<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title</td>
<td>Fixed Point Theorems and Convergence Theorems for Non-self Mappings in Hilbert Spaces (Nonlinear Analysis and Convex Analysis)</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Hojo, Mayumi</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1841: 17-25</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/194986">http://hdl.handle.net/2433/194986</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Fixed Point Theorems and Convergence Theorems for Non-self Mappings in Hilbert Spaces

（ヒルベルト空間における非自己写像の不動点定理と収束定理）

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Abstract. In this article, we first prove fixed point theorems for nonlinear non-self mappings in a Hilbert space. Next, we deal with weak and strong convergence theorems for nonlinear mappings in a Hilbert space. Using these results, we obtain new and well-known fixed point and convergence theorems. For example, we generalize Hojo and Takahashi's mean strong convergence theorem [11] for generalized hybrid mappings.

1 Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty subset of $H$. Kocourek, Takahashi and Yao [19] introduced a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonsparing mappings [21] and hybrid mappings [30]. A mapping $T : C \rightarrow H$ is said to be generalized hybrid [19] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

(1.1)

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. We call such $T$ an $(\alpha, \beta)$-generalized hybrid mapping. An $(\alpha, \beta)$-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, i.e., $\|Tx - Ty\| \leq \|Tx - Ty\|$ for all $x, y \in C$. It is nonsparing for $\alpha = 2$ and $\beta = 1$, i.e., $2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$ for all $x, y \in C$. Furthermore, it is hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e., $3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 + \|y - x\|^2$ for all $x, y \in C$. They proved fixed point theorems and nonlinear ergodic theorems of Baillon's type [3] for generalized hybrid mappings in a Hilbert space; see also Kohsaka and Takahashi [20] and Iemoto and Takahashi [15]. Putting $x = u$ with $u = Tu$ in (1.1), we have that for any $y \in C$,

$$\alpha\|u - Ty\|^2 + (1 - \alpha)\|u - Ty\|^2 \leq \beta\|u - y\|^2 + (1 - \beta)\|u - y\|^2$$

and hence $\|u - Ty\| \leq \|u - y\|$. This means that an $(\alpha, \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Kocourek, Takahashi and Yao [19] also introduced a more broad class of nonlinear mappings which covers generalized hybrid mappings. A mapping
$S : C \to H$ is called super hybrid \cite{19, 34} if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that
\[\alpha\|Sx - Sy\|^2 + (1 - \alpha + \gamma)\|x - Sy\|^2 \leq (\beta + (\beta - \alpha)\gamma)\|Sx - y\|^2 + (1 - (\beta - \beta - 1)\gamma)\|x - y\|^2 + (\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sx\|^2\] (1.2)

for all $x, y \in C$. We call such a mapping an $(\alpha, \beta, \gamma)$-super hybrid mapping. An $(\alpha, \beta, 0)$-super hybrid mapping is $(\alpha, \beta)$-generalized hybrid. So, the class of super hybrid mappings contains generalized hybrid mappings. On the other hand, Hojo, Takahashi and Yao \cite{12} defined the following class of nonlinear mappings which contains generalized hybrid mappings. A mapping $U : C \to H$ is called extended hybrid if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that
\[\alpha(1 + \gamma)\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2\] (1.3)

for all $x, y \in C$. We note that super hybrid mappings and extended hybrid mappings are not quasi-nonexpansive generally. We also know the following relation between generalized hybrid mappings and extended hybrid mappings

**Theorem 1.1.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $\alpha, \beta$ and $\gamma$ be real numbers with $\gamma \neq -1$. Let $T$ and $U$ be mappings of $C$ into $H$ such that $U = \frac{1}{1 + \gamma} T + \frac{\gamma}{1 + \gamma} I$, where $Ix = x$ for all $x \in H$. Then, for $1 + \gamma > 0$, $T : C \to H$ is an $(\alpha, \beta)$-generalized hybrid mapping if and only if $U : C \to H$ is an $(\alpha, \beta, \gamma)$-extended hybrid mapping.

In this article, motivated by these mappings and results, we first prove fixed point theorems for nonlinear non-self mappings in a Hilbert space. Next, we deal with weak and strong convergence theorems for nonlinear mappings in a Hilbert space. Using these results, we obtain new and well-known fixed point and convergence theorems. For example, we generalize Hojo and Takahashi's mean strong convergence theorem \cite{11} for generalized hybrid mappings.

## 2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. Let $H$ be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From \cite{29}, we know the following basic equality: For any $x, y \in H$ and $\lambda \in \mathbb{R}$, we have
\[\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.\] (2.1)

Furthermore, we know that for any $x, y, u, v \in H$
\[2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2.\] (2.2)

Let $C$ be a nonempty closed convex subset of $H$ and let $T$ be a mapping from $C$ into itself. Then, we denote by $F(T)$ the set of fixed points of $T$. A mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$
is called quasi-nonexpansive if \( \|x - Ty\| \leq \|x - y\| \) for all \( x \in F(T) \) and \( y \in C \). Let \( C \) be a nonempty closed convex subset of \( H \) and \( x \in H \). Then, we know that there exists a unique nearest point \( z \in C \) such that \( \|x - z\| = \inf_{y \in C} \|x - y\| \). We denote such a correspondence by \( z = P_C x \). The mapping \( P_C \) is called the metric projection of \( H \) onto \( C \). It is known that \( P_C \) is nonexpansive and \( \langle x - P_C x, y - P_C x \rangle \geq 0 \) for all \( x \in H \) and \( y \in C \). Furthermore, we know that
\[
\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle \tag{2.3}
\]
for all \( x, y \in H \); see [29] for more details. For proving main results in this paper, we also need the following lemmas proved in [31] and [2].

**Lemma 2.1** ([31]). Let \( D \) be a nonempty closed convex subset of \( H \). Let \( P \) be the metric projection from \( H \) onto \( D \). Let \( \{u_n\} \) be a sequence in \( H \). If \( \|u_{n+1} - u\| \leq \|u_n - u\| \) for all \( u \in D \) and \( n \in \mathbb{N} \), then \( \{P u_n\} \) converges strongly to some \( u_0 \in D \).

**Lemma 2.2** ([2]). Let \( \{s_n\} \) be a sequence of nonnegative real numbers, let \( \{\alpha_n\} \) be a sequence of \([0,1]\) with \( \sum_{n=1}^{\infty} \alpha_n = \infty \), let \( \{\beta_n\} \) be a sequence of nonnegative real numbers with \( \sum_{n=1}^{\infty} \beta_n < \infty \), and let \( \gamma_n\) be a sequence of real numbers with \( \limsup_{n \to \infty} \gamma_n \leq 0 \). Suppose that
\[
s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \gamma_n + \beta_n
\]
for all \( n = 1, 2, \ldots \). Then \( \lim_{n \to \infty} s_n = 0 \).

Let \( l^\infty \) be the Banach space of bounded sequences with supremum norm. Let \( \mu \) be an element of \( (l^\infty)^* \) (the dual space of \( l^\infty \)). Then we denote by \( \mu(f) \) the value of \( \mu \) at \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \). Sometimes, we denote by \( \mu_n(x_n) \) the value \( \mu(f) \). A linear functional \( \mu \) on \( l^\infty \) is called a mean if \( \mu(e) = \|\mu\| = 1 \), where \( e = (1, 1, 1, \ldots) \). A mean \( \mu \) is called a Banach limit on \( l^\infty \) if \( \mu_n(x_{n+1}) = \mu_n(x_n) \). We know that there exists a Banach limit on \( l^\infty \). If \( \mu \) is a Banach limit on \( l^\infty \), then for \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \),
\[
\lim_{n \to \infty} \inf x_n \leq \mu_n(x_n) \leq \lim_{n \to \infty} \sup x_n.
\]
In particular, if \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \) and \( x_n \to a \in \mathbb{R} \), then we have \( \mu(f) = \mu_n(x_n) = a \).

See [27] for the proof of existence of a Banach limit and its other elementary properties. Using Banach limits, Kocourek, Takahashi and Yao [19] proved the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

**Theorem 2.3** ([19]). Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \) and let \( T : C \to C \) be a generalized hybrid mapping. Then \( T \) has a fixed point in \( C \) if and only if \( \{T^n z\} \) is bounded for some \( z \in C \).

## 3 Fixed Point Theorem for Non-Self Mappings

In this section, we first prove a fixed point theorem for generalized hybrid non-self mappings in a Hilbert space. For proving it, we need the following lemmas.

**Lemma 3.1.** Let \( H \) be a Hilbert space and let \( C \) be a nonempty subset of \( H \). Let \( \alpha \) and \( \beta \) be in \( \mathbb{R} \). Then, a non-self mapping \( T : C \to H \) is \( (\alpha, \beta) \)-generalized hybrid if and only if it satisfies that
\[
\|Tx - Ty\|^2 \leq (\alpha - \beta)\|x - y\|^2 + 2(\alpha - 1)(x - Tx, y - Ty) - (\alpha - \beta - 1)\|y - Tx\|^2
\]
for all \( x, y \in C \).

Using Lemma 3.1, we have the following result.

**Lemma 3.2.** Let \( H \) be a Hilbert space and let \( C \) be a nonempty bounded subset of \( H \). If a non-self mapping \( T : C \to H \) is generalized hybrid, then \( TC \) is bounded.

The following is a fixed point theorem for non-self generalized hybrid mappings in a Hilbert space.

**Theorem 3.3** ([12]). Let \( C \) be a nonempty bounded closed convex subset of a Hilbert space \( H \) and let \( \alpha \) and \( \beta \) be real numbers. Let \( T \) be an \((\alpha, \beta)\)-generalized hybrid mapping with \( \alpha - \beta \geq 0 \) of \( C \) into \( H \). Suppose that there exists \( m > 1 \) such that for any \( x \in C \), \( Tx = x + t(y - x) \) for some \( y \in C \) and \( t \) with \( 1 \leq t \leq m \). Then, \( T \) has a fixed point in \( C \).

Recently, Hojo, Suzuki and Takahashi [10] also proved a more general fixed point theorem for nonlinear non-self mappings in a Hilbert space.

**Theorem 3.4** ([10]). Let \( C \) be a nonempty, bounded, closed and convex subset of a Hilbert space \( H \) and let \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \). Let \( T : C \to H \) be an \((\alpha, \beta, \gamma, \delta)\)-normal generalized hybrid mapping, i.e., there exist \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \) such that

\[
\alpha \Vert Tx - Ty \Vert^2 + \beta \Vert x - Ty \Vert^2 + \gamma \Vert Tx - y \Vert^2 + \delta \Vert x - y \Vert^2 \leq 0
\]

for all \( x, y \in C \). Suppose that it satisfies the following condition (1) or (2):

1. \( \alpha + \beta + \gamma + \delta \geq 0 \), \( \alpha + \gamma > 0 \) and \( \alpha + \beta \geq 0 \);
2. \( \alpha + \beta + \gamma + \delta \geq 0 \), \( \alpha + \beta > 0 \) and \( \alpha + \gamma \geq 0 \).

Assume that there exists \( m > 1 \) such that for any \( x \in C \), \( Tx = x + t(y - x) \) for some \( y \in C \) and \( t \) with \( 0 < t \leq m \). Then \( T \) has a fixed point in \( C \). In particular, a fixed point of \( T \) is unique in the case of \( \alpha + \beta + \gamma + \delta > 0 \) on the conditions (1) and (2).

For proving this result, Hojo, Suzuki and Takahashi [10] used the following fixed point theorem obtained by Kawasaki and Takahashi [18].

**Theorem 3.5** ([18]). Let \( H \) be a Hilbert space, let \( C \) be a nonempty, closed and convex subset of \( H \) and let \( T \) be an \((\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)\)-widely more generalized hybrid mapping from \( C \) into itself, i.e., there exist \( \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta \in \mathbb{R} \) such that

\[
\alpha \Vert Tx - Ty \Vert^2 + \beta \Vert x - Ty \Vert^2 + \gamma \Vert Tx - y \Vert^2 + \delta \Vert x - y \Vert^2 \\
+ \epsilon \Vert x - Tx \Vert^2 + \zeta \Vert y - Ty \Vert^2 + \eta \Vert (x - Tx) - (y - Ty) \Vert^2 \leq 0
\]

for all \( x, y \in C \). Suppose that it satisfies the following condition (1) or (2):

1. \( \alpha + \beta + \gamma + \delta \geq 0 \), \( \alpha + \gamma + \epsilon + \eta > 0 \) and \( \zeta + \eta \geq 0 \);
2. \( \alpha + \beta + \gamma + \delta \geq 0 \), \( \alpha + \beta + \zeta + \eta > 0 \) and \( \epsilon + \eta \geq 0 \).

Then \( T \) has a fixed point if and only if there exists \( z \in C \) such that \( \{T^nx : n = 0, 1, \ldots\} \) is bounded. In particular, a fixed point of \( T \) is unique in the case of \( \alpha + \beta + \gamma + \delta > 0 \) on the conditions (1) and (2).

Let us give an example of mappings which is related to the conditions in Theorem 3.4. In the case of \( H = \mathbb{R} \), consider a mapping \( T : [0, 1) \to \mathbb{R} \):

\[
Tx = (1 + 2x) \cos x - 2x^2, \quad \forall x \in [0, 1].
\]
Then, we have
\[ Tx = (1 + 2x)(\cos x - x) + x, \quad \forall x \in [0, 1]. \]

Take \( m = 3 \). For any \( x \in [0, 1] \), take \( t = 1 + 2x \) and \( y = \cos x \). Then, we have that \( Tx = t(y - x) + x, y = \cos x \in [0, 1] \) and \( 0 < t = 1 + 2x \leq 3 \).

4 Weak convergence theorems

In this section, using the technique developed by Takahashi [26], we first prove a mean convergence theorem of Baillon's type [3] for super hybrid mappings in a Hilbert space. For proving it, we need the following lemma.

**Lemma 4.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( T \) be a generalized hybrid mapping from \( C \) into itself. Suppose that \( \{T^n x\} \) is bounded for some \( x \in C \). Define \( S_n x = \frac{1}{n} \sum_{k=1}^{n} T^k x \). Then, \( \lim_{n \to \infty} \sup_{x \in C} \|S_n x - TS_n x\| = 0 \). In particular, if \( C \) is bounded, then
\[ \lim_{n \to \infty} \sup_{x \in C} \|S_n x - TS_n x\| = 0. \]

Using Lemma 4.1, we obtain the following mean convergence theorem.

**Theorem 4.2** ([12]). Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( \alpha, \beta \) and \( \gamma \) be real numbers with \( \gamma \geq 0 \) and let \( S : C \to C \) be an \((\alpha, \beta, \gamma)\)-super hybrid mapping with \( F(S) \neq \emptyset \) and let \( P \) be the metric projection of \( H \) onto \( F(T) \). Then, for any \( x \in C \),
\[ S_n x = \frac{1}{n} \sum_{k=1}^{n} (\frac{1}{1 + \gamma} S + \frac{\gamma}{1 + \gamma} I)^k x \]
converges weakly to \( z \in F(S) \), where \( z = \lim_{n \to \infty} P T^n x \) and \( T = \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I \).

Next, we prove a weak convergence theorem of Mann's type [23] for nonlinear non-self mappings in a Hilbert space. For proving the result, we need the following two lemmas.

**Lemma 4.3.** Let \( C \) be a nonempty, closed and convex subset of a Hilbert space \( H \) and let \( T \) be an \((\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)\)-widely more generalized hybrid mapping from \( C \) into \( H \) with \( F(T) \neq \emptyset \) which satisfies the condition (1) or (2):

1. \( \alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \beta > 0 \) and \( \zeta + \eta \geq 0 \);
2. \( \alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \gamma > 0 \) and \( \varepsilon + \eta \geq 0 \).

Then \( T \) is quasi-nonexpansive.

We remark that if \( T : C \to H \) is quasi-nonexpansive, then \( F(T) \) is closed and convex; see Itoh and Takahashi [16]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that \( F(T) \) is closed, take a sequence \( \{z_n\} \subset F(T) \) with \( z_n \to z \). Since \( C \) is weakly closed, we have \( z \in C \). Furthermore, from \( \|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0 \), we have that \( z \) is a fixed point of \( T \) and so \( F(T) \) is closed. Let us show that \( F(T) \) is convex.
For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Then we have from (2.1) that
\begin{align*}
|z - Tz|^2 &= |\alpha x + (1 - \alpha)y - Tz|^2 \\
&= \alpha|x - Tz|^2 + (1 - \alpha)|y - Tz|^2 - \alpha(1 - \alpha)|x - y|^2 \\
&\leq \alpha|x - z|^2 + (1 - \alpha)|y - z|^2 - \alpha(1 - \alpha)|x - y|^2 \\
&= \alpha(1 - \alpha)^2|x - y|^2 + (1 - \alpha)\alpha^2|x - y|^2 - \alpha(1 - \alpha)|x - y|^2 \\
&= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)|x - y|^2 = 0
\end{align*}
and hence $Tz = z$. This implies that $F(T)$ is convex.

**Lemma 4.4.** Let $H$ be a Hilbert space and let $C$ be a nonempty, closed and convex subset of $H$. Let $T : C \to H$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfies the following condition (1) or (2):

1. $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma + \varepsilon + \eta > 0$;
2. $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta + \zeta + \eta > 0$.

If $x_n \to z$ and $x_n - Tx_n \to 0$, then $z \in F(T)$.

Using Lemmas 4.3, 4.4 and the technique developed by Ibaraki and Takahashi [13, 14], we can prove the following weak convergence theorem.

**Theorem 4.5** ([10]). Let $H$ be a Hilbert space and let $C$ be a nonempty, closed and convex subset of $H$. Let $T : C \to H$ be a widely more generalized hybrid mapping with $F(T) \neq \emptyset$ which satisfies the condition (1) or (2):

1. $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$;
2. $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$.

Let $P$ be the metric projection of $H$ onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\lim \inf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and
\begin{equation}
x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n)Tx_n), \quad n \in \mathbb{N}.
\end{equation}
Then $\{x_n\}$ converges weakly to $v \in F(T)$, where $v = \lim_{n \to \infty} Px_n$.

Using Theorem 4.5, we can show the following weak convergence theorem of Mann’s type for generalized hybrid mappings in a Hilbert space.

**Theorem 4.6** ([19]). Let $H$ be a Hilbert space and let $C$ be a nonempty, closed and convex subset of $H$. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\lim \inf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and
\begin{equation}
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}.
\end{equation}
Then the sequence $\{x_n\}$ converges weakly to an element $v \in F(T)$.

**Proof.** Since $T : C \to C$ is a generalized hybrid mapping, there exist $\alpha, \beta \in \mathbb{R}$ such that
\begin{equation}
\alpha||Tx - Ty||^2 + (1 - \alpha)||x - Ty||^2 \leq \beta||Tx - Ty||^2 + (1 - \beta)||x - Ty||^2
\end{equation}
for all $x, y \in C$. We have that this mapping is an $(\alpha, 1 - \alpha, -\beta, -(1 - \beta), 0, 0, 0)$-widely more generalized hybrid mapping which satisfies the condition (2) in Theorem 4.5. Therefore, we have the desired result from Theorem 4.5.
5 Strong Convergence Theorem

In this section, using an idea of mean convergence by Shimizu and Takahashi [24] and [25], we prove a strong convergence theorem of Halpern's type for super hybrid mappings in a Hilbert space.

**Theorem 5.1** ([12]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be a $(\alpha, \beta, \gamma)$-super hybrid mapping with $F(S) \neq \emptyset$ and let $P$ be the metric projection of $H$ onto $F(S)$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and

$$
\begin{aligned}
x_{n+1} &= \alpha_n u + (1 - \alpha_n)z_n, \\
z_n &= \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I \right)^k x_n
\end{aligned}
$$

for all $n = 1, 2, \ldots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $Pu$.

Recently, Hojo, Suzuki and Takahashi [10] also proved the following strong convergence theorem for widely more generalized hybrid mappings in a Hilbert space.

**Theorem 5.2** ([10]). Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $T$ be a widely more generalized hybrid mapping of $C$ into itself which satisfies the following condition (1) or (2):

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$, $\varepsilon + \eta \geq 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$, $\zeta + \eta \geq 0$ and $\varepsilon + \eta \geq 0$.

Let $u \in C$ and define sequences $\{x_n\}$ and $\{z_n\}$ in $C$ as follows: $x_1 = x \in C$ and

$$
\begin{aligned}
x_{n+1} &= \alpha_n u + (1 - \alpha_n)z_n, \\
z_n &= \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n
\end{aligned}
$$

for all $n = 1, 2, \ldots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T) \neq \emptyset$, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu$, where $P$ is the metric projection of $H$ onto $F(T)$.

Using Theorem 5.2, we can show the following result obtained by Hojo and Takahashi [11].

**Theorem 5.3** ([11]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T$ be a generalized hybrid mapping of $C$ into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in $C$ as follows: $x_1 = x \in C$ and

$$
\begin{aligned}
x_{n+1} &= \alpha_n u + (1 - \alpha_n)z_n, \\
z_n &= \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n
\end{aligned}
$$

for all $n = 1, 2, \ldots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu \in F(T)$, where $P$ is the metric projection of $H$ onto $F(T)$. 

\[\text{23}\]
Proof. As in the proof of Theorem 4.6, a generalized hybrid mapping is a widely more generalized hybrid mapping. Therefore, we have the desired result from Theorem 5.2. □

References


