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Kyoto University
Fixed Point Theorems and Convergence Theorems for Non-self Mappings in Hilbert Spaces

(new article)

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Abstract. In this article, we first prove fixed point theorems for nonlinear non-self mappings in a Hilbert space. Next, we deal with weak and strong convergence theorems for nonlinear mappings in a Hilbert space. Using these results, we obtain new and well-known fixed point and convergence theorems. For example, we generalize Hojo and Takahashi’s mean strong convergence theorem [11] for generalized hybrid mappings.

1 Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty subset of $H$. Kocourek, Takahashi and Yao [19] introduced a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonspreading mappings [21] and hybrid mappings [30]. A mapping $T: C \rightarrow H$ is said to be generalized hybrid [19] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx-Ty\|^2 + (1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2 + (1-\beta)\|x-y\|^2 \quad (1.1)$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. We call such $T$ an $(\alpha, \beta)$-generalized hybrid mapping. An $(\alpha, \beta)$-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, i.e., $\|Tx-Ty\| \leq \|Tx-Ty\|$ for all $x, y \in C$. It is nonspreading for $\alpha = 2$ and $\beta = 1$, i.e., $2\|Tx-Ty\|^2 \leq \|x-Ty\|^2 + \|y-Tx\|^2$ for all $x, y \in C$. Furthermore, it is hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e., $3\|Tx-Ty\|^2 \leq \|x-Ty\|^2 + \|y-Tx\|^2 + \|y-x\|^2$ for all $x, y \in C$. They proved fixed point theorems and nonlinear ergodic theorems of Baillon’s type [3] for generalized hybrid mappings in a Hilbert space; see also Kohsaka and Takahashi [20] and Iemoto and Takahashi [15].

Putting $x = u$ with $u = Tu$ in (1.1), we have that for any $y \in C$,

$$\alpha\|u-Ty\|^2 + (1-\alpha)\|u-Ty\|^2 \leq \beta\|u-y\|^2 + (1-\beta)\|u-y\|^2$$

and hence $\|u-Ty\| \leq \|u-y\|$. This means that an $(\alpha, \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Kocourek, Takahashi and Yao [19] also introduced a more broad class of nonlinear mappings which covers generalized hybrid mappings. A mapping
$S : C \to H$ is called super hybrid \cite{19, 34} if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that
\begin{equation}
\alpha \|Sx - Sy\|^2 + (1 - \alpha + \gamma)\|x - Sy\|^2
\leq (\beta + (\beta - \alpha)\gamma)\|Sx - y\|^2 + (1 - (\beta - \beta - 1)\gamma)\|x - y\|^2
+ (\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2
\tag{1.2}
\end{equation}
for all $x, y \in C$. We call such a mapping an $(\alpha, \beta, \gamma)$-super hybrid mapping. An $(\alpha, \beta, 0)$-super hybrid mapping is $(\alpha, \beta)$-generalized hybrid. So, the class of super hybrid mappings contains generalized hybrid mappings. On the other hand, Hojo, Takahashi and Yao \cite{12} defined the following class of nonlinear mappings which contains generalized hybrid mappings. A mapping $U : C \to H$ is called extended hybrid if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that
\begin{equation}
\alpha(1 + \gamma)\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2
\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2
- (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2
\tag{1.3}
\end{equation}
for all $x, y \in C$. We note that super hybrid mappings and extended hybrid mappings are not quasi-nonexpansive generally. We also know the following relation between generalized hybrid mappings and extended hybrid mappings.

**Theorem 1.1.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \neq -1$. Let $T$ and $U$ be mappings of $C$ into $H$ such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$, where $Ix = x$ for all $x \in H$. Then, for $1 + \gamma > 0$, $T : C \to H$ is an $(\alpha, \beta)$-generalized hybrid mapping if and only if $U : C \to H$ is an $(\alpha, \beta, \gamma)$-extended hybrid mapping.

In this article, motivated by these mappings and results, we first prove fixed point theorems for nonlinear non-self mappings in a Hilbert space. Next, we deal with weak and strong convergence theorems for nonlinear mappings in a Hilbert space. Using these results, we obtain new and well-known fixed point and convergence theorems. For example, we generalize Hojo and Takahashi’s mean strong convergence theorem \cite{11} for generalized hybrid mappings.

## 2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. Let $H$ be a (real) Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From \cite{29}, we know the following basic equality: For any $x, y \in H$ and $\lambda \in \mathbb{R}$, we have
\begin{equation}
\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.
\tag{2.1}
\end{equation}
Furthermore, we know that for any $x, y, u, v \in H$
\begin{equation}
2 \langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2.
\tag{2.2}
\end{equation}
Let $C$ be a nonempty closed convex subset of $H$ and let $T$ be a mapping from $C$ into itself. Then, we denote by $F(T)$ the set of fixed points of $T$. A mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$
is called quasi-nonexpansive if \( \|x - Ty\| \leq \|x - y\| \) for all \( x \in F(T) \) and \( y \in C \). Let \( C \) be a nonempty closed convex subset of \( H \) and \( x \in H \). Then, we know that there exists a unique nearest point \( z \in C \) such that \( \|x - z\| = \inf_{y \in C} \|x - y\| \). We denote such a correspondence by \( z = P_Cx \). The mapping \( P_C \) is called the metric projection of \( H \) onto \( C \). It is known that \( P_C \) is nonexpansive and \( \langle x - P_Cx, P_Cx - u \rangle \geq 0 \) for all \( x \in H \) and \( u \in C \). Furthermore, we know that

\[
\|P_Cx - P_Cy\|^2 \leq \langle x - y, P_Cx - P_Cy \rangle
\]

for all \( x, y \in H \); see [29] for more details. For proving main results in this paper, we also need the following lemmas proved in [31] and [2].

**Lemma 2.1** ([31]). Let \( D \) be a nonempty closed convex subset of \( H \). Let \( P \) be the metric projection from \( H \) onto \( D \). Let \( \{u_n\} \) be a sequence in \( H \). If \( \|u_{n+1} - u\| \leq \|u_n - u\| \) for all \( u \in D \) and \( n \in \mathbb{N} \), then \( \{Pu_n\} \) converges strongly to some \( u_0 \in D \).

**Lemma 2.2** ([2]). Let \( \{s_n\} \) be a sequence of nonnegative real numbers, let \( \{\alpha_n\} \) be a sequence of \([0, 1]\) with \( \sum_{n=1}^\infty \alpha_n = \infty \), let \( \{\beta_n\} \) be a sequence of nonnegative real numbers with \( \sum_{n=1}^\infty \beta_n < \infty \), and let \( \{\gamma_n\} \) be a sequence of real numbers with \( \limsup_{n \to \infty} \gamma_n \leq 0 \). Suppose that

\[
s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n
\]

for all \( n = 1, 2, \ldots \). Then \( \lim_{n \to \infty} s_n = 0 \).

Let \( l^\infty \) be the Banach space of bounded sequences with supremum norm. Let \( \mu \) be an element of \( (l^\infty)^* \) (the dual space of \( l^\infty \)). Then we denote by \( \mu(f) \) the value of \( \mu \) at \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \). Sometimes, we denote by \( \mu_n(x_n) \) the value \( \mu(f) \). A linear functional \( \mu \) on \( l^\infty \) is called a mean if \( \mu(e) = \|\mu\| = 1 \), where \( e = (1, 1, 1, \ldots) \). A mean \( \mu \) is called a Banach limit on \( l^\infty \) if \( \mu_n(x_{n+1}) = \mu_n(x_n) \). We know that there exists a Banach limit on \( l^\infty \). If \( \mu \) is a Banach limit on \( l^\infty \), then for \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \),

\[
\liminf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n.
\]

In particular, if \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \) and \( x_n \to a \in \mathbb{R} \), then we have \( \mu(f) = \mu_n(x_n) = a \). See [27] for the proof of existence of a Banach limit and its other elementary properties. Using Banach limits, Kocourek, Takahashi and Yao [19] proved the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

**Theorem 2.3** ([19]). Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \) and let \( T : C \to C \) be a generalized hybrid mapping. Then \( T \) has a fixed point in \( C \) if and only if \( \{T^n z\} \) is bounded for some \( z \in C \).

## 3 Fixed Point Theorem for Non-Self Mappings

In this section, we first prove a fixed point theorem for generalized hybrid non-self mappings in a Hilbert space. For proving it, we need the following lemmas.

**Lemma 3.1.** Let \( H \) be a Hilbert space and let \( C \) be a nonempty subset of \( H \). Let \( \alpha \) and \( \beta \) be in \( \mathbb{R} \). Then, a non-self mapping \( T : C \to H \) is \( (\alpha, \beta) \)-generalized hybrid if and only if it satisfies that

\[
\|Tx - Ty\|^2 \leq (\alpha - \beta)\|x - y\|^2 + 2(\alpha - 1)(x - Tx, y - Ty) - (\alpha - \beta - 1)\|y - Tx\|^2
\]
for all \(x, y \in C\).

Using Lemma 3.1, we have the following result.

**Lemma 3.2.** Let \(H\) be a Hilbert space and let \(C\) be a nonempty bounded subset of \(H\). If a non-self mapping \(T : C \rightarrow H\) is generalized hybrid, then \(TC\) is bounded.

The following is a fixed point theorem for non-self generalized hybrid mappings in a Hilbert space.

**Theorem 3.3** ([12]). Let \(C\) be a nonempty bounded closed convex subset of a Hilbert space \(H\) and let \(\alpha\) and \(\beta\) be real numbers. Let \(T\) be an \((\alpha, \beta)\)-generalized hybrid mapping with \(\alpha - \beta \geq 0\) of \(C\) into \(H\). Suppose that there exists \(m > 1\) such that for any \(x \in C\), \(Tx = x + t(y - x)\) for some \(y \in C\) and \(t\) with \(1 \leq t \leq m\). Then, \(T\) has a fixed point in \(C\).

Recently, Hojo, Suzuki and Takahashi [10] also proved a more general fixed point theorem for nonlinear non-self mappings in a Hilbert space.

**Theorem 3.4** ([10]). Let \(C\) be a nonempty, bounded, closed and convex subset of a Hilbert space \(H\) and let \(\alpha, \beta, \gamma, \delta \in \mathbb{R}\). Let \(T : C \rightarrow H\) be an \((\alpha, \beta, \gamma, \delta)\)-normal generalized hybrid mapping, i.e., there exist \(\alpha, \beta, \gamma, \delta \in \mathbb{R}\) such that

\[
\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0
\]

for all \(x, y \in C\). Suppose that it satisfies the following condition (1) or (2):

1. \(\alpha + \beta + \gamma + \delta > 0\), \(\alpha + \gamma > 0\) and \(\alpha + \beta \geq 0\);  
2. \(\alpha + \beta + \gamma + \delta \geq 0\), \(\alpha + \beta > 0\) and \(\alpha + \gamma \geq 0\).

Assume that there exists \(m > 1\) such that for any \(x \in C\), \(Tx = x + t(y - x)\) for some \(y \in C\) and \(t\) with \(0 < t \leq m\). Then \(T\) has a fixed point in \(C\). In particular, a fixed point of \(T\) is unique in the case of \(\alpha + \beta + \gamma + \delta > 0\) on the conditions (1) and (2).

For proving this result, Hojo, Suzuki and Takahashi [10] used the following fixed point theorem obtained by Kawasaki and Takahashi [18].

**Theorem 3.5** ([18]). Let \(H\) be a Hilbert space, let \(C\) be a nonempty, closed and convex subset of \(H\) and let \(T\) be an \((\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)\)-widely more generalized hybrid mapping from \(C\) into itself, i.e., there exist \(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta \in \mathbb{R}\) such that

\[
\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \epsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0
\]

for all \(x, y \in C\). Suppose that it satisfies the following condition (1) or (2):

1. \(\alpha + \beta + \gamma + \delta \geq 0\), \(\alpha + \gamma + \epsilon + \eta > 0\) and \(\zeta + \eta \geq 0\);  
2. \(\alpha + \beta + \gamma + \delta \geq 0\), \(\alpha + \beta + \zeta + \eta > 0\) and \(\epsilon + \eta \geq 0\).

Then \(T\) has a fixed point if and only if there exists \(z \in C\) such that \(\{T^nz : n = 0, 1, \ldots\}\) is bounded. In particular, a fixed point of \(T\) is unique in the case of \(\alpha + \beta + \gamma + \delta > 0\) on the conditions (1) and (2).

Let us give an example of mappings which is related to the conditions in Theorem 3.4. In the case of \(H = \mathbb{R}\), consider a mapping \(T : [0, 1] \rightarrow \mathbb{R}\):

\[
Tx = (1 + 2x) \cos x - 2x^2, \quad \forall x \in [0, 1].
\]
Then, we have
\[ Tx = (1 + 2x)(\cos x - x) + x, \quad \forall x \in [0, 1]. \]

Take \( m = 3 \). For any \( x \in [0, 1] \), take \( t = 1 + 2x \) and \( y = \cos x \). Then, we have that
\[ Tx = t(y - x) + x, \quad y = \cos x \in [0, 1] \quad \text{and} \quad 0 < t = 1 + 2x \leq 3. \]

4 Weak convergence theorems

In this section, using the technique developed by Takahashi [26], we first prove a mean convergence theorem of Baillon's type [3] for super hybrid mappings in a Hilbert space. For proving it, we need the following lemma.

**Lemma 4.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( T \) be a generalized hybrid mapping from \( C \) into itself. Suppose that \( \{T^n x\} \) is bounded for some \( x \in C \). Define \( S_n x = \frac{1}{n} \sum_{k=1}^{n} T^k x \). Then, \( \lim_{n \to \infty} \|S_n x - TS_n x\| = 0 \). In particular, if \( C \) is bounded, then
\[ \lim_{n \to \infty} \sup_{x \in C} \|S_n x - TS_n x\| = 0. \]

Using Lemma 4.1, we obtain the following mean convergence theorem.

**Theorem 4.2** ([12]). Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( \alpha \), \( \beta \) and \( \gamma \) be real numbers with \( \gamma \geq 0 \) and let \( S : C \to C \) be an \((\alpha, \beta, \gamma)\)-super hybrid mapping with \( F(S) \neq \emptyset \) and let \( P \) be the metric projection of \( H \) onto \( F(T) \). Then, for any \( x \in C \),
\[ S_n x = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1}{1 + \gamma} S + \frac{\gamma}{1 + \gamma} I \right)^k x \]
converges weakly to \( z \in F(S) \), where \( z = \lim_{n \to \infty} PT^n x \) and \( T = \frac{1}{1 + \gamma} S + \frac{\gamma}{1 + \gamma} I \).

Next, we prove a weak convergence theorem of Mann's type [23] for nonlinear non-self mappings in a Hilbert space. For proving the result, we need the following two lemmas.

**Lemma 4.3.** Let \( C \) be a nonempty, closed and convex subset of a Hilbert space \( H \) and let \( T \) be an \((\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)\)-widely more generalized hybrid mapping from \( C \) into \( H \) with \( F(T) \neq \emptyset \) which satisfies the condition (1) or (2):

1. \( \alpha + \beta + \gamma + \delta \geq 0 \), \( \alpha + \beta > 0 \) and \( \zeta + \eta \geq 0 \);
2. \( \alpha + \beta + \gamma + \delta \geq 0 \), \( \alpha + \gamma > 0 \) and \( \epsilon + \eta \geq 0 \).

Then \( T \) is quasi-nonexpansive.

We remark that if \( T : C \to H \) is quasi-nonexpansive, then \( F(T) \) is closed and convex; see Itoh and Takahashi [16]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that \( F(T) \) is closed, take a sequence \( \{z_n\} \subset F(T) \) with \( z_n \to z \). Since \( C \) is weakly closed, we have \( z \in C \). Furthermore, from \( \|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0 \), we have that \( z \) is a fixed point of \( T \) and so \( F(T) \) is closed. Let us show that \( F(T) \) is convex.
For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Then we have from (2.1) that

$$
\|z - Tz\|^2 = \|\alpha x + (1 - \alpha)y - Tz\|^2
= \alpha\|x - Tz\|^2 + (1 - \alpha)\|y - Tz\|^2 - \alpha(1 - \alpha)\|x - y\|^2
\leq \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2
= \alpha(1 - \alpha)^2\|x - y\|^2 + (1 - \alpha)\alpha^2\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2
= \alpha(1 - \alpha)(1 - \alpha + 1 - \alpha - 1)\|x - y\|^2 = 0
$$

and hence $Tz = z$. This implies that $F(T)$ is convex.

**Lemma 4.4.** Let $H$ be a Hilbert space and let $C$ be a nonempty, closed and convex subset of $H$. Let $T : C \rightarrow H$ be an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfies the following condition (1) or (2):

(1) $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma + \epsilon + \zeta + \eta > 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta + \zeta + \eta > 0$.

If $x_n \rightarrow z$ and $x_n - Tx_n \rightarrow 0$, then $z \in F(T)$.

Using Lemmas 4.3, 4.4 and the technique developed by Ibaraki and Takahashi [13, 14], we can prove the following weak convergence theorem.

**Theorem 4.5 ([10]).** Let $H$ be a Hilbert space and let $C$ be a nonempty, closed and convex subset of $H$. Let $T : C \rightarrow H$ be a widely more generalized hybrid mapping with $F(T) \neq \emptyset$ which satisfies the condition (1) or (2):

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\epsilon + \zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$.

Let $P$ be the metric projection of $H$ onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \rightarrow \infty} \alpha_n (1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$
x_{n+1} = P_C (\alpha_n x_n + (1 - \alpha_n)Tx_n), \quad n \in \mathbb{N}.
$$

Then $\{x_n\}$ converges weakly to $v \in F(T)$, where $v = \lim_{n \rightarrow \infty} Px_n$.

Using Theorem 4.5, we can show the following weak convergence theorem of Mann’s type for generalized hybrid mappings in a Hilbert space.

**Theorem 4.6 ([19]).** Let $H$ be a Hilbert space and let $C$ be a nonempty, closed and convex subset of $H$. Let $T : C \rightarrow C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \rightarrow \infty} \alpha_n (1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}.
$$

Then the sequence $\{x_n\}$ converges weakly to an element $v \in F(T)$.

**Proof.** Since $T : C \rightarrow C$ is a generalized hybrid mapping, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$
\alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - Ty\|^2 + (1 - \beta)\|x - Ty\|^2
$$

for all $x, y \in C$. We have that this mapping is an $(\alpha, 1 - \alpha, -\beta, -(1 - \beta), 0, 0, 0)$-widely more generalized hybrid mapping which satisfies the condition (2) in Theorem 4.5. Therefore, we have the desired result from Theorem 4.5. \qed
5 Strong Convergence Theorem

In this section, using an idea of mean convergence by Shimizu and Takahashi [24] and [25], we prove a strong convergence theorem of Halpern’s type for super hybrid mappings in a Hilbert space.

Theorem 5.1 ([12]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $\alpha, \beta$ and $\gamma$ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be a $(\alpha, \beta, \gamma)$-super hybrid mapping with $F(S) \neq \emptyset$ and let $P$ be the metric projection of $H$ onto $F(S)$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and

$$
\begin{align*}
  x_{n+1} &= \alpha_n u + (1 - \alpha_n)z_n, \\
  z_n &= \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1}{1 + \gamma} S + \frac{\gamma}{1 + \gamma} I \right)^k x_n
\end{align*}
$$

for all $n = 1, 2, \ldots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $Pu$.

Recently, Hojo, Suzuki and Takahashi [10] also proved the following strong convergence theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 5.2 ([10]). Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $T$ be a widely more generalized hybrid mapping of $C$ into itself which satisfies the following condition (1) or (2):

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$, $\epsilon + \eta \geq 0$ and $\zeta + \eta \geq 0$;

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$, $\zeta + \eta \geq 0$ and $\epsilon + \eta \geq 0$.

Let $u \in C$ and define sequences $\{x_n\}$ and $\{z_n\}$ in $C$ as follows: $x_1 = x \in C$ and

$$
\begin{align*}
  x_{n+1} &= \alpha_n u + (1 - \alpha_n)z_n, \\
  z_n &= \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n
\end{align*}
$$

for all $n = 1, 2, \ldots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T) \neq \emptyset$, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu$, where $P$ is the metric projection of $H$ onto $F(T)$.

Using Theorem 5.2, we can show the following result obtained by Hojo and Takahashi [11].

Theorem 5.3 ([11]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T$ be a generalized hybrid mapping of $C$ into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in $C$ as follows: $x_1 = x \in C$ and

$$
\begin{align*}
  x_{n+1} &= \alpha_n u + (1 - \alpha_n)z_n, \\
  z_n &= \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n
\end{align*}
$$

for all $n = 1, 2, \ldots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu \in F(T)$, where $P$ is the metric projection of $H$ onto $F(T)$. 
Proof. As in the proof of Theorem 4.6, a generalized hybrid mapping is a widely more generalized hybrid mapping. Therefore, we have the desired result from Theorem 5.2. □

References


