VARIATIONAL CONVERGENCE OVER $p$-UNIFORMLY CONVEX SPACES (Geometric Aspect of Partial Differential Equations and Conservation Laws)

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VARIATIONAL CONVERGENCE OVER $p$-UNIFORMLY
CONVEX SPACES

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Abstract. We establish a variational convergence over $p$-uniformly
convex spaces for $p \geq 2$. Variational convergence for Cheeger type
energy functionals over $L^p$-maps into $p$-uniformly convex space
having NPC property of Busemann type and the existence of $p$-
harmonic map for Cheeger type energy functionals with Dirichlet
boundary condition are also presented.

1. INTRODUCTION AND MAIN RESULT

This article is a summary of a part of the paper [17] under prepa-
ratio. We study a variational convergences over $p$-uniformly convex
spaces having NPC property in the sense of Busemann, where a $p$-
uniformly convex space is a natural generalization of $p$-uniformly con-
 vex Banach space. Typical examples of $p$-uniformly convex spaces are
$L^p$-spaces with $p \geq 2$, CAT(0)-spaces, more concretely, Hadamard
manifolds and trees, and so on. If the target space is a $p$-uniformly
convex space having NPC property in the sense of Busemann, then the
$L^p$-mapping space is also a $p$-uniformly convex geodesic spaces hav-
ing NPC property in the sense of Busemann, and an energy functional
defined in a suitable way becomes convex and lower semi-continuous.
Thus, it is reasonable to consider that $(H_t, d_{H_t})$ and $(H, d_H)$ are all $p$-
uniformly convex geodesic spaces having the weak $L$-convexity of Buse-
mann type instead of such $L^p$-mapping spaces (see Definition 3.1 below
for the weak $L$-convexity), and $E_i : H_t \to [0, \infty]$ and $E : H \to [0, \infty]$ are convex lower semi-continuous functions with $E_i, E \not\equiv +\infty$. For any
$\lambda \geq 0$ and $u \in H$, there exists a unique minimizer, say $J^{E}_i(u) \in H$, of $v \mapsto \lambda^{p-1}E(v) + d_{H_t}^{p}(u, v)$. This defines a map $J^{E}_i : H \to H$, called the resolvent of $E$ (see Theorem 5.2 below and [9, 22, 20] for the case
$p = 2$). The minimum $E^{\lambda}(u) := \min_{v \in H}(\lambda^{p-1}E(v) + d_{H}^{p}(u, v))$ is called the Moreau-Yosida approximation or the Hopf-Lax formula. Note that
if $X$ is a Hilbert space and if $E$ is a closed densely defined symmetric

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mation, Hopf-Lax formula, Mosco convergence, convergence of resolvents.

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quadratic form on $X$, then we have $J_{\lambda}^{E} = (I + \lambda A)^{-1}$, where $A$ is the infinitesimal generator associated with $E$. The one-parameter family $[0, +\infty \ni \lambda \mapsto J_{\lambda}^{E}(u)$ gives a deformation of a given map $u \in H$ to a minimizer of $E$ (or a harmonic map), $\lim_{\lambda \to +\infty} J_{\lambda}^{E}(u)$ (if any). Jost [13] studied convergence of resolvents and Moreau-Yosida approximations. Although his study is only on a fixed CAT(0)-space, we extend it for a sequence of $p$-uniformly convex geodesic spaces having the weak $L$-convexity of Busemann type with an asymptotic relation (Theorem 5.11 below). This is new even on a fixed $p$-uniformly convex geodesic spaces having the weak $L$-convexity of Busemann type.

We can apply our result in the following way. Let $(X_{i}, q_{i}) \to (X, q)$ and $(Y_{i}, o_{i}) \to (Y, o)$ ($i = 1, 2, 3, \ldots$) be two pointed Gromov-Hausdorff convergent sequences of proper metric spaces, where ‘proper’ means that any bounded subset is relatively compact, and let us give a positive Radon measure $m_{i}$ on $X_{i}$ with full support which converge to a positive Radon measure $m$ on $X$ (see the definition for the convergence of measures in [20]). We are interested in the convergence and asymptotic behavior of maps $u_{i}: X_{i} \to Y_{i}$ and also energy functionals $E_{i}$ defined on a family of maps from $X_{i} \to Y_{i}$. We set $L_{i}^{p} := L_{o_{i}}^{p}(X_{i}, Y_{i}, m_{i})$ and $L^{p} := L_{o}^{p}(X, Y, m)$. For $u_{i}, v_{i} \in L_{i}^{p}$ (resp. $u, v \in L^{p}$), we set $d_{L_{i}^{p}}(u_{i}, v_{i}) := \|d_{Y_{i}}(u_{i}, v_{i})\|_{L_{i}^{p}}$ (resp. $d_{L^{p}}(u, v) := \|d_{Y}(u, v)\|_{L^{p}}$), where $\| \cdot \|_{L_{i}^{p}}$ (resp. $\| \cdot \|_{L^{p}}$) is the $L^{p}$-norm with respect to the measure $m_{i}$ (resp. $m$). Consider

$$L^{p} := \bigsqcup_{i} L_{i}^{p} \sqcup L^{p}$$

and endowed the $L^{p}$-topology defined in [20] with $L^{p}$. The $L^{p}$-topology on $L^{p}$ has some nice properties involving the $L^{p}$-metric structure of $L_{i}^{p}$ and $L^{p}$, such as, if $L_{i}^{p} \ni u_{i}, v_{i} \to u, v \in L^{p}$ respectively in $L_{i}^{p}$, then $d_{L_{i}^{p}}(u_{i}, v_{i}) \to d_{L^{p}}(u, v)$. By their properties we present a set of axioms for a topology on $L^{p}$ for $(L_{i}^{p}, d_{L_{i}^{p}})$ and $(L^{p}, d_{L^{p}})$. We call such a topology satisfying the axioms the asymptotic relation between $\{L_{i}^{p}\}$ and $L^{p}$ (see Definition 4.3). Since $L_{i}^{p}$ and $L^{p}$ are typically improper, the asymptotic relation can be thought as a non-uniform variant of Gromov-Hausdorff convergence.

We now assume that $Y_{i}$ and $Y$ are $p$-uniformly convex spaces with common parameter $k \in [0, 2]$ having NPC in the sense of Busemann and satisfying (B) and (C). Then $L_{i}^{p}$ and $L^{p}$ are so. Let $E_{i}$ (resp. $E$) be Cheeger type $p$-energy functional on $H^{1,p}(X_{i}, Y_{i}; m_{i})(\subset L_{i}^{p})$ (resp. $H^{1,p}(X, Y; m)(\subset L^{p})$). Here $H^{1,p}(X_{i}, Y_{i}; m_{i})$ (resp. $H^{1,p}(X, Y; m)$) is the Cheeger-type $p$-Sobolev space for $L^{p}$-maps with respect to $m_{i}$ (resp. $m$) from $X_{i}$ to $Y_{i}$ (resp. $X$ to $Y$) (see Section 6 below). Then $E_{i}$ (resp. $E$) is a lower semi-continuous functional on $L_{i}^{p}$ (resp. $L^{p}$). As a corollary of Theorem 5.11 below, we have the following:
Theorem 1.1. If $E_i$ converges to $E$ in the Mosco sense, then for any 
$\lambda > 0$ we have the following (1) and (2).

1. $E_i^\lambda$ strongly converges to $E^\lambda$.

Under a suitable condition like uniform Ricci lower bound condition
for $X_i$, $X$, we can expect that the Mosco convergence of $\{E_i\}$ to $E$
holds. At present, we are still in progress to deduce it.

As an addendum, we also show the existence of $p$-harmonic map for
Cheeger type energy functionals over $L^p$-maps into $p$-uniformly convex
space having NPC in the sense of Busemann with Dirichlet boundary
condition (see Theorem 6.20 below).

2. $p$-UNIFORMLY CONVEX SPACES

Definition 2.1 (Geodesics). Let $(Y, d)$ be a metric space. A map
$\gamma : I \to Y$ is said to be a curve if it is continuous, where $I = [a, b] \subset \mathbb{R}$
is a closed interval. The length $L(\gamma)$ of a curve $\gamma : I \to Y$ is defined to be

$$L(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \middle| a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \right\}.$$  

A curve $\gamma : I \to Y$ is said to be a minimal geodesic if $L(\gamma|_{[s,t]}) = d(\gamma_s, \gamma_t)$ holds for any $s, t \in I$, $s < t$, equivalently $d(\gamma_t, \gamma_s) + d(\gamma_s, \gamma_r)$ for any $r < s < t$. A curve $\gamma : I \to Y$ is said to be a geodesic if
for any $s, t \in I$, $s < t$ with sufficiently small $|t - s|$, $L(\gamma|_{[s,t]}) = d(\gamma_s, \gamma_t)$
holds. A metric space $(Y, d)$ is called a $R$-geodesic space for $R \in [0, \infty)$ if
any two points in $Y$ whose distance is strictly less than $R$ can be joined
by a minimal geodesic. We simply say that $(Y, d)$ is a geodesic space if
it is an $\infty$-geodesic space. Throughout this paper, for given $x, y \in Y$, denote by $\gamma_{xy} : [0, 1] \to Y$ a minimal geodesics from $x = : \gamma_{xy}(0)$ to
$y = : \gamma_{xy}(1)$ provided $(Y, d)$ is an $R$-geodesic space and $d(x, y) < R$.

For $n \in \mathbb{N}$, we denote by $\mathbb{M}^n(\kappa)$ the $n$-dimensional space form of
constant curvature $\kappa \in \mathbb{R}$. Let $R_\kappa$ be the diameter of $\mathbb{M}^n(\kappa)$, that is,
$R_\kappa := \infty$ if $\kappa \leq 0$ and $R_\kappa := \pi/\sqrt{\kappa}$ if $\kappa > 0$.

Definition 2.2 (CAT($\kappa$)-Inequality, see [2]). Let $(Y, d)$ be a metric
space and $\Delta$ a geodesic triangle in $Y$ with perimeter strictly less than
$2R_\kappa$. Let $\tilde{\Delta}$ be a comparison triangle of $\Delta$ in $\mathbb{M}^2(\kappa)$. We say that $\Delta$
satisfies CAT($\kappa$)-inequality if all $p, q \in \Delta$ and its corresponding points
$\tilde{p}, \tilde{q} \in \tilde{\Delta}$ satisfy

$$d(p, q) \leq d(\tilde{p}, \tilde{q}).$$

Definition 2.3 (CAT($\kappa$)-Space, see [2]). A metric space $(Y, d)$ is said
to be a CAT($\kappa$)-space if $(Y, d)$ is a $R_\kappa$-geodesic space and all geodesic
triangles in $Y$ with perimeter strictly less than $2R_\kappa$ satisfy CAT($\kappa$)-inequality.
\textbf{Definition 2.4} ($p$-Uniformly Convex Geodesic Space; cf. Naor-Silberman [25]). A metric space $(Y,d)$ is called \textit{$p$-uniformly convex with parameter} $k > 0$ if $(Y,d)$ is a geodesic space and for any three points $x,y,z \in Y$, any minimal geodesic $\gamma := (\gamma_t)_{t \in [0,1]}$ in $Y$ with $\gamma_0 = x$, $\gamma_1 = y$, and all $t \in [0,1]$,

\begin{equation}
(2.1) \quad d^p(z,\gamma_t) \leq (1-t)d^p(z,x) + td^p(z,y) - \frac{k}{2}t(1-t)d^p(x,y).
\end{equation}

By definition, putting $z = \gamma_t$, we see $k \in [0,2]$ and $p \in [2,\infty[$. The inequality (2.1) yields the (strict) convexity of $Y \ni x \mapsto d^p(z,x)$ for a fixed $z \in Y$. Any closed convex subset of a $p$-uniformly convex space is again a $p$-uniformly convex space with the same parameter. Any $L^p$ space over a measurable space is $p$-uniformly convex with parameter $k = \frac{8}{4p^p}(\frac{p-1}{p})^{p-1}$ provided $p \geq 2$. Every CAT(0)-space is a $p$-uniformly convex space with parameter $k = \frac{8}{4p^p}(\frac{p-1}{p})^{p-1}$ for $p > 2$ (we can take $k = 2$ if $p = 2$), because $\mathbb{R}^2$ is isometrically embedded into $L^p([0,1])$ for $p > 1$ (see [5],[25]) and any $L^p$-space is $p$-uniformly convex for $p \geq 2$. Ohta [28] proved that for $\kappa > 0$ any CAT($\kappa$)-space $Y$ with $\text{diam}(Y) < \frac{\kappa}{2}$ is a $2$-uniformly convex space with parameter $(\pi - 2\sqrt{\kappa\epsilon})\tan \sqrt{\kappa\epsilon}$ for any $\epsilon \in [0,\frac{\kappa}{2} - \text{diam}(Y)]$.

\textbf{Remark 2.5.} A Banach space $(Y,\| \cdot \|)$ is said to be \textit{uniformly convex} if

$$
\delta_Y(\epsilon) := \inf \left\{ 1 - \frac{\|x+y\|}{2} \mid x,y \in Y, \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon \right\},
$$

the modulus of convexity of $Y$, satisfies $\delta_Y(\epsilon) > 0$ for $\epsilon \in [0,2]$. For $p \geq 2$, $(Y,\| \cdot \|)$ is said to be \textit{$p$-uniformly convex} if there exists $c > 0$ such that $\delta_Y(\epsilon) \geq c\epsilon^p$ for $\epsilon \in [0,2]$. It is known that for $p \geq 2$, $\delta_{L^p}(\epsilon) = 1 - \left[1 - \left(\frac{\epsilon}{2}\right)^p\right]^\frac{1}{p} \geq \frac{1}{2p^p}\epsilon^p$ for $\epsilon \in [0,2]$. By Lemma 2.1 in [29], if a Banach space $(Y,\| \cdot \|)$ is $p$-uniformly convex for $p \geq 2$, then there exists $d = d(c,p) > 0$ such that

$$
\|(1-t)x+ty\|^p \leq (1-t)\|x\|^p + t\|y\|^p - d\{t(1-t)^p + t^p(1-t)\}\|x-y\|^p
$$

for all $x,y \in Y$ and $t \in [0,1]$. Actually, we can take $d = \frac{c}{p}\left(\frac{p-1}{p}\right)^{p-1}$ as an optimal value. Since $\frac{4}{p^p} \leq (1-t)^{p-1} + t^{p-1} \leq 1$ for all $t \in [0,1]$, $p$-uniform convexity of the Banach space implies the $p$-uniform convexity of geodesic space.

The following propositions can be proved in the same way as in [28]. So we omit its proof.

\textbf{Proposition 2.6} (cf. Lemma 2.3 in [28]). Let $(Y,d)$ be a $p$-uniformly convex space. For $x,y,z \in Y$, any minimal geodesic $\gamma := (\gamma_t)_{t \in [0,1]}$ in
with $\gamma_0 = x$, $\gamma_1 = y$, and all $t \in [0, 1]$, we have

\begin{equation}
(2.2) \quad d^p(z, \gamma_t) \leq \frac{2}{k} \cdot \frac{1}{t^{p-1} + (1-t)^{p-1}} \times ((1-t)^{p-1}d^p(z, x) + t^{p-1}d^p(z, y) - (1-t)^{p-1}t^{p-1}d^p(x, y)).
\end{equation}

**Proposition 2.7** (cf. Lemma 2.2 and Proposition 2.4 in [28]). Any two points in a $p$-uniformly convex space can be connected by a unique minimal geodesic and contractible.

**Lemma 2.8** (Projection Map to Convex Set). Let $(Y, d)$ be a complete $p$-uniformly convex space with parameter $k \in [0, 2]$. The following hold:

1. Let $F$ be a closed convex subset of $(Y, d)$. Then for each $x \in Y$, there exists a unique element $\pi_F(x) \in F$ such that $d(x, F) = d(\pi_F(x), x)$ holds. We call $\pi_F : Y \to F$ the projection map to $F$.

2. Let $F$ be as above. Then $\pi_F$ satisfies

\begin{equation}
(2.3) \quad d^p(z, \pi_F(z)) + \frac{k}{2}d^p(\pi_F(z), w) \leq d^p(z, w), \quad \text{for } z \in Y, w \in F,
\end{equation}

in particular, $(\frac{k}{2})^{1/p}d(\pi_F(z), w) \leq d(z, w)$ for $z \in Y, w \in F$.

**Definition 2.9** (Vertical Geodesics). Let $(Y, d)$ be a geodesic space. Take a geodesic $\eta$ with a point $p_0$ on it and another geodesic $\gamma$ through $p_0$. We say that $\gamma$ is vertical to $\eta$ at $p_0$ (write $\gamma \perp_{p_0} \eta$ in short) if for any $x \in \gamma$ and $y \in \eta$,

\[d(x, p_0) \leq d(x, y)\]

holds.

Let $(Y, d)$ be a complete $p$-uniformly convex space with parameter $k \in [0, 2]$. We consider the following conditions:

(A) For any closed convex set $F$ in $(Y, d)$, the projection map $\pi_F : Y \to Y$ satisfies $d(\pi_F(x), y) \leq d(x, y)$ for $x \in Y$, $y \in F$.

(B) Let $\gamma$ and $\eta$ be minimal geodesics among two points such that $\gamma$ intersects $\eta$ at $p_0$. Then $\gamma \perp_{p_0} \eta$ implies $\eta \perp_{p_0} \gamma$.

(C) Let $\sigma$ and $\eta$ be minimal geodesics among two points such that $\sigma$ intersects $\eta$ at $p_0$ and $\sigma \neq \{p_0\}$. Suppose that $\gamma$ is a minimal geodesic among two points which contains $\sigma$. Then $\sigma \perp_{p_0} \eta$ implies $\gamma \perp_{p_0} \eta$.

**Lemma 2.10.** (B) implies (A).

**Remark 2.11.** Theorem 2.13 below shows that the conditions (A), (B), (C) are satisfied for any complete CAT($\kappa$)-space with diameter strictly less than $R_\kappa/2$. For any complete $p$-uniformly convex space $(Y, d)$ with parameter $k \in [0, 2]$ which is also a weakly $L$-convex space in the sense
of Busemann for some \((L_1, L_2)\) satisfying the conditions (A), (B), (C), the space \(L^p_h(X, Y; m)\) of \(L^p\)-maps from \((X, X, m)\) into \(Y\) with a map \(h : X \to Y\) is also a complete \(p\)-uniformly convex space with the same parameter \(k \in [0, 2]\) which is also a weakly \(L\)-convex space in the sense of Busemann for the same \((L_1, L_2)\). and \(L^p_h(X, Y; m)\) satisfies the conditions (A), (B), (C).

**Lemma 2.12.** Take a geodesic triangle \(\triangle ABC\) in \(\mathbb{M}^n(\kappa)\) and set \(a := d_{\mathbb{M}^n(\kappa)}(B, C), b := d_{\mathbb{M}^n(\kappa)}(C, A), c := d_{\mathbb{M}^n(\kappa)}(A, B)\). Assume \(a, b, c < R_\kappa/2\) and \(\angle BAC \geq \pi/2\). Then for any point \(P\) on \(AB\), \(d_{\mathbb{M}^n(\kappa)}(C, A) \leq d_{\mathbb{M}^n(\kappa)}(C, P) \leq d_{\mathbb{M}^n(\kappa)}(C, B)\) holds.

**Theorem 2.13.** Let \(\kappa \in \mathbb{R}\). Any \(\text{CAT}(\kappa)\)-space \((Y, d)\) with \(\text{diam}(Y) < R_\kappa/2\) is a 2-uniformly convex space with some parameter \(k \in [0, 2]\) satisfying the conditions (A), (B), (C).

### 3. \(L\)-convex spaces of Busemann type

**Definition 3.1** (\(L\)-Convexity of Busemann Type, cf. Ohta [28]). Let \(L_1, L_2 \geq 0\). A metric space \((Y, d)\) is called an \(L\)-convex space for \((L_1, L_2)\) in the sense of Busemann if \((Y, d)\) is a geodesic space, and for any three points \(x, y, z \in Y\) and any minimal geodesics \(\gamma := \gamma_{xy} : [0, 1] \to Y\) and \(\eta := \gamma_{xz} : [0, 1] \to Y\), and for all \(t \in [0, 1]\),

\[
(3.1) \quad d(\gamma_t, \eta_t) \leq \left(1 + L_1 \min\{d(x, y) + d(x, z), 2L_2\}\right) \frac{2}{2} td(y, z)
\]

holds. A metric space \((Y, d)\) is called a weakly \(L\)-convex space for \((L_1, L_2)\) in the sense of Busemann if \((Y, d)\) is a geodesic space, and for any three points \(x, y, z \in Y\) and any minimal geodesics \(\gamma := \gamma_{xy} : [0, 1] \to Y\) and \(\eta := \gamma_{xz} : [0, 1] \to Y\), and for all \(t \in [0, 1]\),

\[
(3.2) \quad d(\gamma_t, \eta_t) \leq (1 + L_1 L_2) td(y, z)
\]

holds. A metric space \((Y, d)\) is said to be quasi-\(L\)-convex for \((L_1, L_2)\) in the sense of Busemann if \((Y, d)\) is weakly \(L\)-convex for \((L_1, L_2)\) in the sense of Busemann such that for any \(x \in Y\), any two minimal geodesics \(\gamma\) and \(\eta\) emanating from \(x\) and \(t, s \in [0, \infty[\), the limit

\[
(3.3) \quad \lim_{\epsilon \to 0} \frac{1}{\epsilon} d(\gamma_{te}, \eta_{se})
\]

always exists.

Clearly, any complete separable \(\text{CAT}(0)\)-space is an \(L\)-convex space for \((L_1, L_2)\) with \(L_1 L_2 = 0\) in the sense of Busemann. Let \((Y, d)\) be a \(\text{CAT}(1)\)-space with \(\text{diam}(Y) \leq \pi - \epsilon, \epsilon \in [0, \pi[\) in which no triangle has a perimeter greater than \(2\pi\). Then by Proposition 4.1 in [28], \((Y, d)\) is an \(L\)-convex space for

\[
(L_1, L_2) = \left(\frac{2\{(\pi - \epsilon) - \sin \epsilon\}}{(\pi - \epsilon) \sin \epsilon}, \pi - \epsilon\right).
\]
By Lemma 4.1 in [28], $L$-convexity of Busemann type implies the quasi-$L$-convexity of Busemann type.

Let $(Y, d)$ be a quasi-$L$-convex space for some $(L_1, L_2)$. For $x \in Y$, we define $\Sigma_x$ as the set of unit speed minimal geodesics emanating from $x \in Y$. Then $\gamma, \eta \in \Sigma_x$ and $t, s \in [0, \infty[$, we can define the limit $\lim_{\epsilon \to 0} d(\gamma_{t\epsilon}, \eta_{s\epsilon})/\epsilon$. Define the space of directions $\Sigma_x$ at $x \in X$ by $\Sigma_x := \Sigma_x/\sim$, where $\gamma \sim \eta$ holds if $\lim_{\epsilon \to 0} d(\gamma_{t\epsilon}, \eta_{s\epsilon})/\epsilon = 0$.

Put $K'_x := \Sigma_x \times [0, \infty[/(\sim)$, where $(\gamma, t) \sim (\eta, s)$ holds if $\lim_{\epsilon \to 0} d(\gamma_{t\epsilon}, \eta_{s\epsilon})/\epsilon = 0$.

Then $d_{K'_x}((\gamma, t), (\eta, s)) := \lim_{\epsilon \to 0} \frac{d(\gamma_{t\epsilon}, \eta_{s\epsilon})}{\epsilon}$ gives a distance function on $K'_x$. Define the space of directions $\Sigma_x$ at $x \in X$ by $\Sigma_x := \Sigma_x/\sim$.

The following proposition can be similarly proved as for Proposition 4.2 in [28].

**Proposition 3.2** (cf. Proposition 4.2 in [28]). For a $p$-uniformly convex space $(Y, d)$ having the quasi-$L$-convexity of Busemann type for some $(L_1, L_2)$ and $x \in Y$, the tangent cone $(K_x, d_{K_x})$ is a geodesic space. Moreover, it is weakly $L$-convex in the sense of Busemann with $L_1L_2 = 0$, that is, a Busemann's NPC space.

4. **Weak convergence over $p$-uniformly convex spaces**

Throughout this section, we denote by $i$ any element of a given directed set $\{i\}$. We need the following:

**Proposition 4.1.** Let $\{(H_i, d_{H_i})\}$ be a net of complete $p$-uniformly convex spaces with common parameter $k \in [0, 2]$ and all $(H_i, d_{H_i})$ have the weak $L$-convexity of Busemann type for some $(L_1, L_2)$. Let $x_i \in H_i$ be a net and $\gamma_i^t, \eta_i^t : [0, 1] \to H_i$ a net of minimal segments. Set $\alpha_0 := \varlimsup_{i} d_{H_i}(\gamma_0^i, \eta_0^i)$, $\alpha_1 := \varlimsup_{i} d_{H_i}(\gamma_1^i, \eta_1^i)$ and $A := (1 + L_1L_2)(\alpha_0 + \alpha_1)$. Then

$$\lim_{i} d_{H_i}(\pi_{\gamma_i^t}(x_i), \pi_{\eta_i^t}(x_i)) \leq A + \left(\frac{2p}{k}\right)^{1/p} \left(\sup_j d_j(x_j, y_j) + 2A\right)^{\frac{p-1}{p}} \cdot (2A)^{\frac{1}{p}}$$

or

$$\lim_{i} d_{H_i}(\pi_{\gamma_i^t}(x_i), \pi_{\eta_i^t}(x_i)) \leq A + \left(\frac{2p}{k}\right)^{1/p} \left(\sup_j d_j(x_j, y_j) + 2A\right)^{\frac{p-1}{p}} \cdot (2A)^{\frac{1}{p}}$$

holds.

**Corollary 4.2.** Let $\{(H_i, d_{H_i})\}$ be a net of complete $p$-uniformly convex spaces with common parameter $k \in [0, 2]$ and all $(H_i, d_{H_i})$ have the weak
$L$-convexity of Busemann type for some common $(L_1, L_2)$. Let $x_i \in H_i$ be a net and $\gamma^i, \eta^i : [0, 1] \to H_i$ a net of minimal segments. If
\[
\lim_i d_{H_i}(\gamma^i_0, \eta^i_0) = \lim_i d_{H_i}(\gamma^i_1, \eta^i_1) = 0
\]
holds, then
\[
\lim_i d_{H_i}(\pi_{\gamma^i}(x_i), \pi_{\eta^i}(x_i)) = 0.
\]

Let $\{(H_i, d_{H_i})\}$ be a net of metric spaces and $(H, d_H)$ a metric space. Define
\[
\mathcal{H} := \left( \bigsqcup_i H_i \right) \sqcup H \quad \text{(disjoint union).}
\]

**Definition 4.3** (Asymptotic Relation on $\mathcal{H}$). We call a topology on $\mathcal{H}$ satisfying the following (A1)–(A4) an asymptotic relation between $\{(H_i, d_{H_i})\}$ and $(H, d_H)$.

(A1) $H_i$ and $H$ are all closed in $\mathcal{H}$ and the restricted topology of $\mathcal{H}$ on each of $H_i$ and $H$ coincides with its original topology.

(A2) For any $x \in H$ there exists a net $x_i \in H_i$ converging to $x$ in $\mathcal{H}$.

(A3) If $H_i \ni x_i \to x \in H$ and $H_i \ni y_i \to y \in H$ in $\mathcal{H}$, then we have $d_{H_i}(x_i, y_i) \to d_H(x, y)$.

(A4) If $H_i \ni x_i \to x \in H$ in $\mathcal{H}$ and if $y_i \in H_i$ is a net with $d_{H_i}(x_i, y_i) \to 0$, then $y_i \to x$ in $\mathcal{H}$.

**Definition 4.4** (Asymptotic Compactness of Asymptotic Relation). Assume that $\{(H_i, d_{H_i})\}$ and $(H, d_H)$ have an asymptotic relation. We say that a net $x_i \in H_i$ is bounded if $d_{H_i}(x_i, o_i)$ is bounded for some convergent net $o_i \in H_i$. The asymptotic relation is said to be asymptotically compact if any bounded net $x_i \in H_i$ has a convergent subnet in $\mathcal{H}$ with respect to the asymptotic relation.

Hereafter, strong convergence on $\mathcal{H}$ means the convergence with respect to a given asymptotic relation over $\mathcal{H}$. Assume that an asymptotic relation between metric spaces $\{H_i\}$ and $H$ given. Consider the following condition:

(G) If $\gamma^i : [0, 1] \to H_i$ and $\gamma : [0, 1] \to H$ are minimal geodesics such that $\gamma^i_0 \to \gamma_0$ and $\gamma^i_1 \to \gamma_1$, then $\gamma^i_t \to \gamma_t$ for any $t \in [0, 1]$.

**Proposition 4.5.**

1. If (G) is satisfied and if each $H_i$ is a geodesic space, then $H$ is so.
2. If (G) is satisfied and if each $H_i$ is $p$-uniformly convex with common parameter $k \in [0, 2]$, then $H$ is so.
3. If each $H_i$ is $p$-uniformly convex with common parameter $k \in [0, 2]$ and $H$ is a geodesic space, then (G) is satisfied and $H$ is $p$-uniformly convex with parameter $k \in [0, 2]$.

In the proof of Proposition 4.5, we use Proposition 2.6.

We now define the weak convergence over $\mathcal{H}$, which generalize the notions introduced in [8, 6, 20].
Definition 4.6 (Weak Convergence on $\mathcal{H}$). Let $\{(H_i, d_{H_i})\}$ be a net of complete $p$-uniformly convex spaces with common parameter $k \in [0, 2]$ and $(H, d_H)$ a complete $p$-uniformly convex space with the same parameter $k$. We say that a net $x_i \in H_i$ weakly converges to a point $x \in H$ if for any net of geodesic segments $\gamma^i$ in $H_i$ strongly converging to a geodesic segment $\gamma$ in $H$ with $\gamma_0 = x$, $\pi_\gamma(x_i)$ strongly converges to $x$. Here the strong convergence of $\{\gamma^i\}$ to $\gamma$ means that for any $t \in [0, 1]$, $\gamma^i_t$ strongly converges to $\gamma_t$. It is easy to prove that a strong convergence implies a weak convergence and that a weakly convergent net always has a unique weak limit.

The following proposition is omitted in [20]. We shall give it for completeness.

Proposition 4.7 (Weak Topology on $\mathcal{H}$). The weak convergence over $\mathcal{H}$ of complete $p$-uniformly convex spaces with parameter $k \in [0, 2]$ induces a Hausdorff topology on it. We call it weak topology of $(H, d_H)$.

Remark 4.8. The notion of weak convergence over a fixed CAT(0)-space is proposed by Jost [8]. In [20], we extend it over $\mathcal{H}$ of CAT(0)-spaces. In Kirk-Panyanak [14], they give a different approach on the weak convergence, so-called $\Delta$-convergence, and Espínola and Fernández-León [6] proved the equivalence between the weak convergence and the $\Delta$-convergence over a fixed CAT(0)-space or CAT(1)-space whose diameter strictly less than $\pi/2$ (see Proposition 5.2 in [6]). Such an equivalence is also valid for a fixed $p$-uniformly convex space in the same way as in the proof of Proposition 5.2 in [6].

Lemma 4.9. Let $\{(H_i, d_{H_i})\}$ be a net of complete $p$-uniformly convex space with common parameter $k \in [0, 2]$ and $(H, d_H)$ a complete $p$-uniformly convex space with the same parameter $k$. Suppose that a net $x_i \in H_i$ is weakly convergent to $x \in H$ and a net $y_i \subset H_i$ is strongly convergent to $y \in H$. Then we have the following:

1. Under (A) for all $(H_i, d_{H_i})$, $d_H(x, y) \leq \lim_{i} d_{H_i}(x_i, y_i)$.
2. Under (B) for all $(H_i, d_{H_i})$, $\lim_{i} d_{H_i}(x_i, y_i) = d_H(x, y)$ if and only if $x_i \in H_i$ strongly converges to $x \in H$.

The main result of this section is the following theorem:

Theorem 4.10 (Banach-Alaoglu Type Theorem). Let $\{(H_i, d_{H_i})\}$ be a net of complete $p$-uniformly convex spaces with common parameter $k \in [0, 2]$ and $(H, d_H)$ a complete $p$-uniformly convex space with the same parameter $k$ and all $(H_i, d_{H_i})$ and $(H, d_H)$ have the weak $L$-convexity of Busemann type for some common $(L_1, L_2)$. Suppose one of the following:

1. (B) and (C) hold for $(H, d_H)$ and $(H_i, d_{H_i}) = (H, d_H)$ holds for all $i$.
2. $(H, d_H)$ is separable.
Then every bounded net \( \{x_i\} \subset \mathcal{H} \) has a weakly convergent subsequence.

Combining Theorems 2.13 and 4.10, we obtain the following:

**Corollary 4.11** (Banach-Alaoglu Type Theorem over CAT(\( \kappa \))-Spaces). Let \( \{(H_i, d_{H_i})\} \) be a net of complete CAT(\( \kappa \))-spaces with \( \text{diam}(H_i) < R_{\kappa}/2 - \epsilon \) with \( \epsilon \in [0, R_{\kappa}/2[ \), and \((H, d_H)\) a complete CAT(\( \kappa \))-space with \( \text{diam}(H) < R_{\kappa}/2 - \epsilon \) with \( \epsilon \in [0, R_{\kappa}/2[ \). Assume that \((H_i, d_{H_i}) = (H, d_H)\) for all \( i \) or \((H, d_H)\) is separable. Then every bounded net \( \{x_i\} \subset \mathcal{H} \) has a weakly convergent subsequence.

**Remark 4.12.** The assertion of Theorem 4.10 was proved by Theorem 2.1 in Jost [8] over a fixed complete CAT(0)-space without assuming the separability. In the framework of convergence over CAT(0)-spaces, Lemma 5.5 in [20] extends Theorem 2.1 in [8]. For a fixed CAT(\( \kappa \))-space \((H, d_H)\) with \( \text{diam}(H) < R_{\kappa}/2 - \epsilon \) with \( \epsilon \in [0, R_{\kappa}/2[ \), the assertion of Corollary 4.11 is essentially shown by combining Corollary 4.4 and Remark 5.3 of [6]. Corollary 4.11 also extends the result in [6].

5. **Variational Convergence over \( p \)-Uniformly Convex Spaces**

In this section we fix \( p \geq 2 \).

5.1. **Resolvents.** Throughout this subsection, we fix a complete \( p \)-uniformly convex space \((H, d_H)\) with parameter \( \kappa \in [0, 2[ \). Consider a function \( E : H \to [0, \infty] \) and set \( D(E) := \{ x \in H \mid E(x) < \infty \} \).

**Definition 5.1** (Moreau-Yosida Approximation, [9]). For \( E : H \to [0, +\infty] \) we define \( E^\lambda : H \to [0, +\infty] \) by

\[
E^\lambda(x) := \inf_{y \in H} (\lambda^{p-1}E(y) + d_H^p(y, x)), \quad x \in H, \ \lambda > 0,
\]

and call it the *Moreau-Yosida approximation* or the *Hopf-Lax formula* for \( E \).

**Theorem 5.2** (Existence of Resolvent). If \( E \) is lower semi-continuous, convex and \( E \not\equiv +\infty \), then for any \( x \in H \) there exists a unique point, say \( J^\lambda(x) \in H \), such that

\[
E^\lambda(x) = \lambda^{p-1}E(J^\lambda(x)) + d_H^p(x, J^\lambda(x)).
\]

This defines a map \( J^\lambda : H \to H \), called the resolvent of \( E \).

Note that if \( H \) is a Hilbert space and \( p = 2 \), and if \( E \) is a closed densely defined non-negative quadratic form on \( H \), then we have \( J^\lambda = (I + \lambda A)^{-1} = \frac{1}{\lambda}G_{\frac{1}{\lambda}} \). Here, \( I \) is the identity operator, \( A \) the infinitesimal generator associated with \( E \), i.e., the non-negative self-adjoint operator on \( H \) such that \( D(E) = \sqrt{A} \) and \( E(x) = \langle \sqrt{A}x, \sqrt{A}x \rangle_H \) for any \( x \in D(E) \), where \( \langle \cdot, \cdot \rangle_H \) is the Hilbert inner product on \( H \), and \( G_{\alpha} = (\alpha + A)^{-1}, \ \alpha > 0 \) is the resolvent operator associated with \( A \).
To the end of this subsection, we always assume the convexity of $E$. We have the following lemmas and theorems which are known for the case that $(H, d_H)$ is a CAT(0)-space. The proofs are omitted.

**Lemma 5.3.** For $\lambda, \mu > 0$, we have

$$\frac{1}{\mu^{p-1}} \left( \frac{1}{\lambda^{p-1}} E^\lambda \right)^\mu = \frac{1}{(\lambda + \mu)^{p-1}} E^{\lambda + \mu}.$$

**Lemma 5.4.** Let $E : H \to [0, \infty]$ be a lower semi-continuous function with $E \not= \infty$. For $x \in H$ and $s \in [0, 1]$, we have

$$J_\lambda(x) = J_{(1-s)\lambda}((1-s)x + sJ_\lambda(x)),$$

where $(1-s)x + sJ_\lambda(x)$ is the point on the geodesic joining $x$ to $J_\lambda(x)$ such that $d_H(x, (1-s)x + sJ_\lambda(x)) = sd_H(x, J_\lambda(x))$.

**Lemma 5.5.** Let $J_\lambda : H \to H$, $\lambda > 0$ be the resolvent associated with a lower semi-continuous convex function $E : H \to [0, \infty]$ with $E \not= \infty$. For $x \in \overline{D(E)}$, then

$$\lim_{\lambda \to 0} d_H(J_\lambda(x), x) = 0.$$

**Theorem 5.6.** Let $E : H \to [0, \infty]$ be a lower semi-continuous convex function with $E \not= \infty$. Take $x \in H$ and assume that $(J_{\lambda_n}(x))_{n \in \mathbb{N}}$ is bounded for some sequence $\lambda_n \to \infty$. Then $(J_\lambda(x))_{\lambda > 0}$ converges to a minimizer of $E$.

5.2. **Variational Convergence.** Throughout this subsection, we fix a net $\{(H_i, d_{H_i})\}$ of complete $p$-uniformly convex spaces with common parameter $k \in [0, 2]$ and a complete $p$-uniformly convex space $(H, d)$ with the same parameter $k \in [0, 2]$. Consider a net $\{E_i\}$ of functions $E_i : H_i \to [0, \infty]$ and a function $E : H \to [0, \infty]$.

**Definition 5.7** (Asymptotic Compactness, [24],[20]). The net $\{E_i\}$ of functions is said to be asymptotically compact if for any bounded net $x_i \in H$ with $\lim_i E_i(x_i) < +\infty$ there exists a convergent subnet of $\{x_i\}$.

**Definition 5.8** ($\Gamma$-convergence). We say that $E_i$ $\Gamma$-converges to $E$ if the following (Γ1) and (Γ2) are satisfied:

- (Γ1) For any $x \in H$ there exists a net $x_i \in H_i$ such that $x_i \to x$ and $E_i(x_i) \to E(x)$.
- (Γ2) If $H_i \ni x_i \to x \in H$ then $E(x) \leq \lim_i E_i(x_i)$.

**Definition 5.9** (Mosco convergence). We say that $E_i$ converges to $E$ in the Mosco sense if both (Γ1) in Definition 5.8 and the following ($\Gamma'2$) hold.

- (Γ2') If $H_i \ni x_i \to x \in H$ weakly, then $E(x) \leq \lim_i E_i(x_i)$.

Note that (Γ2') is a stronger condition than (Γ2), so that a Mosco convergence implies a Γ-convergence.

It is easy to prove the following proposition. The proof is omitted.
Proposition 5.10. Assume that \( \{E_i\} \) is asymptotically compact. Then the following (1)–(3) are all equivalent to each other:

1. \( E_i \) converges to \( E \) in the Mosco sense.
2. \( E_i \Gamma \)-converges to \( E \).
3. \( E_i \) compactly converges to \( E \).

In what follows, we assume that all \( H_i \) and \( H \) are \( p \)-uniformly convex spaces with a common parameter \( k \in ]0, 2] \) having the weak \( L \)-convexity of Busemann type, and all functions \( E_i : H_i \to [0, +\infty) \) and \( E : H \to [0, +\infty] \) are all lower semi-continuous, convex, and are not identically equal to \(+\infty\). Let \( J_\lambda^i \) and \( J_\lambda \) be the resolvents of \( E_i \) and \( E \) respectively.

Theorem 5.11. Suppose that all \((H_i, d_{H_i})\) satisfy the condition (B). Assume that \((H_i, d_{H_i}) = (H, d_H)\) for all \( i \) and \((H, d_H)\) satisfies (C), or \((H, d_H)\) is separable. If \( E_i \) converges to \( E \) in the Mosco sense, then for any \( \lambda > 0 \) we have the following (1) and (2).

1. \( E_\lambda^i \) strongly converges to \( E^\lambda \).
2. \( J_\lambda^i \) strongly converges to \( J_\lambda \).

Proposition 5.12. If \( E_\lambda^i \) strongly converges to \( E^\lambda \) for any \( \lambda > 0 \), then \( E_i \Gamma \)-converges to \( E \).

Propositions 5.10, 5.12 and Theorem 5.11 together imply the following

Corollary 5.13. Assume that \( \{E_i\} \) is asymptotically compact and all \((H_i, d_{H_i})\) satisfies the condition (A). Then, the following (1) and (2) are equivalent.

1. \( E_i \) compactly converges to \( E \).
2. \( E_\lambda^i \) strongly converges to \( E^\lambda \) for any \( \lambda > 0 \).

6. Cheeger type Sobolev space over \( L^p \)-maps

In this section, we prepare several notions for our main Theorem 1.1.

6.1. The space of \( L^p \)-maps. Let \((X, \mathcal{X}, m)\) be a \( \sigma \)-finite measure space. Denote by \( \mathcal{X}^m \) the completion of \( \mathcal{X} \) with respect to \( m \). In what follows, we simply say measurable (resp. \( \mathcal{X}^m \)-measurable) for \( \mathcal{X} \)-measurable (resp. \( \mathcal{X}^m \)-measurable). A numerical function \( f \) on \( X \) is a map \( f : X \to [-\infty, \infty] \). For a measurable numerical function \( f \) on \( X \), we set \( \|f\|_p := (\int_X |f(x)|^p m(dx))^{1/p} \), \( \|f\|_\infty := \inf\{\lambda > 0 \mid |f(x)| \leq \lambda \; m\text{-a.e. } x \in X\} \). For \( p \in ]0, \infty[ \), denote by \( L^p(X; m) \) the family of \( m \)-equivalence classes of \( \mathcal{X}^m \)-measurable functions finite with respect to \( \| \cdot \|_p \). Denote by \( L^0(X; m) \) the family of \( m \)-equivalence classes of \( \mathcal{X}^m \)-measurable numerical functions \( f : X \to [-\infty, \infty] \) with \( |f| < \infty \) \( m \)-a.e.
Let $(Y,d)$ be a metric space. For $p \in [0, \infty]$ and measurable maps $f, g : X \to Y$, define a pseudo distance $d_p(f,g)$ by $d_p(f,g) := \|d(f,g)\|_p$. If $p < \infty$, then
\[
d_p(f,g) := \left( \int_X d^p(f(x),g(x))m(dx) \right)^{1/p}.
\]
If $p = \infty$, then $d_\infty(f,g)$ is the $m$-essentially supremum of $x \mapsto d(f(x),g(x))$. We say that $f$ and $g$ are $m$-equivalent if
\[
f(x) = g(x) \text{ m-a.e. } x \in X
\]
and write $f \sim m g$. For a fixed measurable map $h : X \to Y$, we set
\[
L_h^p(X,Y;m) := \{f \in \mathcal{F}(Y) \mid d(f,h) \in L^p(X;m)\} / \sim_m.
\]
The map $h : X \to Y$ is called a base map. If $m(X) < \infty$ and $h : X \to Y$ is bounded, then $L_h^p(X,Y;m)$ is independent of the choice of such $h$.

**Lemma 6.1.** Let $(Y,d)$ be a metric space. For a fixed measurable map $h : X \to Y$ and $p \in [1, \infty]$, we have the following:

1. If $(Y,d)$ is complete (resp. separable), then $(L_h^p(X,Y;m),d_p)$ is so.
2. Suppose that $(Y,d)$ is a geodesic space and any two points can be connected by a unique minimal geodesic. For given $\gamma_0, \gamma_1 \in Y$ and each $t \in [0,1]$, let $\gamma_t$ be the $t$-point in a unique minimal geodesic $\gamma : [0,1] \to Y$ joining $\gamma_0$ to $\gamma_1$. Assume that for each $t \in [0,1]$, $\gamma_t$ is continuous with respect to $(\gamma_0,\gamma_1)$. Then for given $f_0, f_1 \in L_h^p(X,Y;m)$, the map $f_t : X \to Y$ defined by $f_t(x) := (f_0(x)f_1(x))$, belongs to $L_h^p(X,Y;m)$ and forms a minimal geodesic joining $f_0$ to $f_1$ in $L_h^p(X,Y;m)$. In particular, $(L_h^p(X,Y;m),d_p)$ is a geodesic space.

**Theorem 6.2.** Let $(Y,d)$ be a complete $p$-uniformly convex space having the weak $L$-convexity of Busemann type. Fix a measurable map $h : X \to Y$. Then we have the following:

1. $(L_h^p(X,Y;m),d_p)$ is a complete $p$-uniformly convex space having the weak $L$-convexity of Busemann type.
2. Let $\gamma : [0,\infty] \to L_h^p(X,Y;m)$ be a minimal geodesic. Then for each $x \in X$ and $L \in [0,\infty]$, there exists a minimal segment $\widehat{\gamma}^{(L)}(x) : [0,L] \to Y$ such that $d_p(\gamma_t, \widehat{\gamma}^{(L)}_t) = 0$ for all $t \in [0,L]$, where $\widehat{\gamma}^{(L)} : [0,L] \to L_h^p(X,Y;m)$ is a minimal segment defined by $\widehat{\gamma}^{(L)}(x)$.
3. Assume that $(Y,d)$ satisfies the quasi-$L$-convexity of Busemann type for some $(L_1, L_2)$. Then $(L_h^p(X,Y;m),d_p)$ is so.

**Lemma 6.3.** Let $(Y,d)$ be a complete $p$-uniformly convex space having the weak $L$-convexity of Busemann type such that $(Y,d)$ satisfies (A). Let $F$ be a closed convex subset of $(L_h^p(X,Y;m),d_p)$. For each $x \in X$, set $F(x) := \{f(x) \mid f \in F\}$.
(1) For each \(x \in X\), \(F(x)\) is convex in \((Y, d)\).
(2) Take an \(f \in L_h^p(X,Y;m)\). Then \(\pi_F(f) = (\pi_{\overline{F(x)}}(f(x)))_{x \in X}\) in \(L_h^p(X,Y;m)\).

**Theorem 6.4.** Let \((Y, d)\) be a complete \(p\)-uniformly convex space having the weak \(L\)-convexity of Busemann type. The following hold:

1. If \((Y, d)\) satisfies (A), then \((L_h^p(X,Y;m), d_p)\) does so.
2. If \((Y, d)\) satisfies (B), then \((L_h^p(X,Y;m), d_p)\) does so.
3. If \((Y, d)\) satisfies (C), then \((L_h^p(X,Y;m), d_p)\) does so.

**Corollary 6.5.** For \(p \geq 2\), \(L^p(X;m)\) satisfies (A), (B), (C).

**Corollary 6.6.** Let \((Y, d)\) be a complete \(\text{CAT}(\kappa)\)-space with a diameter strictly less than \(R_\kappa/2\). Then we have the following:

1. \((L_h^2(X,Y;m), d_2)\) is a 2-uniformly convex space with the same parameter \(k \in [0,2]\) having the weak \(L\)-convexity of Busemann type.
2. \((L_h^2(X,Y;m), d_2)\) satisfies (A), (B) and (C).

Hereafter, we focus only on the case that \(X\) is a locally compact separable metric space and \(h \equiv 0\), where \(o \in Y\) is a fixed base point. We write \(L_o^r(X,Y;m)\) instead of \(L_h^r(X,Y;m)\) in such a case.

**Definition 6.7 (Lipschitz Maps with Compact Support).** The support \(\text{supp}[u]\) for a measurable map \(u : X \rightarrow Y\) is defined to be the subset of \(X\) satisfying the condition that \(x \in X \setminus \text{supp}[u]\) if and only if there exists an open neighborhood \(U\) of \(x\) such that \(u = o\) on \(U\). Denote by \(C_o^{\text{Lip}}(X,Y)\) the set of Lipschitz continuous maps \(u : X \rightarrow Y\) with compact support \(\text{supp}[u]\).

**Theorem 6.8.** Suppose that \((Y, d)\) is a separable geodesic space. Let \(r \geq 1\). Then \(C_o^{\text{Lip}}(X,Y)\) is a dense subset of \((L_o^r(X,Y;m), d_r)\).

**6.2. Upper gradient and Cheeger’s Sobolev spaces.** In what follows, let \((X, d_X)\) be a metric space, and \(U \subset X\) be an open set, and \(m\) be a Borel regular measure on \(X\) such that any ball with finite positive radius is of finite positive measure. Let \((Y, d)\) be a complete geodesic space.

**Definition 6.9 (Upper Gradient).** A Borel function \(g : U \rightarrow [0, \infty]\) is called an upper gradient for a map \(u : U \rightarrow Y\) if, for any unit speed curve \(c : [0, \ell] \rightarrow U\), we have

\[
\Phi(u(c(0)), u(c(\ell))) \leq \int_0^\ell g(c(s))ds.
\]
**Definition 6.10 (Upper Pointwise Lipschitz Constant Function).** For a map \( u : U \rightarrow Y \) and a point \( z \in U \), we define

\[
\text{Lip} u(z) := \lim_{r \rightarrow 0} \sup_{d_X(z,w) = r} \frac{d(u(z), u(w))}{r},
\]

and we put \( \text{Lip} u(z) = \text{Lip} u(z) = 0 \) if \( z \) is an isolated point. Clearly \( \text{Lip} u \leq \text{Lip} u \) on \( X \). We call \( \text{Lip} u \) the upper pointwise Lipschitz constant function for \( u \).

Cheeger [4] proved that for a locally Lipschitz function \( u : U \rightarrow \mathbb{R} \), then \( \text{Lip} u \), hence \( \text{Lip} u \), is an upper gradient for \( u \). We next define the Cheeger type Sobolev spaces. Fix a point \( o \in Y \) as a base point and \( p \in [1, \infty] \). Let \( L^p_0(U, Y; m) \) be the space of \( L^p \)-maps as defined in the previous section. We write \( L^p(U, Y; m) \) instead of \( L^p_0(U, Y; m) \) for simplicity.

**Definition 6.11 (Cheeger Type Sobolev Space).** For \( u \in L^p(U, Y; m) \), we define the Cheeger type \( p \)-energy of \( u \) as

\[
E_p(u) := \inf \lim_{i \rightarrow \infty} \| g_i \|_{L^p(U; m)}^p,
\]

where the infimum is taken over all sequences \( \{(u_i, g_i)\}_{i=1}^\infty \) such that \( u_i \rightarrow u \) in \( L^p(U, Y; m) \) as \( i \rightarrow \infty \) and \( g_i \) is an upper gradient for \( u_i \) for each \( i \). The **Cheeger type \((1, p)\)-Sobolev space** is defined by

\[
H^{1,p}(U, Y; m) := \{ u \in L^p(U, Y; m) | E_p(u) < \infty \}.
\]

By definition, if \( u = v \) \( m \)-a.e. on \( U \), then \( E_p(u) = E_p(v) \).

The following is proved in [26].

**Theorem 6.12 (Lower Semi Continuity of Energy, see Theorem 2.8 in [26]).** If a sequence \( \{u_i\}_{i=1}^\infty \) converges to \( u \) in \( L^p(U, Y; m) \), then \( E_p(u) \leq \lim_{i \rightarrow \infty} E_p(u_i) \).

**Definition 6.13 (Generalized Upper Gradient).** A function \( g \in L^p(U; m) \) is called a **generalized upper gradient** for \( u \in H^{1,p}(U, Y; m) \) if there exists a sequence \( \{(u_i, g_i)\}_{i=1}^\infty \) such that \( g_i \) is an upper gradient for \( u_i \) and \( u_i \rightarrow u, g_i \rightarrow g \) in \( L^p(U, Y; m), L^p(U; m) \) respectively as \( i \rightarrow \infty \).

From the definition of the \( p \)-energy, \( E_p(u) \leq \| g \|_{L^p(U; m)}^p \) for any generalized upper gradient \( g \in L^p(U; m) \) for \( u \in H^{1,p}(U, Y; m) \).

**Definition 6.14 (Minimal Generalized Upper Gradient).** A generalized upper gradient \( g \in L^p(U; m) \) for a map \( u \in H^{1,p}(U, Y; m) \) is said to be **minimal** if it satisfies \( E_p(u) = \| g \|_{L^p(U; m)}^p \).
Hereafter, we assume that \((Y, d)\) is weakly \(L\)-convex with \(L_1 L_2 = 0\), that is, \((Y, d)\) is a Busemann's NPC space. Then the distance function \(d : Y \times Y \to [0, \infty]\) is convex. We know the following results:

**Lemma 6.15** (See, Lemma 3.1 in [28]). Suppose that \((Y, d)\) is weakly \(L\)-convex with \(L_1 L_2 = 0\). Let \(u_1, u_2 : U \to Y\) be maps. For any upper gradient \(g_1, g_2\) for \(u_1, u_2\) respectively and \(0 \leq \lambda \leq 1\). The function \(g := (1-\lambda)g_1 + \lambda g_2\) is an upper gradient for the map \(v := (1-\lambda)u_1 + \lambda u_2\). In particular, for any \(u_1, u_2 \in H^{1,p}(U,Y;m)\) with \(1 \leq p < \infty\) and for any \(0 \leq \lambda \leq 1\), we have

\[
E_p((1 - \lambda)u_1 + \lambda u_2)^{1/p} \leq (1 - \lambda)E_p(u_1)^{1/p} + \lambda E_p(u_2)^{1/p}.
\]

**Theorem 6.16** (See, Theorem 3.2 in [26]). Let \(p \in [1, \infty]\). Suppose that \((Y, d)\) is weakly \(L\)-convex with \(L_1 L_2 = 0\). Then for any \(u \in H^{1,p}(U,Y;m)\), there exists a unique minimal generalized upper gradient \(g_u\) for \(u\).

For \(p \in [1, \infty]\), we define a distance \(d_{H^{1,p}}\) on \(H^{1,p}(U,Y;m)\): for \(u, v \in H^{1,p}(U,Y;m)\),

\[
d_{H^{1,p}}(u, v) := d_p(u,v) + \|g_u - g_v\|_{L^p(U;m)},
\]

where \(g_u, g_v\) is the minimal generalized upper gradient for \(u, v \in H^{1,p}(U,Y;m)\), respectively. Let \((\overline{H}^{1,p}(U,Y;m), d_{\overline{H}^{1,p}})\) be the completion of \((H^{1,p}(U,Y;m), d_{H^{1,p}})\).

The following assertion is not declared clearly in [26]. We provide its proof for completeness.

**Theorem 6.17.** Let \(p \in [1, \infty]\). We have \(\overline{H}^{1,p}(U,Y;m) = H^{1,p}(U,Y;m)\).

**Remark 6.18.** Theorem 6.17 does not necessarily imply the \(d_{H^{1,p}}\)-completeness of \(H^{1,p}(U,Y;m)\), that is, \(d_{\overline{H}^{1,p}} = d_{H^{1,p}}\) on \(H^{1,p}(U,Y;m)\).

### 6.3. \(p\)-harmonic maps

In this subsection, we still assume that \((Y, d)\) is weakly \(L\)-convex with \(L_1 L_2 = 0\).

**Definition 6.19** (\(p\)-Harmonic Map). For \(v \in H^{1,p}(U,Y;m)\), let \(H^{1,p}_v(U,Y;m)\) be the \(d_{H^{1,p}}\)-closure of

\[
\{ u \in H^{1,p}(U,Y;m) \mid \text{supp } d(u,v) \subseteq U \}.
\]

\(v\) is said to be \(p\)-harmonic if and only if \(E_p(v) = \inf_{u \in H^{1,p}_v(U,Y;m)} E_p(u)\).

**Theorem 6.20.** Suppose \(p \geq 2\). If there exists \(C > 0\) such that for any \(f \in H^{1,p}_0(U)\),

\[
\int_U |f|^p dm \leq C \int_U |g_f|^p dm,
\]

(Poincaré Inequality)

then there exists a \(p\)-harmonic map in \(H^{1,p}_v(U,Y;m)\) for given \(v \in H^{1,p}(U,Y;m)\).
REFERENCES


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