<table>
<thead>
<tr>
<th>Title</th>
<th>Existence and Approximation of Attractive Points for Nonlinear Mappings in Banach Spaces (Nonlinear Analysis and Convex Analysis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Takahashi, Wataru</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1841: 114-122</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/194974">http://hdl.handle.net/2433/194974</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Existence and Approximation of Attractive Points for Nonlinear Mappings in Banach Spaces

東京工業大学, 廣電工業大学, 東京理科大学, 台湾国立中山大学
高橋 拓 (Wataru Takahashi)
Tokyo Institute of Technology, Keio University, Tokyo University of Science, Japan
and National Sun Yat-sen University, Taiwan

Abstract. Let $H$ be a real Hilbert space norm $\| \cdot \|$. Let $C$ be a nonempty subset of $H$ and let $T$ be a mapping of $C$ into $H$. We denote by $F(T)$ the set of fixed points of $T$ and by $A(T)$ the set of attractive points of $T$, i.e.,

(i) $F(T) = \{ z \in C : Tz = z \}$;
(ii) $A(T) = \{ z \in H : \|Tx - z\| \leq \|x - z\|, \forall x \in C \}$.

In this article, we extend the concept of attractive points in a Hilbert space to that in a Banach space and then prove attractive point theorems and mean convergence theorems without convexity for nonlinear mappings in a Banach space.

1 Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. Let $C$ be a nonempty subset of $H$. A mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that if $C$ is a bounded, closed and convex subset of $H$ and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty. Furthermore, from Baillon [4] we know the first nonlinear mean convergence theorem for nonexpansive mappings in a Hilbert space. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping $F$ is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$. Kohsaka and Takahashi [16], and Takahashi [24] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \to H$ is called nonspreading [16] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is called hybrid [24] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. The class of nonspreading mappings was first defined in a smooth, strictly convex and reflexive Banach space. The resolvents of a maximal monotone operator are
nonspreading mappings; see [16] for more details. These three classes of nonlinear mappings are important in the study of the geometry of infinite dimensional spaces. Indeed, by using the fact that the resolvents of a maximal monotone operator are nonspreading mappings, Takahashi, Yao and Kohsaka [27] solved an open problem which is related to Ray’s theorem [19] in the geometry of Banach spaces. Kocourek, Takahashi and Yao [12] defined a broad class of nonlinear mappings containing nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space. A mapping $T : C \to H$ is called generalized hybrid [12] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \| Tx - Ty \|^2 + (1 - \alpha) \| x - Ty \|^2 \leq \beta \| Tx - y \|^2 + (1 - \beta) \| x - y \|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. We call such $T$ an $(\alpha, \beta)$-generalized hybrid mapping; see also [2]. Kocourek, Takahashi and Yao [12] proved a fixed point theorem for such mappings in a Hilbert space.

**Theorem 1.1** ([12]). Let $C$ be a nonempty, closed and convex subset of a Hilbert space $H$ and let $T : C \to C$ be a generalized hybrid mapping. Then $T$ has a fixed point in $C$ if and only if $\{T^n z\}$ is bounded for some $z \in C$.

They also proved a mean convergence theorem which generalizes Baillon’s nonlinear ergodic theorem [4] in a Hilbert space.

**Theorem 1.2** ([12]). Let $H$ be a real Hilbert space, let $C$ be a nonempty, closed and convex subset of $H$, let $T$ be a generalized hybrid mapping from $C$ into itself with $F(T) \neq \emptyset$ and let $P$ be the metric projection of $H$ onto $F(T)$. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in F(T)$, where $p = \lim_{n \to \infty} P T^n x$.

Recently, Takahashi and Takeuchi [25] introduced the concept of attractive points of nonlinear mappings in a Hilbert space and then they proved attractive point and mean convergence theorems without convexity for generalized hybrid mappings.

In this talk, we extend the concept of attractive points in a Hilbert space to that in a Banach space and then prove attractive point theorems and mean convergence theorems without convexity for nonlinear mappings in a Banach space.

## 2 Preliminaries

Let $E$ be a real Banach space with norm $\| \cdot \|$ and let $E^*$ be the topological dual space of $E$. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. The modulus $\delta$ of convexity of $E$ is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for all $\epsilon$ with $0 \leq \epsilon \leq 2$. A Banach space $E$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let $E$ be a Banach space. The duality mapping $J$ from $E$ into $2^{E^*}$ is defined by

$$J x = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \}$$
for all $x \in E$. Let $U = \{ x \in E : \|x\| = 1 \}$. The norm of $E$ is said to be \textit{Gâteaux differentiable} if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists. In the case, $E$ is called \textit{smooth}. We know that $E$ is smooth if and only if $J$ is a single-valued mapping of $E$ into $E^*$. We also know that $E$ is reflexive if and only if $J$ is surjective, and $E$ is strictly convex if and only if $J$ is one-to-one. Therefore, if $E$ is a smooth, strictly convex and reflexive Banach space, then $J$ is a single-valued bijection. The norm of $E$ is said to be \textit{uniformly Gâteaux differentiable} if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be \textit{Fréchet differentiable} if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space $E$ is called \textit{uniformly smooth} if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of $E$ is uniformly Gâteaux differentiable, then $J$ is uniformly norm-to-weak* continuous on each bounded subset of $E$, and if the norm of $E$ is Fréchet differentiable, then $J$ is norm-to-norm continuous. If $E$ is uniformly smooth, $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$. For more details, see [22, 23]. The following result is well known; see [22].

**Lemma 2.1** ([22]). \textit{Let $E$ be a smooth Banach space and let $J$ be the duality mapping on $E$. Then, $(x - y, Jx - Jy) \geq 0$ for all $x, y \in E$. Furthermore, if $E$ is strictly convex and $(x - y, Jx - Jy) = 0$, then $x = y$.}

Let $E$ be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$; see [1] and [11]. We have from the definition of $\phi$ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$$

(2.2)

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w)$$

(2.3)

for all $x, y, z, w \in E$. Let $\phi_* : E^* \times E^* \to \mathbb{R}$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^* \rangle + \|y^*\|^2$$

for all $x^*, y^* \in E^*$, where $J$ is the duality mapping of $E$. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx)$$

(2.4)

for all $x, y \in E$. If $E$ is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \iff x = y.$$

(2.5)

The following results are in Xu [28] and Kamimura and Takahashi [11].

**Lemma 2.2** ([28]). \textit{Let $E$ be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and $\lambda$ with $0 \leq \lambda \leq 1$, where $B_r = \{ z \in E : \|z\| \leq r \}$.
Lemma 2.3 ([11]). Let $E$ be smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that $g(0) = 0$ and $g(||x - y||) \leq \phi(x, y)$ for all $x, y \in B_r$, where $B_r = \{z \in E : ||z|| \leq r\}$.

Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. A mapping $T : C \rightarrow E$ is called generalized nonexpansive [8] if $F(T) \neq \emptyset$ and $\phi(Tx, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in F(T)$. Let $D$ be a nonempty subset of a Banach space $E$. A mapping $R : E \rightarrow D$ is said to be sunny if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \geq 0$. A mapping $R : E \rightarrow D$ is said to be a retraction or a projection if $Rx = x$ for all $x \in D$. A nonempty subset $D$ of a smooth Banach space $E$ is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of $E$ if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) $R$ from $E$ onto $D$; see [8] for more details. The following results are in Ibaraki and Takahashi [8].

Lemma 2.4 ([8]). Let $C$ be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space $E$. Then the sunny generalized nonexpansive retraction from $E$ onto $C$ is uniquely determined.

Lemma 2.5 ([8]). Let $C$ be a nonempty closed subset of a smooth and strictly convex Banach space $E$ such that there exists a sunny generalized nonexpansive retraction $R$ from $E$ onto $C$ and let $(x, z) \in E \times C$. Then the following hold:

(i) $z = Rx$ if and only if $(x - z, Jy - Jz) \leq 0$ for all $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [14] proved the following results:

Lemma 2.6 ([14]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed subset of $E$. Then the following are equivalent:

(a) $C$ is a sunny generalized nonexpansive retract of $E$;
(b) $C$ is a generalized nonexpansive retract of $E$;
(c) $JC$ is closed and convex.

Lemma 2.7 ([14]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed sunny generalized nonexpansive retract of $E$. Let $R$ be the sunny generalized nonexpansive retraction from $E$ onto $C$ and let $(x, z) \in E \times C$. Then the following are equivalent:

(i) $z = Rx$;
(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Let $l^\infty$ be the Banach space of bounded sequences with supremum norm. Let $\mu$ be an element of $(l^\infty)^*$ (the dual space of $l^\infty$). Then we denote by $\mu(f)$ the value of $\mu$ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional $\mu$ on $l^\infty$ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean $\mu$ is called a Banach limit on $l^\infty$ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on $l^\infty$. If $\mu$ is a Banach limit on $l^\infty$, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\liminf_{n \rightarrow \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \rightarrow \infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \rightarrow a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [22] for the proof of existence of a Banach limit and its other elementary properties.
3 Existence of Attractive Points in Banach Spaces

In 2011, Takahashi and Takeuchi [25] proved the following attractive point theorem in a Hilbert space.

**Theorem 3.1** ([25]). Let $H$ be a Hilbert space, let $C$ be a nonempty subset of $H$ and let $T$ be a generalized hybrid mapping of $C$ into itself. Suppose that there exists an element $z \in C$ such that $\{T^n z\}$ is bounded. Then $A(T)$ is nonempty. Additionally, if $C$ is closed and convex, then $F(T)$ is nonempty.

In this section, we first try to extend Takahashi and Takeuchi’s attractive point theorem [25] to Banach spaces. Let $E$ be a smooth Banach space. Let $C$ be a nonempty subset of $E$ and let $T$ be a mapping of $C$ into $E$. We denote by $A(T)$ the set of attractive points [17] of $T$, i.e.,

$$A(T) = \{z \in E : \phi(z, Tx) \leq \phi(z, x), \quad \forall x \in C\}.$$  

From Lin and Takahashi [17], $A(T)$ is a closed and convex subset of $E$. A mapping $T : C \to E$ is called generalized nonspreading [13] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha \phi(Tx, Ty) + (1 - \alpha) \phi(x, Ty) + \gamma \{\phi(Ty, Tx) - \phi(Ty, x)\} \leq \beta \phi(Tx, y) + (1 - \beta) \phi(x, y) + \delta \{\phi(y, Tx) - \phi(y, x)\}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2 \langle x, Jy \rangle + \|y\|^2$ for $x, y \in E$. We call such $T$ an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping. For example, a $(1, 1, 1, 0)$-generalized nonspreading mapping is a nonspreading mapping in the sense of Kohsaka and Takahashi [16], i.e.,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(y, Tx), \quad \forall x, y \in C;$$

see also [15] and [3]. Let $T$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping. Observe that if $F(T) \neq \emptyset$, then $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. Using the technique developed by [20] and [21], we can prove an attractive point theorem for generalized nonspreading mappings in a Banach space.

**Theorem 3.2** (Lin and Takahashi [17]). Let $E$ be a smooth and reflexive Banach space. Let $C$ be a nonempty subset of $E$ and let $T$ be a generalized nonspreading mapping of $C$ into itself. Then, the following are equivalent:

(a) $A(T) \neq \emptyset$;

(b) $\{T^n x\}$ is bounded for some $x \in C$.

Additionally, if $E$ is strictly convex and $C$ is closed and convex, then the following are equivalent:

(a) $F(T) \neq \emptyset$;

(b) $\{T^n x\}$ is bounded for some $x \in C$. 
4 Skew-Attractive Point Theorems

Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. Let $T : C \to E$ be a generalized nonspreading mapping; see (3.1). This mapping has the property that if $u \in F(T)$ and $x \in C$, then $\phi(u, Tx) \leq \phi(u, x)$. This property can be revealed by putting $x = u \in F(T)$ in (3.1). Similarly, putting $y = u \in F(T)$ in (3.1), we obtain that for any $x \in C$,

$$
\alpha \phi(Tx, u) + (1 - \alpha) \phi(x, u) + \gamma \{\phi(u, Tx) - \phi(u, x)\} \\
\leq \beta \phi(Tx, u) + (1 - \beta) \phi(x, u) + \delta \{\phi(u, Tx) - \phi(u, x)\}
$$

(4.1)

and hence

$$
(\alpha - \beta) \{\phi(Tx, u) - \phi(x, u)\} + (\gamma - \delta) \{\phi(u, Tx) - \phi(u, x)\} \leq 0.
$$

(4.2)

Therefore, we have that $\alpha > \beta$ together with $\gamma \leq \delta$ implies $\phi(Tx, u) \leq \phi(x, u)$. Motivated by this property of $T$ and $F(T)$, we give the following definition. Let $E$ be a smooth Banach space. Let $C$ be a nonempty subset of $E$ and let $T$ be a mapping of $C$ into $E$. We denote by $B(T)$ the set of skew-attractive points of $T$, i.e.,

$$
B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z), \ \forall x \in C\}.
$$

The following result was proved by Lin and Takahashi [17].

**Lemma 4.1** ([17]). Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. Let $T$ be a mapping from $C$ into $E$. Then $B(T)$ is closed.

Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty subset of $E$. Let $T$ be a mapping of $C$ into $E$. Define a mapping $T^*$ as follows:

$$
T^*x^* = JTJ^{-1}x^*, \ \forall x^* \in JC,
$$

where $J$ is the duality mapping on $E$ and $J^{-1}$ is the duality mapping on $E^*$. A mapping $T^*$ is called the adjoint mapping of $T$; see also [26] and [6]. It is easy to show that if $T$ is a mapping of $C$ into itself, then $T^*$ is a mapping of $JC$ into itself. In fact, for $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$. So, we have $T^*x^* = JTJ^{-1}x^* \in JC$. Then, $T^*$ is a mapping of $JC$ into itself. We can prove the following result in a Banach space which was proved by Lin and Takahashi [17].

**Lemma 4.2** ([17]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty subset of $E$. Let $T$ be a mapping of $C$ into $E$ and let $T^*$ be the duality mapping of $T$. Then, the following hold:

1. $JB(T) = A(T^*)$;
2. $JA(T) = B(T^*)$.

In particular, $JB(T)$ is closed and convex.

Using these results, we can discuss skew-attractive point theorems in Banach spaces. Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. A mapping $T : C \to E$ is called skew-generalized nonspreading [7] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$
\alpha \phi(Ty, Tx) + (1 - \alpha) \phi(Ty, x) + \gamma \{\phi(Ty, Ty) - \phi(x, Ty)\} \\
\leq \beta \phi(y, Tx) + (1 - \beta) \phi(y, x) + \delta \{\phi(Tx, y) - \phi(x, y)\}
$$

(4.3)
for all \( x, y \in C \). We call such \( T \) an \((\alpha, \beta, \gamma, \delta)\)-skew-generalized nonspreading mapping. For example, a \((1,1,1,0)\)-skew-generalized nonspreading mapping is a skew-nonspreading mapping in the sense of Ibaraki and Takahashi [9], i.e.,

\[
\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx), \quad \forall x, y \in C.
\]

The following theorem was proved by Lin and Takahashi [17].

**Theorem 4.3 ([17]).** Let \( E \) be a smooth, strictly convex and reflexive Banach space and let \( C \) be a nonempty subset of \( E \). Let \( T \) be a skew-generalized nonspreading mapping of \( C \) into itself. Then, the following are equivalent:

1. \( B(T) \neq \emptyset \);
2. \( \{T^{n}x\} \) is bounded for some \( x \in C \).

Additionally, if \( C \) is closed and \( JC \) is closed and convex, then the following are equivalent:

1. \( F(T) \neq \emptyset \);
2. \( \{T^{n}x\} \) is bounded for some \( x \in C \).

### 5 Mean Convergence Theorems in Banach Spaces

In this section, we can prove a mean convergence theorem without convexity for generalized nonspreading mappings in a Banach space. Before proving it, we state the following lemmas.

**Lemma 5.1 ([20, 5]).** Let \( E \) be a reflexive Banach space, let \( \{x_{n}\} \) be a bounded sequence in \( E \) and let \( \mu \) be a mean on \( l^{\infty} \). Then there exists a unique point \( z_{0} \in \overline{co}\{x_{n} : n \in \mathbb{N}\} \) such that

\[
\mu_{n} \langle x_{n}, y^{*} \rangle = \langle z_{0}, y^{*} \rangle, \quad \forall y^{*} \in E^{*}.
\]

(5.1)

A unique point \( z_{0} \in \overline{co}\{x_{n} : n \in \mathbb{N}\} \) satisfying (5.1) is called the mean vector of \( \{x_{n}\} \) for \( \mu \).

**Lemma 5.2 ([18]).** Let \( E \) be a smooth, strictly convex and reflexive Banach space with the duality mapping \( J \) and let \( D \) be a nonempty, closed and convex subset of \( E \). Let \( \{x_{n}\} \) be a bounded sequence in \( D \) and let \( \mu \) be a mean on \( l^{\infty} \). If \( g : D \to \mathbb{R} \) is defined by

\[
g(z) = \mu_{n} \phi(x_{n}, z), \quad \forall z \in D,
\]

then the mean vector \( z_{0} \) of \( \{x_{n}\} \) for \( \mu \) is a unique minimizer in \( D \) such that

\[
g(z_{0}) = \min\{g(z) : z \in D\}.
\]

**Lemma 5.3 ([18]).** Let \( E \) be a smooth and reflexive Banach space and let \( C \) be a nonempty subset of \( E \). Let \( T \) be a generalized nonspreading mapping of \( C \) into itself. Suppose that \( \{T^{n}x\} \) is bounded for some \( x \in C \). Define

\[
S_{n}x = \frac{1}{n} \sum_{k=0}^{n-1} T^{k}x, \quad \forall n \in \mathbb{N}.
\]

If a subsequence \( \{S_{n_{i}}x\} \) of \( \{S_{n}x\} \) converges weakly to a point \( u \), then \( u \in A(T) \). Additionally, if \( E \) is strictly convex and \( C \) is closed and convex, then \( u \in F(T) \).
Lemma 5.4 ([18]). Let \( E \) be a uniformly convex and smooth Banach space. Let \( C \) be a nonempty subset of \( E \) and let \( T : C \rightarrow C \) be a mapping such that \( B(T) \neq \emptyset \). Then, there exists a unique sunny generalized nonexpansive retraction \( R \) of \( E \) onto \( B(T) \). Furthermore, for any \( x \in C \), \( \lim_{n \to \infty} R T^n x \) exists in \( B(T) \).

Using these lemmas, we prove the following mean convergence theorem for generalized nonspreading mappings in a Banach space.

Theorem 5.5 (Lin and Takahashi [17]). Let \( E \) be a uniformly convex Banach space with a Fréchet differentiable norm and let \( C \) be a nonempty subset of \( E \). Let \( T : \rightarrow C \) be a generalized nonspreading mapping such that \( A(T) = B(T) \neq \emptyset \). Let \( R \) be the sunny generalized nonexpansive retraction of \( E \) onto \( B(T) \). Then, for any \( x \in C \), the sequence \( \{S_n x\} \) of Cesàro means

\[
S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x
\]

converges weakly to an element \( q \) of \( A(T) \), where \( q = \lim_{n \to \infty} R T^n x \).

Using Theorem 5.5, we obtain the following theorems.

Theorem 5.6 (Kocourek, Takahashi and Yao [13]). Let \( E \) be a uniformly convex Banach space with a Fréchet differentiable norm. Let \( T : E \rightarrow E \) be an \((\alpha, \beta, \gamma, \delta)\)-generalized nonspreading mapping such that \( \alpha > \beta \) and \( \gamma \leq \delta \). Assume that \( F(T) \neq \emptyset \) and let \( R \) be the sunny generalized nonexpansive retraction of \( E \) onto \( F(T) \). Then, for any \( x \in E \), the sequence \( \{S_n x\} \) of Cesàro means

\[
S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x
\]

converges weakly to an element \( q \) of \( F(T) \), where \( q = \lim_{n \to \infty} R T^n x \).

Proof. We also know that \( \alpha > \beta \) together with \( \gamma \leq \delta \) implies that \( \phi(Tx, u) \leq \phi(x, u) \) for all \( x \in E \) and \( u \in F(T) \). We also note that \( A(T) = F(T) \) and \( B(T) = F(T) \). So, we have the desired result from Theorem 5.5. \( \square \)

References