r)}).
\]

(1.33)

By (1.32), (1.32) and the claim (1), we get the claim (3).

The claim (4) follows immediately from the claims (1) and (2).

Using Lemma 1.4, we have

**Lemma 1.5.** Let \( \Pi \) be the \( D_{N} \)-type standard root basis. Let \( w_{o} \) be the longest element of \((W(\Pi), S(\Pi))\). Let \( s_{k} := s_{e_{k} - e_{k+1}} \in S(\Pi) \) for \( k \in J_{1,N-1} \) and let \( s_{N} := s_{e_{k} + e_{k+1}} \in S(\Pi) \). For \( k \in J_{1,N-1} \), let

\[
(1.34) \quad d_{(k)} := s_{k} \cdots s_{N-2} s_{N-1} s_{N}^{N-k}. 
\]

Then

\[
(1.35) \quad \ell(d_{(k)}) = (N - k)(N - k + 1)
\]

and

\[
(1.36) \quad d_{(k)} = \begin{cases} 
  b_{(k,N)} & \text{if } N - k \text{ is odd,} \\
  b_{(k,N-1)} & \text{if } N - k \text{ is even.}
\end{cases}
\]

In particular,

\[
(1.37) \quad w_{o} = d_{(1)}.
\]
Proof. By (1.6), (1.7) and (1.8), we have

\begin{equation}
\begin{aligned}
s_{N-1}s_{N} &= \left\{ \begin{array}{ll}
N - 1 & N \\
-(N - 1) & -N
\end{array} \right\} = s_{e_{N-1}}s_{e_{N}}.
\end{aligned}
\end{equation}

Then we have

\[
\text{RHS of (1.34)} = (s_{(k,N-1)}s_{e_{N-1}}s_{e_{N}})^{N-k} \quad \text{(by (1.38))}
\]

\begin{equation}
\begin{aligned}
&= (s_{(k,N-1)}s_{e_{N-1}}s_{e_{N}})^{N-k}s_{e_{N}}^{N-k} \quad \text{(by (1.6) and (1.10))} \\
&= b_{(k,N-1)}s_{e_{N}}^{N-k} \quad \text{(by (1.26))} \\
&= \text{RHS of (1.36)}
\end{aligned}
\end{equation}

By (1.36), we have

\[
\mathcal{L}(d_{(k)}) = \{ e_{t} + ce_{t'} \mid c \in \{-1, 1\}, t \in J_{k,r}, t' \in J_{t',N} \}.
\]

Hence by (1.2), we have (1.35) and (1.37). This completes the proof. \(\square\)

2 Weyl groupoids of super C\(D\)-type

Let \(m \in J_{1,N-1}\). Let \(\mathcal{D}_{m|N-m}\) be the set of maps \(a : J_{1,n} \rightarrow J_{0,1}\) with \(|a^{-1}(\{0\})| = m\).

Let \(a \in \mathcal{D}_{m|N-m}\). Let \((\cdot, \cdot)^{a} : \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow \mathbb{R}\) be the \(\mathbb{R}\)-bilinear map defined by \((e_{i}, e_{j})^{a} := \delta_{ij} \cdot (-1)^{a(i)}\). For \(v \in \mathbb{R}^{N}\) with \((v, v)^{a} \neq 0\), define \(s_{v} \in \text{GL}_{N}(\mathbb{R})\) by \(s_{v}^{a}(u) := u - \frac{2(u,v)^{a}}{(v,v)^{a}}v\) \((u \in \mathbb{R}^{N})\).

Let \(\dot{\mathcal{D}}_{m|N-m} := \{ (a, d) \in \mathcal{D}_{m|N-m} \times J_{0,1} \mid d \in J_{0,a(N)} \}\).
For $i \in J_{1,N}$, define the bijection $\tau_i : \dot{\mathcal{D}}_{m|N-m} \to \dot{\mathcal{D}}_{m|N-m}$ by

$$\tau_i(a, d) := \begin{cases} (a \circ s_{e_i-e_{i+1}}, d) & \text{if } i \in J_{1,N-2} \text{ and } a(i) \neq a(i+1), \\ (a \circ s_{e_{N-1}-e_N}, d) & \text{if } i = N - 1, d = 0 \text{ and } a(N-1) \neq b(N), \\ (a \circ s_{e_{N-1}-e_N}, 1) & \text{if } i = N, a(N-1) = 1, a(N) = 0, \\ (a \circ s_{e_{N-1}-e_N}, 0) & \text{if } i = N, a(N-1) = 0, a(N) = 1 \text{ and } d = 1, \\ (a, d) & \text{otherwise.} \end{cases}$$

Then $\tau_i^2 = id_{\mathbb{R}^N}$.

Let $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$. Let

$$R^{(a,d)}_+ := \{e_x + te_y | x, y \in J_{1,N}, x < y, t \in \{1, -1\}\} \cup \{2e_z | z \in J_{1,N}, a(z) = 1\},$$

and $R^{(a,d)} := R^{(a,d)}_+ \cup -R^{(a,d)}_+$. Then

$$(2.1) \quad |R^{(a,d)}_+| = N(N - 1) + (N - m) = N^2 - m.$$  

For $i \in J_{1,N}$, let

$$\alpha^{(a,d)}_i := \begin{cases} e_i - e_{i+1} & \text{if } i \in J_{1,N-2}, \\ e_{N-1} - e_N & \text{if } i = N - 1 \text{ and } d = 0, \\ 2e_N & \text{if } i = N - 1 \text{ and } d = 1, \\ e_{N-1} + e_N & \text{if } i = N, a(N) = 0 \text{ and } d = 0, \\ 2e_N & \text{if } i = N, a(N) = 1 \text{ and } d = 0, \\ e_{N-1} - e_N & \text{if } i = N, d = 1. \end{cases}$$

Let $\Pi^{(a,d)} := \{\alpha_i^{(a,d)} | i \in J_{1,N}\}$. Then $\Pi^{(a,d)}$ is an $\mathbb{R}$-basis of $\mathbb{R}^N$. Moreover

$$\Pi^{(a,d)} \subset R^{(a,d)}_+ \subset (\bigoplus_{t=1}^{N} \mathbb{Z}_{\geq 0}\alpha_i^{(a,d)}) \setminus \{0\}.$$
Note that
\[ \tau_i(a, d) = (a, d) \text{ if and only if } (\alpha_{i}^{(a,d)}, \alpha_{i}^{(a,d)})^a \neq 0. \]

For \(i \in J_{1,N}\), define \(s_{i}^{(a,d)} \in GL_N(\mathbb{R})\) by
\[
s_{i}^{(a,d)}(\alpha_{i}^{(a,d)}) := \begin{cases} 
-\alpha_{i}^{\tau_i(a,d)} & \text{if } i = j, \\
\sigma_{\alpha_{i}^{\tau_i(a,d)}}^a(\alpha_{j}^{\tau_i(a,d)}) & \text{if } i \neq j \text{ and } (\alpha_{i}^{(a,d)}, \alpha_{i}^{(a,d)})^a \neq 0, \\
\alpha_{j}^{\tau_i(a,d)} & \text{if } i \neq j \text{ and } (\alpha_{i}^{(a,d)}, \alpha_{i}^{(a,d)})^a = (\alpha_{i}^{(a,d)}, \alpha_{j}^{(a,d)})^a = 0, \\
\alpha_{j}^{\tau_i(a,d)} + \alpha_{i}^{\tau_i(a,d)} & \text{if } i \neq j, (\alpha_{i}^{(a,d)}, \alpha_{i}^{(a,d)})^a = 0 \text{ and } (\alpha_{i}^{(a,d)}, \alpha_{j}^{(a,d)})^a \neq 0.
\end{cases}
\]

We can directly see

\textbf{Lemma 2.1.} Let \((a, d) \in \mathcal{D}_{m|N-m}\) and \(i \in J_{1,N}\). Assume that \(d = 0\).
Assume that \(i \in J_{1,N-1}\) if \(a(N-1) = 1\) and \(a(N) = 0\). Then \(s_{i}^{(a,d)} = s_{\alpha_{i}^{(a,d)}}\), where \(s_{\alpha_{i}^{(a,d)}}\) is the one of Definition 1.1.

\textbf{Notation.} Let \((a, d) \in \mathcal{D}_{m|N-m}\). Let \(\text{Map}^N_0\) be a set with \(|\text{Map}^N_0| = 1\).
For \(r \in \mathbb{N}\), let \(\text{Map}^N_r\) be the set of all maps from \(J_{1,r}\) to \(J_{1,N}\). Let \(\text{Map}^N_\infty\) be
the set of all maps from \(\mathbb{N}\) to \(J_{1,N}\). For \(r \in \mathbb{Z}_{\geq 0}\), \(f \in \text{Map}^N_r \cup \text{Map}^N_\infty\) and \(t \in J_{1,r}\), let
\[
(a, d)_{f,0} := (a, d), \quad 1^{(a,d)} s_{f,0} := \text{id}_{\mathbb{R}^N}
\]
\[
(a, d)_{f,t} := \tau_i((a, d)_{f,t-1}), \quad 1^{(a,d)} s_{f,t} := 1^{(a,d)} s_{f,t-1} s_{f(t)}^{(a,d)_{f,t}}.
\]

\textbf{Proposition 2.2.} Let \((a, d) \in \mathcal{D}_{m|N-m}\) be such that \(d = 0\), \(b(z) = 1\) (\(z \in J_{1,N-m}\)) and \(b(z') = 0\) (\(z' \in J_{N-m+1,N}\)). Let \(n := |\mathcal{D}_{m|N}\|\). Define \(f \in \text{Map}^N_n\) by
\[
f(t) := \begin{cases} 
N - m + t & (\text{if } t \in J_{1,m}), \\
f(t - m) & (\text{if } t \in J_{m+1,m(m-1)}), \\
t - m(m-1) & (\text{if } t \in J_{m(m-1)+1,m(m-1)+N}), \\
2N + m(m-1) - t & (\text{if } t \in J_{m(m-1)+N+1,m^2+N}), \\
f(t - (N + m)) & (\text{if } t \in J_{m^2+N+1,n}).
\end{cases}
\]
Then

\[(2.3) \quad 1^{(a,d)} s_{f,n} = \begin{cases} 
  b_{(1,N)} & \text{if } m \text{ is odd,} \\
  b_{(1,N-1)} & \text{if } m \text{ is even.}
\end{cases}\]

Proof. For \( y \in J_{1,m} \), define \( a^{(y)} \in \mathcal{D}_{m|N-m} \) by

\[ a^{(y)}(z) := \begin{cases} 
  1 & \text{if } z \in J_{1,N-m-1} \cup \{N-m+y\}, \\
  0 & \text{if } z \in J_{N-m,N-m+y-1} \cup J_{N-m+y+1,N}.
\end{cases} \]

Then we can directly see that for \( t \in J_{1,n} \),

\[(a, d)_{f,t} = \begin{cases} 
  (a, d) & \text{if } t \in J_{1,m(m-1)+N-m-1}, \\
  (a^{t-(N-m-1)}, 0) & \text{if } t \in J_{m(m-1)+N-m(m-1)+N-1}, \\
  (a^{(m-(t-(m(m-1)+N))-1)}, 0) & \text{if } t \in J_{m(m-1)+N,m(m-1)+N+m}, \\
  (a, d)_{f,t-(N+m)} & \text{if } t \in J_{m^{2}+N+1,n}.
\end{cases} \]

So we see that for \( t \in J_{1,n} \),

\[(2.4) \quad s_{f(t)}^{(a,d)} = \begin{cases} 
  s_{e_{f(t)}-e_{f(t)+1}} & \text{if } f(t) \in J_{1,N-1}, \\
  s_{e_{N-1}+e_{N}} & \text{if } t \in J_{1,m(m-1)} \text{ and } f(t) = N, \\
  s_{2e_{N}} (= s_{e_{N}}) & \text{if } t \in J_{m(m-1)+1,n} \text{ and } f(t) = N.
\end{cases} \]

Define \( f' \in \text{Map}_{n-m(m-1)}^{N} \) by \( f'(t) := f(t+m(m-1)) \), so

\[(2.5) \quad 1^{(a,d)} s_{f,n} = 1^{(a,d)} s_{f,m(m-1)} 1^{(a,d)} f', \quad n-m(m-1). \]

By (1.29) and (1.36), \( 1^{(a,d)} s_{f,m(m-1)} \) equals \( b_{(N-m+1,N)} \) (resp. \( b_{(N-m+1,N-1)} \)) if \( m \) is odd (resp. even). By (1.29) and (2.4), \( 1^{(a,d)} s_{f',n-m(m-1)} = b_{(1,N-m)} \). Hence by (1.22) and (2.5), we have (2.3), as desired. \( \square \)

For \( (a, d) \in \dot{\mathcal{D}}_{m|N-m} \) and \( i, j \in J_{1,N} \), define \( C^{(a,d)} = [c_{ij}^{(a,d)}]_{i,j \in J_{1,N}} \in M_{N}(\mathbb{Z}) \) by

\[ s_{i}^{(a,d)} (\alpha_{j}^{(a,d)}) = \alpha_{j}^{\tau(a,d)} - c_{ij}^{(a,d)} \alpha_{i}^{\tau(a,d)}. \]
Then $C^{(a,d)}$ is a generalized Cartan matrix, i.e., (M1) and (M2) below hold.

(M1) $c^{(a,d)}_{ii} = 2$ (i $\in J_{1,N}$).
(M2) $c^{(a,d)}_{jk} \leq 0$, $\delta_{c^{(a,d)}_{jk},0} = \delta_{c^{(a,d)}_{kj},0}$ (j, k $\in J_{1,N}$, j $\neq$ k).

Then the data

\[ \hat{C}_{m|N-m} := C(J_{1,N}, \hat{D}_{m|N-m}, (\tau_{i})_{i \in J_{1,N}}, (C^{(a,d)})_{(a,d) \in \hat{D}_{m|N-m}}) \]

is a (rank-$N$) Cartan scheme, i.e., (C1) and (C2) below hold.

(C1) $\tau_{i}^{2} = \text{id}_{\hat{D}_{m|N-m}}$ (i $\in J_{1,N}$).
(C2) $c^{\tau_{i}(a,d)}_{ij} = c^{(a,d)}_{ij}$ (i $\in J_{1,N}$).

Note that

\[-c^{(a,d)}_{ij} = |R^{(a,d)}_{+} \cap (\mathbb{Z}\alpha_{i}^{(a,d)} \oplus \mathbb{Z}\alpha_{j}^{(a,d)})| \quad (i, j \in J_{1,N}, i \neq j).\]

The data

\[ \hat{R}_{m|N-m} := R(\hat{C}_{m|N-m}, (R^{(a,d)}_{+})_{(a,d) \in \hat{D}_{m|N-m}}). \]

is a generalized root system of type $C$, i.e., (R1)-(R4) below hold.

(R1) $R^{(a,d)} = R^{(a,d)}_{+} \cup -R^{(a,d)}_{+}$ ((a, d) $\in \hat{D}_{m|N-m}$).
(R2) $R^{(a,d)} \cap \mathbb{Z}\alpha_{i} = \{ \alpha_{i}, -\alpha_{i} \}$ ((a, d) $\in \hat{D}_{m|N-m}$, i $\in J_{1,N}$).
(R3) $s_{i}^{(a,d)}(R^{(a,d)}) = R^{\tau_{i}(a,d)}$ ((a, d) $\in \hat{D}_{m|N-m}$, i $\in J_{1,N}$).
(R4) $(\tau_{i}\tau_{j})^{-c^{(a,d)}_{ij}}(a, d) = (a, d)$ ((a, d) $\in \hat{D}_{m|N-m}$, i, j $\in J_{1,N}$).

For (a, d) $\in \hat{D}_{m|N-m}$, let

\[ W^{(a,d)} := \{ 1^{(a,d)}s_{f,r} \in GL_{N}(\mathbb{R}) \mid r \in \mathbb{Z}_{\geq 0}, f \in \text{Map}_{r}^{N} \}, \]

and define the map $\ell^{(a,d)} : W^{(a,d)} \to \mathbb{Z}_{\geq 0}$ by

\[ \ell^{(a,d)}(w) := \min\{ r \in \mathbb{Z}_{\geq 0} \mid \exists f \in \text{Map}_{r}^{N}, w = 1^{(a,d)}s_{f,r} \}. \]

By [HY08, Lemma 8 (iii)], we see that

\[ 1^{(a,d)}s_{f,r} = 1^{(a,d)}s_{f',r'} \text{ implies } (a, d)_{f,r} = (a, d)_{f',r'}, \]
and that

\[(2.7) \quad \ell^{(a,d)}(w) = |w^{-1}(R_{+}^{(a,d)}) \cap \mathbb{Z}_{\geq 0}\alpha_i|.
\]

For \((a, d) \in \mathcal{D}_{m|N-m}, w \in W^{(a,d)}\) and \(f \in \text{Map}^{N}_{\ell^{(a,d)}}(w),\) if \(w = 1^{(a,d)}s_{f,\ell^{(a,d)}}(w),\)
we call \(f\) a reduced word map of \(w\).

By (2.6) and (2.7), we have formulas for \(W^{(a,d)}\) similar to (1.3) and (1.4). In particular, for each \((a, d) \in \mathcal{D}_{m|N-m},\) there exists a unique \(w_{0}^{(a,d)} \in W^{(a,d)}\) such that

\[\ell^{(a,d)}(w_{0}^{(a,d)}) = |R_{+}^{(a,d)}|,
\]
and we call \(w_{0}^{(a,d)}\) the longest element of \(W^{(a,d)}\).

By Proposition 2.2, we have

**Theorem 2.3.** Let \((a, d) \in \mathcal{D}_{m|N-m}\) be such that \(d = 0, a(z) = 1 (z \in J_{1,N-m})\) and \(a(z') = 0 (z' \in J_{N-m+1,N}).\) Then a reduced word map of \(w_{0}^{(a,d)}\) is given by (2.2). Moreover,

\[(2.8) \quad w_{0}^{(a,d)} = \begin{cases} \ b_{(1,N)} & \text{if } m \text{ is odd}, \\ b_{(1,N-1)} & \text{if } m \text{ is even}. \end{cases}
\]

**Definition 2.4.** For \((a, d), (a', d') \in \mathcal{D}_{m|N-m},\) let \(W^{(a,d)}_{(a',d')}\) be the subset of \(W^{(a,d)}\) composed of all the elements \(1^{(a,d)}s_{f,r}\) with \(r \in \mathbb{Z}_{\geq 0}, f \in \text{Map}^{N}_{r}\) and \((a, d)_{f,r} = (a', d'),\) and \(H^{(a,d)}_{(a',d')} := \{(a, d)\} \times W^{(a,d)}_{(a',d')} \times \{(a', d')\}(\subset \mathcal{D}_{m|N-m} \times \text{GL}_{N}(\mathbb{R}) \times \mathcal{D}_{m|N-m}).\) Let

\[(\mathcal{W}_{m|N-m})' := \bigcup_{(a,d),(a',d')\in\mathcal{D}_{m|N-m}} H_{(a',d')}^{(a,d)},
\]
and \(\mathcal{W}_{m|N-m} := (\mathcal{W}_{m|N-m})' \cup \{0\},\) where \(0\) is an element such that \(0 \notin (\mathcal{W}_{m|N-m})'.\) We regard \(\mathcal{W}_{m|N-m}\) as the semigroup by \(\omega 0 := 0 \omega := 0 (\omega \in \mathcal{W}_{m|N-m})\) and

\[((a_1, d_1), w_1, (a_2, d_2))((a_3, d_3), w_2, (a_4, d_4))
:= \begin{cases} ((a_1, d_1), w_1 w_2, (a_4, d_4)) & \text{if } (a_2, d_2) = (a_3, d_3), \\ 0 & \text{if } (a_2, d_2) \neq (a_3, d_3). \end{cases}
\]

We call \(\mathcal{W}_{m|N-m}\) the Weyl groupoid of the Lie superalgebra osp\((2m|2(N-m)).\)
For \((a, d) \in \dot{\mathcal{D}}_{m|N-m}\), let 
\[\epsilon^{(a,d)} := ((a, d), \text{id}_{\mathbb{R}^N}, (a, d)) \in \mathcal{H}_{(a,d)}^{(a,d)}.\]

For \((a, d) \in \dot{\mathcal{D}}_{m|N-m}\) and \(i \in J_{1,N}\), let 
\[\sigma_i^{(a,d)} := (\tau_i(a, d), s_i^{(a,d)}, (a, d)) \in \mathcal{H}_{\tau_i(a,d)}^{(a,d)}.\]

For \(r \in \mathbb{Z}_{\geq 0}\), \(t \in J_{0,r}\) and \(f \in \text{Map}_r^N\), let 
\[1^{(a,d)}s_{f,r} := ((a, d), 1^{(a,d)}s_{f,r}, (a, d)) \in \mathcal{H}_{(a,d)_{f,r}}^{(a,d)}(t \in \mathbb{N}).\]

Let \((a, d) \in \dot{\mathcal{D}}_{m|N-m}, r \in \mathbb{Z}_{\geq 0}\) and \(f, f' \in \text{Map}_r^N\). We write 
\[f \sim_r^{(a,d)} f'\]
if there exist \(i, j \in J_{1,N}\) and \(t \in J_{0,r}\) such that \(i \neq j, t - c_{ij}^{(a,d)} \leq r, f(k_1) = f'(k_1) (k_1 \in J_{1,t} \cup J_{t-c_{ij}^{(a,d)}+1,r}), f(k_2) = i, f'(k_2) = j (k_2 \in J_{t+1,t-c_{ij}^{(a,d)}+1,r} \cap 2\mathbb{N}).\]

By [HY08, Theorem 1], we have

**Theorem 2.5.** The semigroup \(\dot{\mathcal{W}}_{m|N-m}\) can also be defined by the generators

\[0, \epsilon^{(a,d)}, \sigma_i^{(a,d)} ((a, d) \in \dot{\mathcal{D}}_{m|N-m}, i \in J_{1,N}),\]
and relations

\[\alpha \omega = \omega \alpha = \omega (\omega \in \dot{\mathcal{W}}_{m|N-m}),\]
\[\epsilon^{(a,d)}(a,d) = \epsilon^{(a,d)(a',d')} = \epsilon^{(a,d)} (a, d) \neq (a', d'),\]
\[\epsilon^{\tau_i(a,d)} \sigma_i^{(a,d)} = \sigma_i^{(a,d)} \epsilon^{(a,d)} = \sigma_i^{(a,d)}, \sigma_i^{\tau_i(a,d)} \sigma_i^{(a,d)} = \epsilon^{(a,d)},\]
\[1^{(a,d)} \sigma_{f_{ij},-2c_{ij}^{(a,d)}}(a,d) = \epsilon^{(a,d)} (i \neq j).\]

We write \(f \sim_r^{(a,d)} f'\) if \(f = f'\) or there exists \(t \in \mathbb{N}\) and \(f_k \in \text{Map}_r^N (k \in J_{1,t})\) such that \(f \sim_r^{(a,d)} f_1, f_k \sim_r^{(a,d)} f_{k+1}\) and \(f_{t} \sim_r^{(a,d)} f'.\)

By [HY08, Theorem 5, Corollary 6], we have

**Theorem 2.6.** Let \((a, d) \in \dot{\mathcal{D}}_{m|N-m}\) and \(w \in W^{(a,d)}\).

(1) Let \(f, f' \in \text{Map}_{\ell(a,d)^\prime}^N(w)\) be such that 
\[1^{(a,d)}s_{f,\ell(a,d)^\prime}(w) = 1^{(a,d)}s_{f',\ell(a,d)^\prime}(w) = w.\]
Then \(f \sim_{\ell(a,d)} f'\).

(2) Let \(r \in \mathbb{N}\) and \(f \in \text{Map}_r^N\) be such that \(r > \ell(a,d)(w)\) and 
\[1^{(a,d)}s_{f,r}(w) = w.\]
Then there exist \(f' \in \text{Map}_r^N\) and \(t \in J_{1,r-1}\) such that \(f \sim_r^{(a,d)} f'\) and 
\[f(t) = f'(t + 1).\]
References

