A geometric constant induced by the Dunkl-Williams inequality

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1 Introduction

In this note, we mainly consider about the Dunkl-Williams constant. In particular, we describe some recent results obtained in [19].

Throughout this note, the term “normed linear space” always means a real normed linear space which has two or more dimension. For a normed linear space $X$, let $B_X$ and $S_X$ denote the unit ball and the unit sphere of $X$, respectively. In 1964, Dunkl and Williams [7] showed the following simple inequalities: Let $X$ be a normed linear space. Then the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x - y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$, and if $X$ is an inner product space, the stronger inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x - y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$. These inequalities are so called the Dunkl-Williams inequality. In the same paper, it was also proved that for any $\varepsilon > 0$ there exist $x, y \in (\mathbb{R}^2, \|\cdot\|_1)$ such that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| > (4 - \varepsilon) \frac{\|x - y\|}{\|x\| + \|y\|}.$$

This means that the constant 4 is the best possible choice for the Dunkl-Williams inequality in the space $(\mathbb{R}^2, \|\cdot\|_1)$. There are many result related to this inequality (cf. [1, 4, 5, 6, 16, 17, 20, 21, 22, 23, 24], and so on).
2 The Dunkl-Williams inequality

In this section, we list some results related to the Dunkl-Williams inequality. First, we see the original proof of the inequality.

**Theorem 2.1** (The Dunkl-Williams inequality). Let $X$ be a normed linear space. Then, the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\| + \|y\|}$$

holds for all $x,y \in X \setminus \{0\}$, and if $X$ is an inner product space, the stronger inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\| + \|y\|}$$

holds for all $x,y \in X \setminus \{0\}$.

**Proof.** Let $x$ and $y$ be two nonzero elements of $X$. Then we have

$$\|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|x\|} \right\| + \|x\| \left\| \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\|$$

$$= \|x-y\| + \|x\| - \|y\|$$

$$\leq 2\|x-y\|.$$ 

Replacing $x$ with $y$, we also have

$$\|y\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq 2\|x-y\|.$$ 

Therefore we obtain

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\| + \|y\|}.$$ 

Next, we assume that $X$ is an inner product space. Then, for each nonzero elements $x, y \in X$, we obtain

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = 2 - 2\text{Re} \left( \frac{x}{\|x\|}, \frac{y}{\|y\|} \right)$$

$$= \frac{1}{\|x\| \|y\|} \left( 2\|x\| \|y\| - 2\text{Re} \langle x, y \rangle \right)$$

$$= \frac{1}{\|x\| \|y\|} \left( \|x+y\|^2 - (\|x\| - \|y\|)^2 \right).$$

Hence we have

$$\|x-y\|^2 - \left( \frac{\|x\| + \|y\|}{2} \right)^2 \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2$$

$$= \left( \frac{\|x\| - \|y\|}{\|x\| \|y\|} \right)^2 (\|x\|^2 + \|y\|^2) - \|x-y\|^2 \geq 0,$$

and so the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\| + \|y\|}$$

holds. \qed
Dunkl and Williams asked in their paper [7] whether the second inequality in Theorem 2.1 characterizes inner product spaces. A bit later, Kirk and Smiley [14] solved this problem affirmatively. They used the following result of Lorch [15].

**Lemma 2.2** (Lorch, 1948). Let X be a normed linear space. Then, X is an inner product space if and only if $x, y \in X$ and $\|x\| = \|y\|$ implies $\|\alpha x + \alpha^{-1}y\| \geq \|x + y\|$ for all $\alpha > 0$.

Now, we show the result of Kirk and Smiley.

**Theorem 2.3** (Kirk-Smiley, 1964). Let X be a normed linear space. Then, X is an inner product space if the inequality

$$\frac{x - y}{\|x\| - \|y\|} \leq \frac{2\|x - y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$.

**Proof.** Let x and y be nonzero elements of X such that $\|x\| = \|y\|$, and let $\alpha > 0$. Applying the inequality for $\alpha x$ and $\alpha^{-1}y$, we have

$$\|\alpha x + \alpha^{-1}y\| \geq \frac{\|\alpha x\| + \|\alpha^{-1}y\|}{2} \frac{x}{\|x\|} + \frac{y}{\|y\|}$$

$$= \frac{\alpha + \alpha^{-1}}{2} \|x + y\|$$

$$\geq \|x + y\|.$$  

Thus, X is an inner product space by Lemma 2.2. $\square$

As a consequence of Theorems 2.1 and 2.3, it turns out that a normed linear space X is an inner product space if and only if the inequality

$$\frac{x - y}{\|x\| - \|y\|} \leq \frac{2\|x - y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$. Thus, the best possible choice for the Dunkl-Williams inequality measures “how much” the space is close (or far) to be an inner product space. Motivated by this fact, Jiménez-Melado et al. [13] defined the Dunkl-Williams constant $DW(X)$ of a normed linear space X as the best constant for the Dunkl-Williams inequality, that is,

$$DW(X) = \sup \left\{ \frac{\|x\| + \|y\|}{\|x - y\|} : x, y \in X \setminus \{0\}, x \neq y \right\}.$$  

As was mentioned in Section 1, $DW((\mathbb{R}^2, \| \cdot \|_1)) = 4$, and Theorems 2.1 and 2.3 are restated as follows: Let X be a normed linear space. Then,

(i) $2 \leq DW(X) \leq 4$.

(ii) X is an inner product space if and only if $DW(X) = 2$.

Furthermore, it is known that $DW(X) = 4$ if and only if the space X is not uniformly non-square. Recall that a normed linear space X is said to be uniformly non-square if there exists $\delta > 0$ such that $x, y \in S_X$ and $\|x - y\| > 2(1 - \delta)$ implies $\|x + y\| \leq 2(1 - \delta)$. However, the Dunkl-Williams constant is very hard to calculate. In fact, except the case of $DW(X) = 2$ or 4, there have been probably no other example of the space X for which $DW(X)$ is determined precisely.
3 A calculation method for $DW(X)$

In [19], we constructed a new calculation method for the Dunkl-Williams constant. In this section, we describe the calculation method. As an application, we determine the precise value of $DW(\ell_2-\ell_\infty)$, where $\ell_2-\ell_\infty$ is the Day-James space defined as the space $\mathbb{R}^2$ endowed with the norm $\|\cdot\|_{2,\infty}$ given by

$$\| (a, b) \|_{2,\infty} = \begin{cases} \| (a, b) \|_2 & \text{if } ab \geq 0, \\ \| (a, b) \|_\infty & \text{if } ab \leq 0. \end{cases}$$

for all $(a, b) \in \mathbb{R}^2$.

When constructing a method, the notion of Birkhoff orthogonality plays an important role. We recall that for two elements $x, y$ of a normed linear space $X$, $x$ is said to be Birkhoff orthogonal to $y$, denoted by $x \perp_B y$, if $\| x + \lambda y \| \geq \| x \|$ for all $\lambda \in \mathbb{R}$. Obviously, Birkhoff orthogonality is always homogeneous, that is, $x \perp_B y$ implies $\alpha x \perp_B \beta y$ for all $\alpha, \beta \in \mathbb{R}$. More details about Birkhoff orthogonality can be found in Birkhoff [3], Day [8, 9] and James [10, 11, 12].

To construct a calculation method, we introduce some notations. Suppose that $X$ is a normed linear space. For each $x \in S_X$, let $V(x)$ be a subset of $X$ defined by $V(x) = \{ y \in X : x \perp_B y \}$. For each $x \in S_X$ and each $y \in V(x)$, we define $\Gamma(x, y)$ and $m(x, y)$ by

$$\Gamma(x, y) = \left\{ \frac{\lambda + \mu}{2} : \lambda \leq 0 \leq \mu, \| x + \lambda y \| = \| x + \mu y \| \right\}$$

and

$$m(x, y) = \sup \{ \| x + \gamma y \| : \gamma \in \Gamma(x, y) \},$$

respectively. Furthermore, let

$$M(x) = \sup \{ m(x, y) : y \in V(x) \}.$$

Using these notions, we obtain a new calculation method for the Dunkl-Williams constant.

**Theorem 3.1 ([19]).** Let $X$ be a normed linear space. Then,

$$DW(X) = 2 \sup \{ M(x) : x \in S_X \}.$$

If $\dim X = 2$, we have the following improvement of the preceding theorem.

**Theorem 3.2 ([19]).** Let $X$ be a normed linear space with $\dim X = 2$. Then,

$$DW(X) = 2 \sup \{ M(x) : x \in \text{ext}(B_X) \},$$

where $\text{ext}(B_X)$ denotes the set of all extreme points of $B_X$.

When we put this theory into practice, the following results are needed.

**Proposition 3.3.** Let $X$ be a normed linear space. Suppose that $x \in S_X$ and $y \in V(x)$. Then, the following hold:

(i) $0 \in V(x)$.

(ii) $\alpha y \in V(x)$ for all $\alpha \in \mathbb{R}$.

(iii) $m(x, 0) = 1 \leq m(x, y)$. 


(iv) $m(x, \alpha y) = m(x, y)$ for all $\alpha \in \mathbb{R} \setminus \{0\}.$

**Proposition 3.4.** Let $X, Y$ be normed linear spaces and let $x \in S_X$ and $y \in V(x)$. Suppose that $T$ is an isometric isomorphism from $X$ onto $Y$. Then, the following hold:

(i) $m(Tx, Ty) = m(x, y)$.

(ii) $M(Tx) = M(x)$.

**Proposition 3.5.** Let $X$ be a normed linear space and let $x \in S_X$ and $y \in V(x) \setminus \{0\}$. Suppose that $T$ is an isometric isomorphism from $X$ onto $Y$. Then, the following hold:

(i) $m(Tx, Ty) = m(x, y)$.

(ii) $M(Tx) = M(x)$.

**Theorem 3.6.** Let $X$ be a normed linear space and let $x \in S_X$ and $y \in V(x)$. Suppose that $\{x_n\}$ is a sequence in $S_X$ which converges to $x$. If the sequence $\{y_n\}$ satisfies $y_n \in V(x_n)$ for each $n \in \mathbb{N}$ and converges to $y$, then

$$m(x, y) \leq \liminf_{n \to \infty} m(x_n, y_n).$$

All of these results can be found in [19].

4 The Dunkl-Williams constant of the space $\ell_2-\ell_\infty$

Applying Theorem 3.2, we obtain the following example.

**Theorem 4.1** ([19]). $DW(\ell_2-\ell_\infty) = 2\sqrt{2}$.

To prove Theorem 4.1, we need a lot of works. First, one can easily show that

$$\text{ext}(B_{\ell_2-\ell_\infty}) = \{(a, b) \in \mathbb{R}^2 : ab \geq 0, a^2 + b^2 = 1\} \cup \{(1, -1), (-1, 1)\}.$$

Now, let $M_0 = \sup\{M((a, b)) : 0 < b < a, a^2 + b^2 = 1\}$. Then, we have the following lemma by Theorems 3.2 and 3.6, and Proposition 3.4.

**Lemma 4.2.** $DW(\ell_2-\ell_\infty) = 2 \max\{M_0, M((1, -1))\}$.

We remark that $0 < b < a$ and $a^2 + b^2 = 1$ implies $b < 1/\sqrt{2} < a$. Next, to calculate $M(x)$, we find the set $V(x)$ for each $x$.

**Lemma 4.3.** Suppose that $0 < b < a$ and $a^2 + b^2 = 1$. Then,

$$V((a, b)) = \{\alpha(b, -a) \in \mathbb{R}^2 : \alpha \in \mathbb{R}\}.$$ 

**Lemma 4.4.** $V((1, -1)) = \{(a, b) \in \mathbb{R}^2 : ab \geq 0\}.$

To reduce the amount of computation, we make use of Proposition 3.3.

**Lemma 4.5.** Suppose that $0 < b < a$ and $a^2 + b^2 = 1$. Then,

$$M((a, b)) = m((a, b), (b, -a)).$$

**Lemma 4.6.** $M((1, -1)) = \sup\{m((1, -1), (a, b)) : 0 < b < a, a^2 + b^2 = 1\}.$
We need to determine the set $\Gamma(x, y)$ to calculate the value of $m(x, y)$.

**Lemma 4.7.** Suppose that $0 < b < a$ and $a^2 + b^2 = 1$. Then,

$$\Gamma((a, b), (b, -a)) = \begin{cases} [0, b/a] & \text{if } a \leq 2b, \\ [0, (a + b - \sqrt{2ab})/(a - b)] & \text{if } a > 2b. \end{cases}$$

**Lemma 4.8.** Suppose that $0 < b < a$ and $a^2 + b^2 = 1$. Then,

$$\Gamma((1, -1), (a, b)) = [b-a, 0].$$

Now we prove Theorem 4.1. Proposition 3.5 is used in this phase.

**Proof of Theorem 4.1.** Suppose that $0 < b < a$ and $a^2 + b^2 = 1$. First, we assume that $a \leq 2b$. Then, by Proposition 3.5 and Lemma 4.7, we have

$$M((a, b)) = m((a, b), (b, -a)) = \max \left\{ \| (a, b) \|_{2, \infty}, \left\| (a, b) + \frac{b}{a}(b, -a) \right\|_{2, \infty} \right\} = \frac{1}{a}.$$

On the other hand, if $0 < b < a$ and $a^2 + b^2 = 1$, then $a \leq 2b$ if and only if $a \leq 2/\sqrt{5}$. Hence we obtain

$$\{M(a, b) : 0 < b < a \leq 2b, a^2 + b^2 = 1\} = \{1/a : 1/\sqrt{2} < a \leq 2/\sqrt{5}\} = [\sqrt{5}/2, \sqrt{2}).$$

Next, we suppose that $a > 2b$. Then we have

$$0 < \frac{a + b - \sqrt{2ab}}{a - b} < 1.$$

Since the function $t \mapsto \|(a, b) + t(b, -a)\|$ is convex and increasing on $[0, \infty)$, we obtain

$$M((a, b)) = m((a, b), (b, -a)) = \left\| (a, b) + \frac{a + b - \sqrt{2ab}}{a - b}(b, -a) \right\|_{2, \infty} \leq \|(a, b) + (b, -a)\|_{2, \infty} = \|(a + b, b - a)\|_{2, \infty} = a + b \leq \sqrt{2(a^2 + b^2)}/2 = \sqrt{2}$$

by Proposition 3.5 and Lemma 4.7. Thus, we have

$$M_0 = \sup \{M((a, b)) : 0 < b < a, a^2 + b^2 = 1\} = \sqrt{2}.$$
Finally, by Proposition 3.5 and Lemma 4.8, we obtain
\[
m((1, -1), (a, b)) = \|((1, -1) + (b - a)(a, b))\|_{2,\infty}
\]
\[
= \|(a^2 + b^2, -a^2 - b^2) + (ab - a^2, b^2 - ab)\|_{2,\infty}
\]
\[
= \|(ab + b^2, -a^2 - ab)\|_{2,\infty}
\]
\[
= (a + b)\|((b, -a)\|_{2,\infty}
\]
\[
= a(a + b) \leq \sqrt{2}.
\]
This implies that \(M((1, -1)) \leq \sqrt{2} = M_0\).
Thus, by Lemma 4.2, we have
\[DW(\ell_2-\ell_{\infty}) = 2\max\{M_0, M((1, -1))\} = 2M_0 = 2\sqrt{2}. \]

References


