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Minimax Programming Problems with Complex Variables

By

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Abstract

Complex programming problems have various types. In this talk, we would take an overview concerning programming problems by complex variables. We introduce these problems start from: linear and fractional linear complex problems with complex constrained to a typical general complex programming.

The main purpose of a complex programming could integrate to be three situations. It includes to establish the necessary optimality conditions for a solution, then to constitute the existence of optimal solution from the converse of necessary conditions with extra assumptions.

Many authors make efforts to find each request extra conditions for supplementary to deduce the sufficient optimally conditions. It follows that the constitution for sufficient optimalities are various.

In order to complete the work, one will often consider the dual part relative to primal problem, thus the duality problem in optimization theory also plays an important part in solving the existence of optimal solution for primal problem. We only present the necessary and sufficient optimality condition for typical nonsmooth complex programming.

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§ 1. Introduction

A typical minimax complex programming may be considered as the following problem:

$$(\mathrm{FP}_c) \qquad \min_{\zeta\in X} \sup_{\eta\in Y} \frac{Re\,[f(\zeta,\eta)+(z^H Az)^{1/2}]}{Re\,[g(\zeta,\eta)-(z^H Bz)^{1/2}]}$$

$$\text{subject to } X=\left\{\zeta=(z,\overline{z})\in\mathbb{C}^{2n} \;\middle|\; -h(\zeta)\in S\subset\mathbb{C}^p\right\}$$

and Y is a compact subset of $\mathbb{C}^{2m}$, S is a polyhedral cone. It is easy to see that: all linear, fractional linear, nonfractional nonlinear, or fractional nonlinear programing problems are deduced from (FP_c) as the special cases.

In the earliest complex linear problem is firstly given by Levinson in 1966 as the form:

$$(\mathrm{C}_1) \qquad \min\; Re\,\left(a^T z+b\right)$$

$$\text{subject to } Az=d,\; A\in\mathbb{C}^{n\times n},\; z\in\mathbb{C}^n, \text{ and } a,\, b,\, d\in\mathbb{C}^n.$$

Shorter after in 1970, Swarap and Sharma consider the problem as:

$$(\mathrm{C}_2) \qquad \min\; Re\,\left(a^T z+b\right)/Re\,\left(c^T z+d\right)$$

$$\text{subject to } Az=\alpha,\; A\in\mathbb{C}^{n\times n}, \text{ and } a,\, b,\, c,\, d,\, \alpha\in\mathbb{C}^n.$$

Hence after, Mond and Craven in 1975, they consider:

$$(\mathrm{C}_3) \qquad \min\; Re\, f(z)/Re\, g(z)$$

$$\text{subject to } f,\, g \text{ are complex continuous on } z\in\mathbb{C}^n,$$

while nonfractional complex programming problem as the form:

$$(\mathrm{C}_4) \qquad \min\; Re[f(\zeta)+(z^H Az)^{1/2}]$$

$$\text{subject to } \zeta=(z,\overline{z})\in\mathbb{C}^{2n}, \text{ and}$$

$$-h(\zeta)\in S, \text{ a polyhedral cone in } \mathbb{C}^P$$

where f is analytic on $\Omega=\{\zeta=(z,\overline{z})\in\mathbb{C}^{2n}\}\subset\mathbb{C}^{2n}$, A is positive semidefinite Hermitian metrix $\mathbb{C}^{n\times n}$.

Note that the objective in C_4 contains complex function $f(\zeta)$ and a positive valued function $(z^H Az)^{1/2}$. This type of function often occurs in engineering.

§ 2. Preliminary of Complex Minimax Programming

We note that in our topic, the convexity (or generalized convexity) always plays an important role. But Ferrero (cf. [2]) explored that nonlinear analytic function $f(\zeta)$, $\zeta \in \mathbb{C}^n$ has no real convex part, so we consider that in many complex programming, the function f often varies on $z \in \mathbb{C}^n$ together with its conjugate complex $\overline{z} \in \mathbb{C}^n$ in the discussed problem. One can see Parkush et al. (cf. ,1984, ZAMM), first consider the complex programming:

$$\begin{aligned} (\mathrm{P}_0) \qquad & \min \frac{Re\,[f(z,\overline{z}) + (z^H A z)^{1/2}]}{Re\,[g(z,\overline{z}) - (z^H B z)^{1/2}]} \\ & \text{subject to } \; -h(z,\overline{z}) \in S \subset \mathbb{C}^p, \;\; z \in \mathbb{C}^n, \end{aligned}$$

and established sufficient optimality conditions under the assumptions in advanced that f and $-g$ have convex real part, and $-h$ is S-convex. The reason comes from that $f(\zeta)$ and $-g(\zeta)$, $\zeta \in \mathbb{C}^n$, may have no convex real parts if $\zeta \neq (z,\overline{z})$.

Hence after, many works in complete programming, like Lai and Liu (cf. [12]), investigate the complex fractional programming involving generalized quasi / pseudo convexity function as the form (P_0). It is natural to extend the one complex variable $\zeta = (z,\overline{z})$ to two independent complex variable functions $g(\zeta,\eta)$ and $f(\zeta,\eta)$ in a complex fractional programming problem as the following form:

$$\begin{aligned} (\mathrm{FP}_c) \qquad & \min_{\zeta\in X} \sup_{\eta\in Y} \frac{Re\,[f(\zeta,\eta) + (z^H A z)^{1/2}]}{Re\,[g(\zeta,\eta) - (z^H B z)^{1/2}]} \\ & \text{subject to } \; X = \left\{\zeta = (z,\overline{z}) \in \mathbb{C}^{2n} \mid -h(\zeta) \in S \subset \mathbb{C}^p\right\} \\ & \qquad\qquad\; Y \text{ is a compact subset of } \mathbb{C}^{2m} \end{aligned}$$

where S is a polyhedral cone in $\mathbb{C}^p$; $A, B \in \mathbb{C}^{n\times n}$ are positive semidefinite Hermitian matrices; $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuous maps on $\mathbb{C}^{2n} \times \mathbb{C}^{2m}$ to $\mathbb{C}$, and for each $\eta \in Y \subset \mathbb{C}^{2m}$, $f(\cdot,\eta)$, $g(\cdot,\eta)$ and $h(\cdot)$ are analytic functions on

$$\zeta \in Q = \{(z,\overline{z}) \mid z \in \mathbb{C}^n\} \subset \mathbb{C}^{2n}.$$

This Q is a linear manifold over real field $\mathbb{R}$.

There are many complex programming fractional as well as nonfractional are special cases of problem (FP_c) including differentiable or nondifferentiable.

The main tasks in study of problem (FP_c) have three tasks including:

1. To find the necessary optimality conditions.
2. To establish the sufficient optimality condition.
3. To constitute some duality models and prove the duality theorem holds for the dual and primal problems have the same optimal values, that is, there is no duality gap between primal problem and duality problem.

Usually, the sufficient optimality conditions are deduced from the converse of necessary conditions with extra assumptions, and hence many researchers always effort to search the extra conditions. Cause to this reason, the sufficient conditions are various, and the duality models are based on the sufficient conditions, thus one may have many duality forms. Sometimes in order to get the solution, one may propose some approximation procedure to get the approximate solution. The technics or methods are also important evaluation for solution.

In this paper, we propose theoretical approach. In order to explain some mathematical methods, we start from [6, 10, 14, 15] for the necessary theorem for minimax problem as following.

At first, we consider a nonfractional minimax programming as the form (cf. [6, 10, 14, 15]):

$$(\mathrm{P}_c) \qquad \min_{\zeta\in X}\sup_{\eta\in Y} \; Re\,\left[f(\zeta,\eta)+\left(z^HAz\right)^{1/2}\right]$$

$$\text{subject to } \; X=\left\{\zeta=(z,\bar z)\in\mathbb{C}^{2n} \;\middle|\; -h(\zeta)\in S\subset\mathbb{C}^p\right\}$$

$$Y\subset\mathbb{C}^{2m} \;\text{ is a compact subset.}$$

For convenience, we denote a set depending the (P_c)-optimal ζ_0 by:

$$Z_{\bar\eta}(\zeta_0)=\left\{\zeta\in\mathbb{C}^{2n}\;\middle|\;\begin{array}{l}-h'_\zeta(\zeta_0)\,\zeta\in S(-h(\zeta_0)),\;\zeta=(z,\bar z)\in Q,\\ \text{and } Re\left[\sum_{i=1}^k\lambda_i f'_\zeta(\zeta_0,\eta_i)\zeta+\langle Az,z\rangle^{1/2}\right]<0\end{array}\right\}.$$

Thus, if $Z_{\bar\eta}(\zeta_0)=\emptyset$, then it follows that the following conditions hold:

$$\text{(I)}\;\sum_{i=1}^k\lambda_i\left[\overline{\nabla_z f(\zeta_0,\eta_i)}+\nabla_{\bar z}f(\zeta_0,\eta_i)+Au\right]+\left[\mu^T\overline{\nabla_z h(\zeta_0)}+\mu^H\nabla_{\bar z}h(\zeta_0)\right]=0,$$

$$\text{(II)}\;Re\,\langle h(\zeta_0),\mu\rangle=0,\;\; u^HAu\le 1,$$

$$\text{(III)}\;\left(z_0^HAz_0\right)^{1/2}=Re\,\left(z_0^HAu\right),$$

where $\lambda_i > 0$, $\sum_{i=1}^{k} \lambda_i = 1$ and

$$\eta_i \in Y(\zeta_0) = \left\{ \eta \in Y \;\middle|\; Re\, f(\zeta_0, \eta) = \sup_{\nu \in Y} Re\, f(\zeta_0, \nu) \right\}, \text{ for } i = 1, \dots, k.$$

Therefore we obtain the following theorem.

Theorem 2.1. *Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be a (P_c)-optimal such that $\langle Az_0, z_0 \rangle = 0$, and $Z_{\widetilde{\eta}}(\zeta_0) = \emptyset$ with some maximum point $\widetilde{\eta} \in Y(\zeta_0)$. Then the above conditions (I), (II) and (III) hold.*

In order to establish the sufficient optimality conditions, we introduce the generalized convexity (strictly) at $\zeta = \zeta_0 \in Q \subset \mathbb{C}^{2n}$, involving the pseudo convex as well as the quasi convex at $\zeta = \zeta_0 \in Q$ for analytic functions and analytic mapping $h(\cdot) : \mathbb{C}^{2n} \longrightarrow \mathbb{C}^p$ in papers [6, 7] by Lai and Huang.

Consequently, we can treat with the necessary conditions as (P_c) in Theorem 2.1, and the optimality conditions for nonsmooth minimax fractional programming (FP_c) which established by Lai and Huang in Theorem 3 of [7].

Under $z_0^H A z_0 = 0$ or $z_0^H B z_0 = 0$, Problem (FP_c) is nondifferentiable at the optimal solution $\zeta_0 = (z_0, \overline{z_0})$, it is worthy to remark here that, if we define the set:

$$Z_{\widetilde{\eta}}(\zeta_0) = \left\{ \zeta \in \mathbb{C}^{2n} \;\middle|\; -h'_\zeta(\zeta_0)\zeta \in S(-h(\zeta_0)),\ \zeta = (z, \overline{z}) \in Q \right\},$$

with any one of the next conditions (i), (ii), and (iii) holds:

(i) $$Re \left\{ \sum_{i=1}^{k} \lambda_i \left[f'_\zeta(\zeta_0, \eta_i) - v^* g'_\zeta(\zeta_0, \eta_i) \right] \zeta + \frac{\langle Az_0, z \rangle}{\langle Az_0, z_0 \rangle^{1/2}} + \langle (v^*)^2 Bz, z \rangle^{1/2} \right\} < 0,$$

if $z_0^H A z_0 > 0$ and $z_0^H B z_0 = 0$;

(ii) $$Re \left\{ \sum_{i=1}^{k} \lambda_i \left[f'_\zeta(\zeta_0, \eta_i) - v^* g'_\zeta(\zeta_0, \eta_i) \right] \zeta + \langle Az, z \rangle^{1/2} + \frac{\langle v^* B z_0, z \rangle}{\langle Bz_0, z_0 \rangle^{1/2}} \right\} < 0,$$

if $z_0^H A z_0 = 0$ and $z_0^H B z_0 > 0$;

(iii) $$Re \left\{ \sum_{i=1}^{k} \lambda_i \left[f'_\zeta(\zeta_0, \eta_i) - v^* g'_\zeta(\zeta_0, \eta_i) \right] \zeta + \langle [A + (v^*)^2 B] z, z \rangle^{1/2} \right\} < 0,$$

if $z_0^H A z_0 = 0$ and $z_0^H B z_0 = 0$.

It then can be deduced a nondifferentiable Problem (FP_c) in nonsmooth optimality conditions as following in which the set $Z_{\widetilde{\eta}}(\zeta_0) = \emptyset$ plays a very important role, and the optimality conditions, in general, can be stated as the following theorem.

Theorem 2.2. *Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be (FP$_c$)-optimal with optimal value v^*. If Problem (FP$_c$) possesses the constraint qualification at ζ_0 and $Z_{\overline{\eta}}(\zeta_0) = \emptyset$. Then there exist $\mu^* \in S^* \subset \mathbb{C}^p$ and vectors $u_1, u_2 \in \mathbb{C}^n$ such that the Kuhn Tucker type necessary conditions hold as in Theorem 2.1.*

Theorem 2.3. *Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be a feasible solution of Problem (FP$_c$). Then by the compactness of $Y(\zeta_0)$, there are*

$$i = 1, \ldots, k, \;\; \textit{with} \; \sum_{i=1}^{k} \lambda_i = 1, \;\; \textit{and} \; v^* \in \mathbb{R}^+$$

such that it exists $0 \neq \mu \in S^*$, the dual polyhedral cone of $S \subset \mathbb{C}^p$, and u_1, $u_2 \in \mathbb{C}^n$ satisfy the necessary conditions of Theorem 2.1 for $\langle Az_0, z_0 \rangle > 0$, $\langle Bz_0, z_0 \rangle > 0$ and for Theorem 2.2 provided $Z_{\overline{\eta}}(\zeta_0) = \emptyset$ holds. Moreover, assume that any one of these three conditions (i), (ii) and (iii) stated in Theorem 4 of [7] holds. Then the feasible solution $\zeta_0 = (z_0, \overline{z_0})$ is an optimal solution in Q.

For detail results, one can see the recent works given in references there in.

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