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On graphical image of the value of payoff function for a vector matrix game

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Abstract

In this paper, we consider to visualize the set of values of some payoff function for a specific two-person zero-sum game with three strategies and two objectives, that is, each payoff function can be represented by one pair of two 3×3 skew symmetric matrices. Moreover, we give a characterization for each pair of two matrices above based on the observation for the image set of the payoff function defined by its pair.

1 Introduction

The famous "minimax theorem" says, in scalar-valued two-person zero-sum games, if the payoff function has a saddle-point then minimax and maximin values coincide and the value attains the saddle-value. In some vector-valued cases, however, the existence of vectorial saddle-points does not always remain this property. So, in [1, 2] Tanaka considers how many properties on minimax and maximin values and saddle-points remains in vector-valued cases. Moreover, [3, 4] give some characterizations for each pair of two 2×2 matrices based on the observation for the image set of the payoff function defined by its pair. On the other hand, the equivalence between a vector-valued linear programming problem and a multi-criteria two-person skew symmetric matrix game has been shown in [5]. In consequence, the study of properties of payoff functions for multi-criteria two-person 3×3 or more large size skew symmetric matrix games are required.

In the paper, we study shapes of each image set of payoff functions for bicriteria two-person skew symmetric matrix games. We clarify some relationship between a payoff matrix and the image set, and classify payoff matrices of the game by the shape of image set.

Notations

For each n, we denote an n-dimensional Euclidean space by $\mathbb{R}^n$ and the origin of $\mathbb{R}^n$ by θ. For x and $y \in \mathbb{R}^n$, we denote the line segment joining x and y by $[x, y]$. T stands for the transpose operation. $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the nonnegative cone and the positive cone in $\mathbb{R}^n$, respectively. $x \geq y$ iff $x - y \in \mathbb{R}^n_+$. $x > y$ iff $x - y \in \mathbb{R}^n_{++}$. Let X be a subset of $\mathbb{R}^n$. $\mathrm{co}\,X$ stands for the convex hull of the set X. $x \times y$ denotes the outer product of two vectors x and $y \in \mathbb{R}^n$. $\|x\|$ stands for the norm of $x \in \mathbb{R}^n$.

2 Classification of matrices for bicriteria matrix game with 3×3 skew symmetric matrices

Let X and Y be the following two strategies sets of Player 1 and Player 2, respectively:

$$X = Y = \mathrm{co}\{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\}.$$

Let A and B be two 3×3 skew symmetric matrices and f the payoff function of Player 1 from $X \times Y$ to $\mathbb{R}^2$ defined by

$$f(x,y) = \left(x^T Ay, x^T By\right)$$

and $-f$ the payoff function of Player 2.

The rest of the paper, let $A = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{pmatrix}$. Let $P_1 = (a_1, b_1)^T, P_2 = (a_2, b_2)^T, P_3 = (a_3, b_3)^T$.

In this section, we consider each shape of image sets of payoff functions i.e., the shape of the following set:

$$\mathcal{S} := f(X,Y) = \bigcup_{(x,y)\in X\times Y} \left\{\left(x^T Ay, x^T By\right)^T\right\}.$$

Let

$$f(X,y) := \bigcup_{x\in X} \left(x^T Ay, x^T By\right)^T \text{ for any fixed } y \in Y \text{ and}$$

$$f(x,Y) := \bigcup_{y\in Y} \left(x^T Ay, x^T By\right)^T \text{ for any fixed } x \in X.$$

Now, we see that every element of $\mathcal{S}$ is a convex combination of $\theta, \pm P_i, i = 1,2,3$. So we have the following proposition.

Proposition 1. $\mathcal{S} \subset \mathcal{P} := \mathrm{co}\{\theta, P_1, P_2, P_3, -P_1, -P_2, -P_3\}$.

Because A and B are skew-symmetric matrices, we see that the following proposition.

Proposition 2. *$\mathcal{S}$ is origin symmetry.*

2.1 Singleton

When $P_1 = P_2 = P_3 = \theta$, obviously $\mathcal{S} = \{\theta\}$, i.e., singleton.

2.2 Line segment

When the linear hull of $\mathcal{P}$ is a subspace of $\mathbb{R}^2$ with one dimension, i.e.,

$$\|P_i \times P_j\| = 0 \text{ for all } i,j \in \{1,2,3\}$$

and

$$\max_{i=1,2,3} \|P_i\| \neq 0,$$

$\mathcal{S}$ is a line segment.

Proof. Without loss of generality, we assume that $\|P_1\| = \max_{i=1,2,3} \|P_i\|$. From Proposition 1, $\mathcal{S} \subset [-P_1, P_1]$. For any $\lambda \in [0,1]$, $\lambda P_1 = f(x,y)$ when $x^T = (1,0,0)$, $y^T = ((1-\lambda), \lambda, 0)$. Thus, by Proposition 2, $\mathcal{S} \supset [-P_1, P_1]$. □

2.3 Hexagonal shape

When P_1, P_2, and P_3 are affinely independent and there exist $\lambda > 0$ and $0 < \mu < 1$ such that

$$P_2 = \lambda(P_1 + P_3) + \mu P_3 + (1 - \mu)P_1.$$

Then $\mathcal{S}$ is hexagonal shape.

Proof. We see that $\operatorname{co}\{\pm P_i,\ i = 1,2,3\}$ is the hexagonal shape with vertices $\pm P_i, i = 1,2,3$. Hence $\mathcal{S}$ is a subset of the hexagon. Conversely, when $x = (1,0,0)^T$, we see that $f(x,Y) = \operatorname{co}\{\theta, P_1, P_2\}$. When $x = (0,1,0)^T$, $f(x,Y) = \operatorname{co}\{-P_1, \theta, P_3\}$. When $x = (0,0,1)^T$, $f(x,Y) = \operatorname{co}\{-P_2, -P_3, \theta\}$. Similarly, when $y = (1,0,0)^T, (0,1,0)^T$ and $(0,0,1)^T$, $f(X,y) = \operatorname{co}\{\theta, -P_1, -P_2\}, \operatorname{co}\{P_1, \theta, -P_3\}$ and $\operatorname{co}\{P_2, P_3, \theta\}$, respectively. Thus $\mathcal{S}$ covers the hexagon $P_1, P_2, P_3, -P_1, -P_2, -P_3$. Therefore, $\mathcal{S}$ is hexagonal shape. □

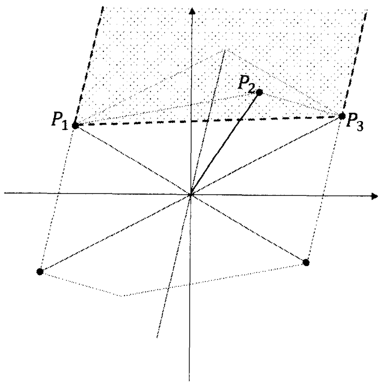


Figure 1: Illustration of the above condition

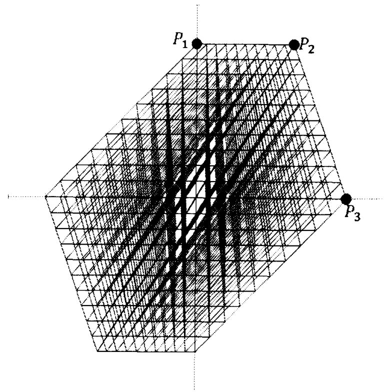


Figure 2: Hexagonal shape

Example 1. Let $A = \begin{pmatrix} 0 & 0 & 1.3 \\ 0 & 0 & 2 \\ -1.3 & -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & 2 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$. Then $P_1 = (0,2)^T, P_2 = (1.3,2)^T$, and $P_3 = (2,0)^T$. So,

$$P_2 = P_1 + \frac{1.3}{2}P_3 = \frac{1.3}{4}(P_1 + P_3) + \frac{2.7}{4}P_1 + (1 - \frac{2.7}{4})P_3.$$

Hence $\mathcal{S}$ is hexagonal shape. Indeed, the graph is Figure 2.

2.4 Tetragon

When θ, P_1, P_2 and P_3 are not on any same straight line and satisfy one of the following three conditions:
(i) $P_2 \in [P_1, P_3]$,
(ii) $P_3 \in [P_2, -P_1]$,
(iii) $P_1 \in [P_2, -P_3]$.
Then $\mathcal{S}$ is square.

Proof. We can consider Tetragon as a special case of hexagonal shape. By similar argument, we see that $\mathcal{S}$ is square. □

Example 2. Let $A = \begin{pmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 2 \\ -0.5 & -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & 1.5 \\ -2 & 0 & 0 \\ -1.5 & 0 & 0 \end{pmatrix}$. Then $P_1 = (0,2)^T, P_2 = (0.5,1.5)^T$, and $P_3 = (2,0)^T$. So,

$$P_2 = \frac{3}{4}P_1 + \frac{1}{4}P_3, \text{ i.e., } P_2 \in [P_1, P_3].$$

Hence $\mathcal{S}$ is Tetragon. Indeed, the graph is Figure 4.

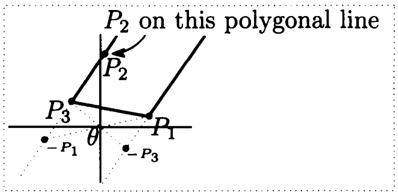


Figure 3: Illustration of condition (ii)

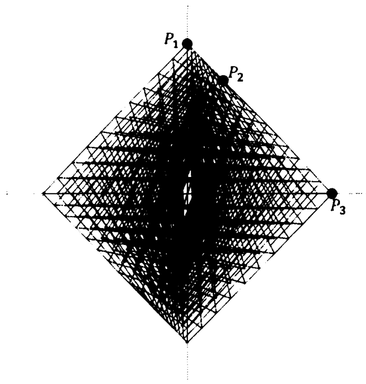


Figure 4: Tetragon

2.5 Envelope

When θ, P_1, P_2 and P_3 are not on any same straight line and satisfy one of the following three conditions:

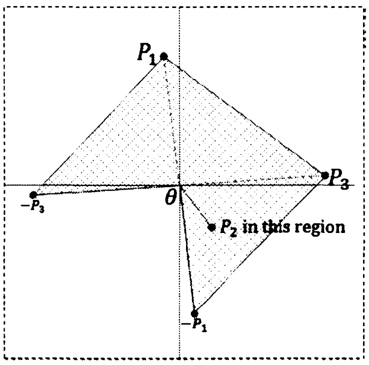
(i) $P_2 \in \mathrm{co}\{P_1, P_3, \theta, (1-\lambda)(-P_3), \lambda(-P_1)+(1-\lambda)(-P_2)\}$ for some $\lambda \in [0,1]$,

(ii) $-P_1 \in \mathrm{co}\{P_3, -P_2, \theta, (1-\lambda)P_2, \lambda(-P_3)+(1-\lambda)P_1\}$ for some $\lambda \in [0,1]$, or

(iii) $-P_3 \in \mathrm{co}\{-P_2, P_1, \theta, (1-\lambda)(-P_1), \lambda P_2+(1-\lambda)P_3)\}$ for some $\lambda \in [0,1]$.

Then $\mathcal{S}$ has envelope.

Proof. Assume that (i) are satisfied. Let $\bar{\mathcal{S}}$ be the union of six triangles consisting of $\mathrm{co}\{\theta, P_1, P_2\}$, $\mathrm{co}\{-P_1, \theta, P_3\}$, $\mathrm{co}\{-P_2, -P_3, \theta\}$, $\mathrm{co}\{\theta, -P_1, -P_2\}$, $\mathrm{co}\{P_1, \theta, -P_3\}$, and $\mathrm{co}\{P_2, P_3, \theta\}$. Then $\bar{\mathcal{S}} \subset \mathcal{S} \subset \mathrm{co}\{\pm P_i, i=1,2,3\}$. If we focus sub-matrices $\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}$ and $\begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix}$, we see that $\mathcal{S}$ has an envelope curve in the intersection of two triangles $\mathrm{co}\{\theta, P_1, P_3\}$ and $\mathrm{co}\{P_1, P_2, P_3\}$; see [6]. Then $\mathcal{S}$ has envelope curves. □



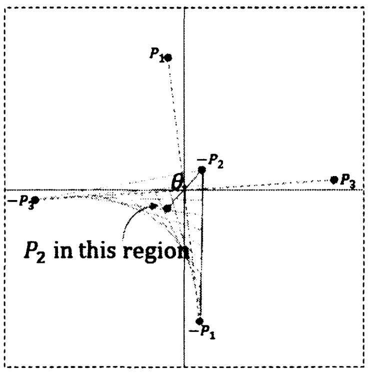


Figure 5: ↖Illustration of condition (i).↗ Figure 6:

Example 3. Let $A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -0.4 \\ -2 & 0.4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0.4 \\ 0 & 0.4 & 0 \end{pmatrix}$. Then $P_1 = (0,2)^T, P_2 = (2,0)^T$, and $P_3 = (-0.4, 0.4)^T$. Assume that $\lambda = 0.5$, the set of above condition (iii) is as follows:

$$\mathrm{co}\left\{\begin{pmatrix}-2\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}0.8\\-0.2\end{pmatrix}.\right\}$$

We see that $\begin{pmatrix}0.5\\-0.5\end{pmatrix} \in \left[\begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}0.8\\-0.2\end{pmatrix}\right]$ and $\begin{pmatrix}0.4\\-0.4\end{pmatrix} \in \left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0.5\\-0.5\end{pmatrix}\right]$. Thus, $-P_3 \in \{-P_2, P_1, \theta, (1-0.5)(-P_1), 0.5P_2 + (1-0.5)P_3\}$. Hence S has an envelope. Indeed, the graph is Figure 7.

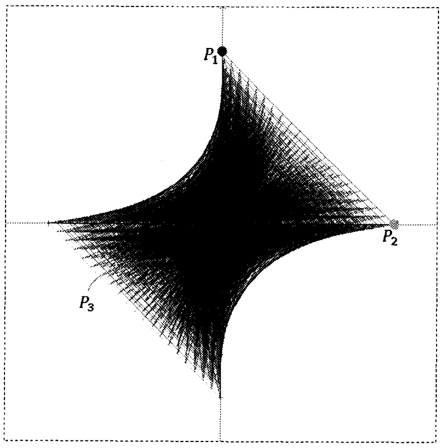


Figure 7: Envelope

2.6 The other patterns

The other patterns are combining envelope and butterfly.

3 Analysis of solution by the graphical approach

A point $\bar{x} \in X$ is said to be a vector solution of bicriteria 3×3 skew symmetric matrix game, if

$$(\bar{x}^T A x, \bar{x}^T B x)^T \not\leq (\bar{x}^T A \bar{x}, \bar{x}^T B \bar{x})^T \not\leq (x^T A \bar{x}, x^T B \bar{x})^T \quad \text{for all } x \in X,$$

i.e.,

$$\bar{f}(x, Y) \cap (-\mathbb{R}^2_{++}) = \emptyset.$$

We see that $x \in X$ is a solution of bicriteria 3×3 skew symmetric matrix game, if one of the following three conditions are satisfied:

(i) $P_1, P_2 \notin (-\mathbb{R}^2_{++})$;

(ii) $-P_1, P_3 \notin (-\mathbb{R}^2_{++})$; and

(iii) $-P_2, P_3 \notin (-\mathbb{R}^2_{++})$.

Proof. Assume (i) is satisfied. Then at least one of the following three conditions are held:

(a) the triangle $\mathrm{co}\{\theta, P_1, P_2\} \cap (-\mathbb{R}^2_{++}) = \emptyset$;

(b) the triangle $\mathrm{co}\{\theta, -P_1, P_3\} \cap (-\mathbb{R}^2_{++}) = \emptyset$; and

(c) the triangle $\mathrm{co}\{\theta, -P_2, -P_3\} \cap (-\mathbb{R}^2_{++}) = \emptyset$.

If (a) is held, for $x = (1,0,0)^T$, $f(x,Y) \cap (-\mathbb{R}^2_{++}) = \emptyset$. If (b) or (c) is held, for $x = (0,1,0)^T$ or $x = (0,0,1)^T$, $f(x,Y) \cap (-\mathbb{R}^2_{++}) = \emptyset$. When (ii) or (iii) are satisfied, by the same way, we see that $f(x,Y) \cap (-\mathbb{R}^2_{++}) = \emptyset$ for $x = (1,0,0)^T$, $x = (0,1,0)^T$, or $x = (0,0,1)^T$. □

Example 4. Let $A = \begin{pmatrix} 0 & 0 & -0.2 \\ 0 & 0 & -1 \\ 0.2 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0.2 \\ -1 & 0 & -0.4 \\ -0.2 & 0.4 & 0 \end{pmatrix}$. Then $P_1 = (0,1)^T$ and $P_2 = (-0.2, 0.2)^T$. So, $(1,0,0)^T$ is a solution.

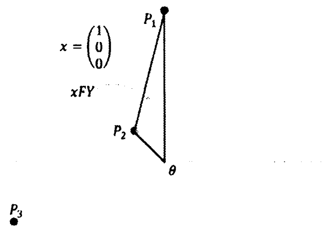


Figure 8:

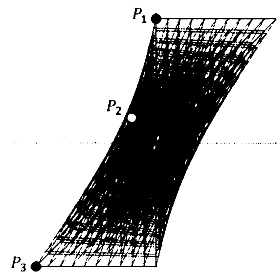


Figure 9:

References

[1] T. Tanaka, Some Minimax Problems of Vector-Valued Functions, *Journal of Optimization Theory and Applications*, vol.59, pp.505–524, 1988.

[2] T. Tanaka, Vector-Valued Minimax Theorems in Multi-Criteria Games, *New Frontiers of Decision Making for the Information Technology Era*, World Scientific, pp.75–99, 2000.

[3] M. Higuchi and T. Tanaka, Classification of Matrices by Means of Envelops for Bicriteria Matrix Games, *International Journal Mathematics, Game Theory, and Algebra*, Nova Science Publishers, New York, vol.12, pp.371–378, 2002.

[4] M. Higuchi and T. Tanaka, On Minimax and Maximin Values in Multi-Criteria Games, *Multi-Objective Programming and Goal-Programming: Theory and Applications*, Springer-Verlag, Berlin, pp.141–146, 2003.

[5] J. M. Hong, M. H. Kim and G. M. Lee, On Linear Vector Program and Vector Matrix Game Equivalence, *Optimization Letters*, vol.6, pp.231–240, 2012.

[6] M. Higuchi, K. Kimura, and T. Tanaka, Classification of Matrices for Tricriteria Two-Person Zero-Sum Matrix Game, *RIMS Kokyuroku* vol.1821, pp.206–213, 2013.

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