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Fixed point and nonlinear ergodic theorems for new nonlinear mappings in Hilbert spaces

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Abstract

In this paper we introduce a broad class of nonlinear mappings which contains the class of constructive mappings and the class of generalized hybrid mappings in a Hilbert space. Then we prove a fixed point theorem for such mappings in a Hilbert space. Furthermore, we prove a nonlinear ergodic theorem of Baillon's type in a Hilbert space. Their results generalize the fixed point theorem and the nonlinear ergodic theorem proved by Kocourek, Takahashi and Yao [10].

1 Introduction

Let $H$ be a real Hilbert space. A mapping $T$ from $H$ into $H$ is said to be constructive if there exists a real number $\alpha$ with $0 < \alpha < 1$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|$$

for any $x, y \in H$. By Banach [2] it is known that any constructive mapping has a unique fixed point. Let $C$ be a non-empty subset of $H$. A mapping $T$ from $C$ into $H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for any $x, y \in C$. By Baillon [1] we know the following nonlinear ergodic theorem in a Hilbert space.

Theorem 1.1. Let $H$ be a real Hilbert space, let $C$ be a non-empty closed convex subset of $H$ and let $T$ be a nonexpansive mapping from $C$ into $C$ with a fixed point. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point of $T$. 
An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping $T$ from $C$ into $H$ is said to be firmly nonexpansive if

$$
\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle
$$

for any $x, y \in C$; see Browder [4] and Goebel and Kirk [6]. It is known that a firmly nonexpansive mapping can be deduced from an equilibrium problem in a Hilbert space; see Blum and Oettli [3] and Combettes and Hirstoaga [5]. Recently Kohsaka and Takahashi [12], and Takahashi [16] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T$ from $C$ into $H$ is said to be nonspreading if

$$
2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2
$$

for any $x, y \in C$. A mapping $T$ from $C$ into $H$ is said to be hybrid if

$$
3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2
$$

for any $x, y \in C$. Motivated by these mappings, Kocourek, Takahashi and Yao [10] defined a class of nonlinear mappings in a Hilbert space. A mapping $T$ from $C$ into $H$ is said to be generalized hybrid if there exist real numbers $\alpha$ and $\beta$ such that

$$
\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2
$$

for any $x, y \in C$. We call such a mapping an $(\alpha, \beta)$-generalized hybrid mapping. We observe that the class of the mappings above covers the classes of well-known mappings. For example, an $(\alpha, \beta)$-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [11] and Iemoto and Takahashi [7]. Moreover Kocourek, Takahashi and Yao [10] proved the following nonlinear ergodic theorem.

**Theorem 1.2.** Let $H$ be a real Hilbert space, let $C$ be a non-empty closed convex subset of $H$, let $T$ be a generalized hybrid mapping from $C$ into $C$ which has a fixed point, and let $P$ be the metric projection of $H$ onto the set of fixed points of $T$. Then for any $x \in C$,

$$
S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x
$$

is weakly convergent to a fixed point $p$ of $T$, where $p = \lim_{n \to \infty} PT^n x$.

In this paper we introduce a broad class of nonlinear mappings $T$ from $C$ into $H$ which contains the class of constructive mappings and the class of generalized hybrid mappings. Then we prove a fixed point theorem for such mappings in a Hilbert space. Furthermore, we prove a nonlinear ergodic theorem of Baillon's type in a Hilbert space. There results generalize the fixed point theorem and the nonlinear ergodic theorem proved by Kocourek, Takahashi and Yao [10].
2 Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. Let $A$ be a nonempty subset of $H$. We denote by $\overline{\partial}A$ the closure of the convex hull of $A$. In a Hilbert space, it is known that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \tag{1}$$

for any $x, y \in H$ and for any $\alpha \in \mathbb{R}$; see [15]. Furthermore, in a Hilbert space, we have that

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2}$$

for any $x, y, z, w \in H$. Let $C$ be a nonempty subset of $H$ and let $T$ be a mapping from $C$ into $H$. We denote by $F(T)$ the set of fixed points of $T$. A mapping $T$ from $C$ into $H$ with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for any $x \in F(T)$ and for any $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping $T$ is closed and convex; see Ito and Takahashi [8]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \rightarrow z$. Since $C$ is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \rightarrow 0,$$

$z$ is a fixed point of $T$ and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Then, we have from (1) that

$$\|z - Tz\|^2 = \|\alpha x + (1 - \alpha)y - Tz\|^2
\leq \alpha\|x - z\|^2 + (1 - \alpha)\|y - z\|^2 - \alpha(1 - \alpha)\|x - y\|^2
= \alpha(1 - \alpha)^2\|x - y\|^2
= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2
= 0.$$  

This implies $Tz = z$. So, $F(T)$ is convex. Let $C$ be a nonempty closed convex subset of $H$ and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_Cx$. The mapping $P_C$ is called the metric projection of $H$ onto $C$. It is known that $P_C$ is nonexpansive and

$$\langle x - P_Cx, P_Cx - u \rangle \geq 0$$

for any $x \in H$ and for any $u \in C$; see [15] for more details. For proving a nonlinear ergodic theorem in this paper, we also need the following lemma proved by Takahashi and Toyoda [17].
Lemma 2.1. Let \( D \) be a nonempty closed convex subset of \( H \). Let \( P \) be the metric projection from \( H \) onto \( D \). Let \( \{u_n\} \) be a sequence in \( H \). If \( \|u_{n+1} - u\| \leq \|u_n - u\| \) for any \( u \in D \) and for any \( n \in \mathbb{N} \), then \( \{Pu_n\} \) converges strongly to some \( u_0 \in D \).

Let \( l^\infty \) be the Banach space of bounded sequences with supremum norm. Let \( \mu \) be an element of \((l^\infty)^* \) (the dual space of \( l^\infty \)). Then, we denote by \( \mu(f) \) the value of \( \mu \) at \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \). Sometimes, we denote by \( \mu_n(x_n) \) the value \( \mu(f) \). A linear functional \( \mu \) on \( l^\infty \) is called a mean if \( \mu(e) = \|\mu\| = 1 \), where \( e = (1, 1, 1, \ldots) \). A mean \( \mu \) is called a Banach limit on \( l^\infty \) if \( \mu_n(x_{n+1}) = \mu_n(x_n) \). We know that there exists a Banach limit on \( l^\infty \). If \( \mu \) is a Banach limit on \( l^\infty \), then for \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \),

\[
\lim_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n.
\]

In particular, if \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \) and \( x_n \to a \in \mathbb{R} \), then we have \( \mu(f) = \mu_n(x_n) = a \). See [14] for the proof of existence of a Banach limit and its other elementary properties.

Using means and the Riesz theorem, we can obtain the following result; see [13] and [14].

Lemma 2.2. Let \( H \) be a Hilbert space, let \( \{x_n\} \) be a bounded sequence in \( H \) and let \( \mu \) be a mean on \( l^\infty \). Then there exists a unique point \( z_0 \in \overline{co}\{x_n \mid n \in \mathbb{N}\} \) such that

\[
\mu_n(x_n, y) = \langle z_0, y \rangle, \quad \forall y \in H.
\]

3 Fixed point theorems

Let \( H \) be a real Hilbert space and let \( C \) be a nonempty subset of \( H \). A mapping \( T \) from \( C \) into \( H \) is said to be widely generalized hybrid if there exist \( \alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \mathbb{R} \) such that

\[
\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \mu\|x - Ty\|^2 \leq 0
\]

for any \( x, y \in C \); see [9]. Such a mapping \( T \) is called \((\alpha, \beta, \gamma, \delta, \epsilon, \zeta)\)-widely generalized hybrid. An \((\alpha, \beta, \gamma, \delta, \epsilon, \zeta)\)-widely generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [10] if \( \alpha + \beta = -\gamma - \delta = 1 \) and \( \epsilon = \zeta = 0 \). We first prove a fixed point theorem for widely generalized hybrid mappings in a Hilbert space.

Theorem 3.1. Let \( H \) be a real Hilbert space, let \( C \) be a non-empty closed convex subset of \( H \) and let \( T \) be an \((\alpha, \beta, \gamma, \delta, \epsilon, \zeta)\)-widely generalized hybrid mapping from \( C \) into itself which satisfies the following conditions (1) and (2):

1. \( \alpha + \beta + \gamma + \delta \geq 0 \);
2. \( \epsilon + \alpha + \gamma > 0 \) or \( \zeta + \alpha + \beta > 0 \).
Then $T$ has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots \}$ is bounded. In particular, a fixed point of $T$ is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition (1).

Remark 3.2. We can also prove Theorem 3.1 by using the following condition instead of the condition (2):

(2') $\epsilon - \beta - \delta > 0$, or $\zeta - \gamma - \delta > 0$.

In the case of the condition $\epsilon - \beta - \delta > 0$, we obtain from (1) that

$$\epsilon - \beta - \delta \leq \epsilon + \alpha + \gamma.$$

Thus we obtain the desired result by Theorem 3.1. Similarly, for the case of $\zeta - \gamma - \delta > 0$, we can obtain the result by using the case of $\zeta + \alpha + \beta > 0$.

As a direct consequence of Theorem 3.1, we obtain the following.

**Theorem 3.3.** Let $H$ be a real Hilbert space, let $C$ be a non-empty bounded closed convex subset of $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$-widely generalized hybrid mapping from $C$ into itself which satisfies the following conditions (1) and (2):

1. $\alpha + \beta + \gamma + \delta \geq 0$;
2. $\epsilon + \alpha + \gamma > 0$, or $\zeta + \alpha + \beta > 0$.

Then $T$ has a fixed point. In particular, a fixed point of $T$ is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition (1).

Note that an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$-widely generalized hybrid mapping $T$ above with $\alpha = 1$, $\beta = \gamma = \epsilon = \zeta = 0$ and $-1 < \delta < 0$ is a contractive mapping. Using Theorem 3.1, we can show the Banach fixed point theorem in a Hilbert space.

**Theorem 3.4 (the Banach fixed point theorem).** Let $H$ be a real Hilbert space and let $T$ be a contractive mapping from $H$ into $H$, that is, there exists a real number $\alpha$ with $0 < \alpha < 1$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|$$

for any $x, y \in H$. Then $T$ has a unique fixed point.

Using Theorem 3.1, we can show the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

**Theorem 3.5 (Kocourek, Takahashi and Yao [10]).** Let $H$ be a real Hilbert space, let $C$ be a non-empty closed convex subset of $H$ and let $T$ be a generalized hybrid mapping from $C$ into $C$, that is, there exist real numbers $\alpha$ and $\beta$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta) \|x - y\|^2$$

for any $x, y \in C$. Then $T$ has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots \}$ is bounded.
Example 3.6. Let $H$ be the real line and let $T$ be a mapping from $H$ into $H$ defined by $Tx=2x$ for any $x \in H$. Taking $\alpha=1$, $\beta=\gamma=-2$, $\delta=4$ and $\epsilon=\zeta=2$, we have that

\[
\alpha\|Tx-Ty\|^2 + \beta\|x-Ty\|^2 + \gamma\|Tx-y\|^2 + \delta\|x-y\|^2 + \max\{\epsilon\|x-Tx\|^2, \zeta\|y-Ty\|^2\} = 8|x-y|^2 - 2|x-2y|^2 - 2|2x-y|^2 + \max\{2x^2, 2y^2\}.
\]

for any $x, y \in H$. Furthermore, since $\{T^n0 | n=0, 1, \ldots\} = \{0\}$, (1) $\alpha + \beta + \gamma + \delta = 1 > 0$ and (2) $\epsilon + \alpha + \gamma = 1 > 0$, we have from Theorem 3.1 that $T$ has a unique fixed point. However $T$ is not a contractive mapping. Moreover, taking $x=0$ and $y=1$, we have that for any real numbers $\alpha$ and $\beta$,

\[
\alpha\|Tx-Ty\|^2 + (1-\alpha)\|x-Ty\|^2 - \beta\|Tx-y\|^2 - (1-\beta)\|x-y\|^2 = 4\alpha + 4(1-\alpha) - \beta - (1-\beta) = 3 > 0.
\]

Thus $T$ is not generalized hybrid.

4 Nonlinear ergodic theorem

In this section, using the technique developed by Takahashi [13], we prove a nonlinear ergodic theorem of Baillon's type in a Hilbert space. Before proving the result, we need the following lemmas.

Lemma 4.1. Let $H$ be a real Hilbert space, let $C$ be a non-empty closed convex subset of $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$-widely generalized hybrid mapping from $C$ into $C$ which has a fixed point and satisfies the condition:

(2) $\epsilon + \alpha + \gamma > 0$, or $\zeta + \alpha + \beta > 0$.

Then the set of fixed points of $T$ is closed.

Lemma 4.2. Let $H$ be a real Hilbert space, let $C$ be a non-empty closed convex subset of $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$-widely generalized hybrid mapping from $C$ into $C$ which has a fixed point and satisfies the conditions (1) and (2):

(1) $\alpha + \beta + \gamma + \delta \geq 0$;

(2) $\epsilon + \alpha + \gamma > 0$, or $\zeta + \alpha + \beta > 0$.

Then the set of fixed points of $T$ is convex.

Lemma 4.3. Let $H$ be a real Hilbert space, let $C$ be a non-empty closed convex subset of $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$-widely generalized hybrid mapping from $C$ into $C$ which has a fixed point and satisfies the conditions (1) and (3):
Then $T$ is quasi-nonexpansive.

**Theorem 4.4.** Let $H$ be a real Hilbert space, let $C$ be a non-empty closed convex subset of $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$-widely generalized hybrid mapping from $C$ into $C$ which has a fixed point and satisfies the conditions (1) and (2):

1. $\alpha + \beta + \gamma + \delta \geq 0$;
2. $\epsilon + \alpha + \gamma > 0$, or $\zeta + \alpha + \beta > 0$;
3. $\alpha + \gamma > 0$, or $\alpha + \beta > 0$;

respectively. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point $p$ of $T$, where $P$ is the metric projection from $H$ onto $F(T)$ and $p = \lim_{n \to \infty} PT^n x$.

Using Theorem 4.4, we can show the following nonlinear ergodic theorem for generalized hybrid mappings in a Hilbert space.

**Theorem 4.5 (Kocourek, Takahashi and Yao [10]).** Let $H$ be a real Hilbert space, let $C$ be a non-empty closed convex subset of $H$ and let $T$ be a generalized hybrid mapping from $C$ into $C$, that is, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for any $x, y \in C$. Suppose that $F(T)$ is nonempty. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point $p$ of $T$, where $p = \lim_{n \to \infty} PT^n x$ and $P$ is the metric projection from $H$ onto $F(T)$. 
References


