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Kyoto University
Existence and Approximation of Attractive Points for Nonlinear Mappings in Banach Spaces

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Abstract. Let $H$ be a real Hilbert space norm $\| \cdot \|$. Let $C$ be a nonempty subset of $H$ and let $T$ be a mapping of $C$ into $H$. We denote by $F(T)$ the set of fixed points of $T$ and by $A(T)$ the set of attractive points of $T$, i.e.,

(i) $F(T) = \{ z \in C : Tz = z \};$

(ii) $A(T) = \{ z \in H : \| Tx - z \| \leq \| x - z \|, \forall x \in C \}.$

In this article, we extend the concept of attractive points in a Hilbert space to that in a Banach space and then prove attractive point theorems and mean convergence theorems without convexity for nonlinear mappings in a Banach space.

1 Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $C$ be a nonempty subset of $H$. A mapping $T : C \to H$ is said to be nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in C$. We know that if $C$ is a bounded, closed and convex subset of $H$ and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty. Furthermore, from Baillon [4] we know the first nonlinear mean convergence theorem for nonexpansive mappings in a Hilbert space. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping $F$ is said to be firmly nonexpansive if

$$\| Fx - Fy \|^2 \leq \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$. Kohsaka and Takahashi [16], and Takahashi [24] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \to H$ is called nonspreading [16] if

$$2\| Tx - Ty \|^2 \leq \| Tx - y \|^2 + \| Ty - x \|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is called hybrid [24] if

$$3\| Tx - Ty \|^2 \leq \| x - y \|^2 + \| Tx - y \|^2 + \| Ty - x \|^2$$

for all $x, y \in C$. The class of nonspreading mappings was first defined in a smooth, strictly convex and reflexive Banach space. The resolvents of a maximal monotone operator are
nonspreading mappings; see [16] for more details. These three classes of nonlinear mappings are important in the study of the geometry of infinite dimensional spaces. Indeed, by using the fact that the resolvents of a maximal monotone operator are nonspreading mappings, Takahashi, Yao and Kohsaka [27] solved an open problem which is related to Ray's theorem [19] in the geometry of Banach spaces. Kocourek, Takahashi and Yao [12] defined a broad class of nonlinear mappings containing nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space. A mapping $T : C \to H$ is called generalized hybrid [12] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. We call such $T$ an $(\alpha, \beta)$-generalized hybrid mapping; see also [2]. Kocourek, Takahashi and Yao [12] proved a fixed point theorem for such mappings in a Hilbert space.

**Theorem 1.1** ([12]). Let $C$ be a nonempty, closed and convex subset of a Hilbert space $H$ and let $T : C \to C$ be a generalized hybrid mapping. Then $T$ has a fixed point in $C$ if and only if $\{T^nx\}$ is bounded for some $x \in C$.

They also proved a mean convergence theorem which generalizes Baillon's nonlinear ergodic theorem [4] in a Hilbert space.

**Theorem 1.2** ([12]). Let $H$ be a real Hilbert space, let $C$ be a nonempty, closed and convex subset of $H$, let $T$ be a generalized hybrid mapping from $C$ into itself with $F(T) \neq \emptyset$ and let $P$ be the metric projection of $H$ onto $F(T)$. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in F(T)$, where $p = \lim_{n \to \infty} PT^n x$.

Recently, Takahashi and Takeuchi [25] introduced the concept of attractive points of nonlinear mappings in a Hilbert space and then they proved attractive point and mean convergence theorems without convexity for generalized hybrid mappings.

In this talk, we extend the concept of attractive points in a Hilbert space to that in a Banach space and then prove attractive point theorems and mean convergence theorems without convexity for nonlinear mappings in a Banach space.

2 Preliminaries

Let $E$ be a real Banach space with norm $\| \cdot \|$ and let $E^*$ be the topological dual space of $E$. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. The modulus $\delta$ of convexity of $E$ is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for all $\epsilon$ with $0 \leq \epsilon \leq 2$. A Banach space $E$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let $E$ be a Banach space. The duality mapping $J$ from $E$ into $2^{E^*}$ is defined by

$$Jx = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \}$$
for all $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of $E$ is said to be \textit{Gâteaux differentiable} if for each $x, y \in U$, the limit
\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]
exists. In the case, $E$ is called \textit{smooth}. We know that $E$ is smooth if and only if $J$ is a single-valued mapping of $E$ into $E^*$. We also know that $E$ is reflexive if and only if $J$ is surjective, and $E$ is strictly convex if and only if $J$ is one-to-one. Therefore, if $E$ is a smooth, strictly convex and reflexive Banach space, then $J$ is a single-valued bijection. The norm of $E$ is said to be \textit{uniformly Gâteaux differentiable} if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be \textit{Fréchet differentiable} if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space $E$ is called \textit{uniformly smooth} if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of $E$ is uniformly Gâteaux differentiable, then $J$ is uniformly norm-to-weak* continuous on each bounded subset of $E$, and if the norm of $E$ is Fréchet differentiable, then $J$ is norm-to-norm continuous. If $E$ is uniformly smooth, $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$. For more details, see [22, 23]. The following result is well known; see [22].

\textbf{Lemma 2.1} ([22]). Let $E$ be a smooth Banach space and let $J$ be the duality mapping on $E$. Then, $(x - y, Jx - Jy) \geq 0$ for all $x, y \in E$. Furthermore, if $E$ is strictly convex and $(x - y, Jx - Jy) = 0$, then $x = y$.

Let $E$ be a smooth Banach space. The function $\phi : E \times E \rightarrow \mathbb{R}$ is defined by
\[
\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2
\]
for all $x, y \in E$; see [1] and [11]. We have from the definition of $\phi$ that
\[
\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle
\]
for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:
\[
2\langle x - y, Jz - Ju \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w)
\]
for all $x, y, z, w \in E$. Let $\phi_* : E^* \times E^* \rightarrow \mathbb{R}$ be the function defined by
\[
\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^* \rangle + \|y^*\|^2
\]
for all $x^*, y^* \in E^*$, where $J$ is the duality mapping of $E$. It is easy to see that
\[
\phi(x, y) = \phi_*(Jy, Jx)
\]
for all $x, y \in E$. If $E$ is additionally assumed to be strictly convex, then
\[
\phi(x, y) = 0 \iff x = y.
\]

The following results are in Xu [28] and Kamimura and Takahashi [11].

\textbf{Lemma 2.2} ([28]). Let $E$ be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and
\[
\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)
\]
for all $x, y \in B_r$ and $\lambda$ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$. 

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Lemma 2.3 ([11]). Let $E$ be smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(||x - y||) \leq \phi(x, y)$ for all $x, y \in B_r$, where $B_r = \{z \in E : ||z|| \leq r\}$.

Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. A mapping $T : C \to E$ is called generalized nonexpansive [8] if $F(T) \neq \emptyset$ and $\phi(Tx, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in F(T)$. Let $D$ be a nonempty subset of a Banach space $E$. A mapping $R : E \to D$ is said to be sunny if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a retraction or a projection if $Rx = x$ for all $x \in D$. A nonempty subset $D$ of a smooth Banach space $E$ is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of $E$ if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) $R$ from $E$ onto $D$; see [8] for more details. The following results are in Ibaraki and Takahashi [8].

Lemma 2.4 ([8]). Let $C$ be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space $E$. Then the sunny generalized nonexpansive retraction from $E$ onto $C$ is uniquely determined.

Lemma 2.5 ([8]). Let $C$ be a nonempty closed subset of a smooth and strictly convex Banach space $E$ such that there exists a sunny generalized nonexpansive retraction $R$ from $E$ onto $C$ and let $(x, z) \in E \times C$. Then the following hold:

(i) $z = Rx$ if and only if $(x - z, Jy - Jz) \leq 0$ for all $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [14] proved the following results:

Lemma 2.6 ([14]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed subset of $E$. Then the following are equivalent:

(a) $C$ is a sunny generalized nonexpansive retract of $E$;
(b) $C$ is a generalized nonexpansive retract of $E$;
(c) $JC$ is closed and convex.

Lemma 2.7 ([14]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed sunny generalized nonexpansive retract of $E$. Let $R$ be the sunny generalized nonexpansive retraction from $E$ onto $C$ and let $(x, z) \in E \times C$. Then the following are equivalent:

(i) $z = Rx$;
(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Let $l^\infty$ be the Banach space of bounded sequences with supremum norm. Let $\mu$ be an element of $(l^\infty)^*$ (the dual space of $l^\infty$). Then we denote by $\mu(f)$ the value of $\mu$ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional $\mu$ on $l^\infty$ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean $\mu$ is called a Banach limit on $l^\infty$ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on $l^\infty$. If $\mu$ is a Banach limit on $l^\infty$, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$
\lim_{n \to \infty} \inf x_n \leq \mu_n(x_n) \leq \lim_{n \to \infty} \sup x_n.
$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [22] for the proof of existence of a Banach limit and its other elementary properties.
3 Existence of Attractive Points in Banach Spaces

In 2011, Takahashi and Takeuchi [25] proved the following attractive point theorem in a Hilbert space.

**Theorem 3.1** ([25]). Let $H$ be a Hilbert space, let $C$ be a nonempty subset of $H$ and let $T$ be a generalized hybrid mapping of $C$ into itself. Suppose that there exists an element $z \in C$ such that $\{T^nz\}$ is bounded. Then $A(T)$ is nonempty. Additionally, if $C$ is closed and convex, then $F(T)$ is nonempty.

In this section, we first try to extend Takahashi and Takeuchi’s attractive point theorem [25] to Banach spaces. Let $E$ be a smooth Banach space. Let $C$ be a nonempty subset of $E$ and let $T$ be a mapping of $C$ into $E$. We denote by $A(T)$ the set of attractive points [17] of $T$, i.e.,

$$A(T) = \{z \in E : \phi(z, Tx) \leq \phi(z, x), \forall x \in C\}.$$  

From Lin and Takahashi [17], $A(T)$ is a closed and convex subset of $E$. A mapping $T : C \to E$ is called generalized nonspreading [13] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha \phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\}$$

$$\leq \beta \phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\}$$

(3.1)

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$ for $x, y \in E$. We call such $T$ an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping. For example, a $(1,1,0,0)$-generalized nonspreading mapping is a nonspreading mapping in the sense of Kohsaka and Takahashi [16], i.e.,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x), \forall x, y \in C;$$

see also [15] and [3]. Let $T$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping. Observe that if $F(T) \neq \emptyset$, then $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. Using the technique developed by [20] and [21], we can prove an attractive point theorem for generalized nonspreading mappings in a Banach space.

**Theorem 3.2** (Lin and Takahashi [17]). Let $E$ be a smooth and reflexive Banach space. Let $C$ be a nonempty subset of $E$ and let $T$ be a generalized nonspreading mapping of $C$ into itself. Then, the following are equivalent:

(a) $A(T) \neq \emptyset$;
(b) $\{T^nx\}$ is bounded for some $x \in C$.

Additionally, if $E$ is strictly convex and $C$ is closed and convex, then the following are equivalent:

(a) $F(T) \neq \emptyset$;
(b) $\{T^nx\}$ is bounded for some $x \in C$. 
4 Skew-Attractive Point Theorems

Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. Let $T : C \to E$ be a generalized nonspreading mapping; see (3.1). This mapping has the property that if $u \in F(T)$ and $x \in C$, then $\phi(u, Tx) \leq \phi(u, x)$. This property can be revealed by putting $x = u \in F(T)$ in (3.1). Similarly, putting $y = u \in F(T)$ in (3.1), we obtain that for any $x \in C$,

$$
\alpha \phi(Tx, u) + (1 - \alpha)\phi(x, u) + \gamma \{\phi(u, Tx) - \phi(u, x)\}
\leq \beta \phi(Tx, u) + (1 - \beta)\phi(x, u) + \delta \{\phi(u, Tx) - \phi(u, x)\}
$$

(4.1)

and hence

$$
(\alpha - \beta)\{\phi(Tx, u) - \phi(x, u)\} + (\gamma - \delta)\{\phi(u, Tx) - \phi(u, x)\} \leq 0.
$$

(4.2)

Therefore, we have that $\alpha > \beta$ together with $\gamma \leq \delta$ implies $\phi(Tx, u) \leq \phi(x, u)$. Motivated by this property of $T$ and $F(T)$, we give the following defintition. Let $E$ be a smooth Banach space. Let $C$ be a nonempty subset of $E$ and let $T$ be a mapping of $C$ into $E$. We denote by $B(T)$ the set of skew-attractive points of $T$, i.e.,

$$
B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z), \forall x \in C\}.
$$

The following result was proved by Lin and Takahashi [17].

**Lemma 4.1** ([17]). Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. Let $T$ be a mapping from $C$ into $E$. Then $B(T)$ is closed.

Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty subset of $E$. Let $T$ be a mapping of $C$ into $E$. Define a mapping $T^*$ as follows:

$$
T^*x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,
$$

where $J$ is the duality mapping on $E$ and $J^{-1}$ is the duality mapping on $E^*$. A mapping $T^*$ is called the adjoint mapping of $T$; see also [26] and [6]. It is easy to show that if $T$ is a mapping of $C$ into itself, then $T^*$ is a mapping of $JC$ into itself. In fact, for $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$. So, we have $T^*x^* = JTJ^{-1}x^* \in JC$. Then, $T^*$ is a mapping of $JC$ into itself. We can prove the following result in a Banach space which was proved by Lin and Takahashi [17].

**Lemma 4.2** ([17]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty subset of $E$. Let $T$ be a mapping of $C$ into $E$ and let $T^*$ be the duality mapping of $T$. Then, the following hold:

1. $JBT(T) = A(T^*)$;
2. $JAT(T) = B(T^*)$.

In particular, $JB(T)$ is closed and convex.

Using these results, we can discuss skew-attractive point theorems in Banach spaces. Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. A mapping $T : C \to E$ is called skew-generalized nonspreading [7] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$
\alpha \phi(Ty, Tx) + (1 - \alpha)\phi(Tx, y) + \gamma \{\phi(Tx, Ty) - \phi(x, Ty)\}
\leq \beta \phi(Ty, Tx) + (1 - \beta)\phi(y, x) + \delta \{\phi(Tx, y) - \phi(x, y)\}
$$

(4.3)
for all $x, y \in C$. We call such $T$ an $(\alpha, \beta, \gamma, \delta)$-skew-generalized nonspraying mapping. For example, a $(1,1,1,0)$-skew-generalized nonspraying mapping is a skew-nonspraying mapping in the sense of Ibaraki and Takahashi [9], i.e.,

$$
\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx), \quad \forall x, y \in C.
$$

The following theorem was proved by Lin and Takahashi [17].

**Theorem 4.3 ([17]).** Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty subset of $E$. Let $T$ be a skew-generalized nonspraying mapping of $C$ into itself. Then, the following are equivalent:

(a) $B(T) \neq \emptyset$;

(b) $\{T^n x\}$ is bounded for some $x \in C$.

Additionally, if $C$ is closed and $JC$ is closed and convex, then the following are equivalent:

(a) $F(T) \neq \emptyset$;

(b) $\{T^n x\}$ is bounded for some $x \in C$.

## 5 Mean Convergence Theorems in Banach Spaces

In this section, we can prove a mean convergence theorem without convexity for generalized nonspraying mappings in a Banach space. Before proving it, we state the following lemmas.

**Lemma 5.1 ([20, 5]).** Let $E$ be a reflexive Banach space, let $\{x_n\}$ be a bounded sequence in $E$ and let $\mu$ be a mean on $l^\infty$. Then there exists a unique point $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$ such that

$$
\mu_n(x_n, y^*) = \langle z_0, y^* \rangle, \quad \forall y^* \in E^*.
$$

(5.1)

A unique point $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$ satisfying (5.1) is called the mean vector of $\{x_n\}$ for $\mu$.

**Lemma 5.2 ([18]).** Let $E$ be a smooth, strictly convex and reflexive Banach space with the duality mapping $J$ and let $D$ be a nonempty, closed and convex subset of $E$. Let $\{x_n\}$ be a bounded sequence in $D$ and let $\mu$ be a mean on $l^\infty$. If $g : D \to \mathbb{R}$ is defined by

$$
g(z) = \mu_n \phi(x_n, z), \quad \forall z \in D,
$$

then the mean vector $z_0$ of $\{x_n\}$ for $\mu$ is a unique minimizer in $D$ such that

$$
g(z_0) = \min\{g(z) : z \in D\}.
$$

**Lemma 5.3 ([18]).** Let $E$ be a smooth and reflexive Banach space and let $C$ be a nonempty subset of $E$. Let $T$ be a generalized nonspraying mapping of $C$ into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Define

$$
S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x, \quad \forall n \in \mathbb{N}.
$$

If a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ converges weakly to a point $u$, then $u \in A(T)$. Additionally, if $E$ is strictly convex and $C$ is closed and convex, then $u \in F(T)$. 
**Lemma 5.4** ([18]). Let $E$ be a uniformly convex and smooth Banach space. Let $C$ be a nonempty subset of $E$ and let $T : C \to C$ be a mapping such that $B(T) \neq \emptyset$. Then, there exists a unique sunny generalized nonexpansive retraction $R$ of $E$ onto $B(T)$. Furthermore, for any $x \in C$, $\lim_{n \to \infty} RT^nx$ exists in $B(T)$.

Using these lemmas, we prove the following mean convergence theorem for generalized nonspreading mappings in a Banach space.

**Theorem 5.5** (Lin and Takahashi [17]). Let $E$ be a uniformly convex Banach space with a Fréchet differentiable norm and let $C$ be a nonempty subset of $E$. Let $T : C \to C$ be a generalized nonspreading mapping such that $A(T) = B(T) \neq \emptyset$. Let $R$ be the sunny generalized nonexpansive retraction of $E$ onto $B(T)$. Then, for any $x \in C$, the sequence $\{S_nx\}$ of Cesàro means

$$S_nx = \frac{1}{n} \sum_{k=0}^{n-1} T^kx$$

converges weakly to an element $q$ of $A(T)$, where $q = \lim_{n \to \infty} RT^nx$.

Using Theorem 5.5, we obtain the following theorems.

**Theorem 5.6** (Kocourek, Takahashi and Yao [13]). Let $E$ be a uniformly convex Banach space with a Fréchet differentiable norm. Let $T : E \to E$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping such that $\alpha > \beta$ and $\gamma \leq \delta$. Assume that $F(T) \neq \emptyset$ and let $R$ be the sunny generalized nonexpansive retraction of $E$ onto $F(T)$. Then, for any $x \in E$, the sequence $\{S_nx\}$ of Cesàro means

$$S_nx = \frac{1}{n} \sum_{k=0}^{n-1} T^kx$$

converges weakly to an element $q$ of $F(T)$, where $q = \lim_{n \to \infty} RT^nx$.

**Proof.** We also know that $\alpha > \beta$ together with $\gamma \leq \delta$ implies that $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in E$ and $u \in F(T)$. We also note that $A(T) = F(T)$ and $B(T) = F(T)$. So, we have the desired result from Theorem 5.5.

\[ \square \]

**References**


