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VARIATIONAL CONVERGENCE OVER $p$-UNIFORMLY CONVEX SPACES

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ABSTRACT. We establish a variational convergence over $p$-uniformly convex spaces for $p \geq 2$. Variational convergence for Cheeger type energy functionals over $L^p$-maps into $p$-uniformly convex space having NPC property of Busemann type and the existence of $p$-harmonic map for Cheeger type energy functionals with Dirichlet boundary condition are also presented.

1. INTRODUCTION AND MAIN RESULT

This article is a summary of a part of the paper [17] under preparation. We study a variational convergences over $p$-uniformly convex spaces having NPC property in the sense of Busemann, where a $p$-uniformly convex space is a natural generalization of $p$-uniformly convex Banach space. Typical examples of $p$-uniformly convex spaces are $L^p$-spaces with $p \geq 2$, CAT(0)-spaces, more concretely, Hadamard manifolds and trees, and so on. If the target space is a $p$-uniformly convex space having NPC property in the sense of Busemann, then the $L^p$-mapping space is also a $p$-uniformly convex geodesic spaces having NPC property in the sense of Busemann, and an energy functional defined in a suitable way becomes convex and lower semi-continuous. Thus, it is reasonable to consider that $(H, d_H)$ and $(H, d_H)$ are all $p$-uniformly convex geodesic spaces having the weak $L$-convexity of Busemann type instead of such $L^p$-mapping spaces (see Definition 3.1 below for the weak $L$-convexity), and $E_i : H_i \rightarrow [0, \infty]$ and $E : H \rightarrow [0, \infty]$ are convex lower semi-continuous functions with $E_i, E \not\equiv +\infty$. For any $\lambda \geq 0$ and $u \in H$, there exists a unique minimizer, say $J^E_{\lambda}(u) \in H$, of $v \mapsto \lambda^{p-1}E(v) + d^p_H(u, v)$. This defines a map $J^E_{\lambda} : H \rightarrow H$, called the resolvent of $E$ (see Theorem 5.2 below and [9, 22, 20] for the case $p = 2$). The minimum $E^{\lambda}(u) := \min_{v \in H}(\lambda^{p-1}E(v) + d^p_H(u, v))$ is called the Moreau-Yosida approximation or the Hopf-Lax formula. Note that if $X$ is a Hilbert space and if $E$ is a closed densely defined symmetric

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quadratic form on $X$, then we have $J^E_\lambda = (I + \lambda A)^{-1}$, where $A$ is the infinitesimal generator associated with $E$. The one-parameter family $[0, +\infty) \ni \lambda \mapsto J^E_\lambda(u)$ gives a deformation of a given map $u \in H$ to a minimizer of $E$ (or a harmonic map), $\lim_{\lambda \to +\infty} J^E_\lambda(u)$ (if any). Jost [13] studied convergence of resolvents and Moreau-Yosida approximations. Although his study is only on a fixed CAT(0)-space, we extend it for a sequence of $p$-uniformly convex geodesic spaces having the weak $L$-convexity of Busemann type with an asymptotic relation (Theorem 5.11 below). This is new even on a fixed $p$-uniformly convex geodesic spaces having the weak $L$-convexity of Busemann type.

We can apply our result in the following way. Let $(X_i, q_i) \to (X, q)$ and $(Y_i, o_i) \to (Y, o)$ ($i = 1, 2, 3, \ldots$) be two pointed Gromov-Hausdorff convergent sequences of proper metric spaces, where ‘proper’ means that any bounded subset is relatively compact, and let us give a positive Radon measure $m_i$ on $X_i$ with full support which converge to a positive Radon measure $m$ on $X$ (see the definition for the convergence of measures in [20]). We are interested in the convergence and asymptotic behavior of maps $u_i : X_i \to Y_i$ and also energy functionals $E_i$ defined on a family of maps from $X_i \to Y_i$. We set $L^p_i := L^p(X_i, Y_i, m_i)$ and $L^p := L^p(X, Y, m)$. For $u_i, u_i \in L^p_i$ (resp. $u, v \in L^p$), we set $d_{L^p_i}(u_i, v_i) := \|d_{Y_i}(u_i, v_i)\|_{L^p_i}$ (resp. $d_{L^p}(u, v) := \|d_Y(u, v)\|_{L^p}$), where $\|\cdot\|_{L^p_i}$ (resp. $\|\cdot\|_{L^p}$) is the $L^p$-norm with respect to the measure $m_i$ (resp. $m$). Consider

$$L^p := \bigcup_i L^p_i \cup L^p$$

and endowed the $L^p$-topology defined in [20] with $L^p$. The $L^p$-topology on $L^p_i$ has some nice properties involving the $L^p$-metric structure of $L^p_i$ and $L^p$, such as, if $L^p_i \ni u_i, v_i \to u, v \in L^p$ respectively in $L^p$, then $d_{L^p_i}(u_i, v_i) \to d_{L^p}(u, v)$. By their properties we present a set of axioms for a topology on $L^p$ for $(L^p_i, d_{L^p_i})$ and $(L^p, d_{L^p})$. We call such a topology satisfying the axioms the asymptotic relation between $\{L^p_i\}$ and $L^p$ (see Definition 4.3). Since $L^p_i$ and $L^p$ are typically improper, the asymptotic relation can be thought as a non-uniform variant of Gromov-Hausdorff convergence.

We now assume that $Y_i$ and $Y$ are $p$-uniformly convex spaces with common parameter $k \in [0, 2]$ having NPC in the sense of Busemann and satisfying (B) and (C). Then $L^p_i$ and $L^p$ are so. Let $E_i$ (resp. $E$) be Cheeger type $p$-energy functional on $H^{1,p}(X_i, Y_i; m_i)(\subset L^p_i)$ (resp. $H^{1,p}(X, Y; m)(\subset L^p)$). Here $H^{1,p}(X_i, Y_i; m_i)$ (resp. $H^{1,p}(X, Y; m)$) is the Cheeger-type $p$-Sobolev space for $L^p$-maps with respect to $m_i$ (resp. $m$) from $X_i$ to $Y_i$ (resp. $X$ to $Y$) (see Section 6 below). Then $E_i$ (resp. $E$) is a lower semi-continuous functional on $L^p_i$ (resp. $L^p$). As a corollary of Theorem 5.11 below, we have the following:
Theorem 1.1. If $E_i$ converges to $E$ in the Mosco sense, then for any $\lambda > 0$ we have the following (1) and (2).

1. $E^\lambda_i$ strongly converges to $E^\lambda$.

Under a suitable condition like uniform Ricci lower bound condition for $X_i$, $X$, we can expect that the Mosco convergence of $\{E_i\}$ to $E$ holds. At present, we are still in progress to deduce it.

As an addendum, we also show the existence of $p$-harmonic map for Cheeger type energy functionals over $L^p$-maps into $p$-uniformly convex space having NPC in the sense of Busemann with Dirichlet boundary condition (see Theorem 6.20 below).

2. $p$-UNIFORMLY CONVEX SPACES

Definition 2.1 (Geodesics). Let $(Y, d)$ be a metric space. A map $\gamma : I \rightarrow Y$ is said to be a curve if it is continuous, where $I = [a, b] \subset \mathbb{R}$ is a closed interval. The length $L(\gamma)$ of a curve $\gamma : I \rightarrow Y$ is defined to be

$$L(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \left| a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \right. \right\}.$$  

A curve $\gamma : I \rightarrow Y$ is said to be a minimal geodesic if $L(\gamma|_{[s,t]}) = d(\gamma_s, \gamma_t)$ holds for any $s, t \in I$, $s < t$, equivalently $d(\gamma_r, \gamma_t) = d(\gamma_r, \gamma_s) + d(\gamma_s, \gamma_t)$ for any $r < s < t$. A curve $\gamma : I \rightarrow Y$ is said to be a geodesic if for any $s, t \in I$, $s < t$ with sufficiently small $|t-s|$, $L(\gamma|_{[s,t]}) = d(\gamma_s, \gamma_t)$ holds. A metric space $(Y, d)$ is called a $R$-geodesic space for $R \in [0, \infty)$ if any two points in $Y$ whose distance is strictly less than $R$ can be joined by a minimal geodesic. We simply say that $(Y, d)$ is a geodesic space if it is an $\infty$-geodesic space. Throughout this paper, for given $x, y \in Y$, denote by $\gamma_{xy} : [0, 1] \rightarrow Y$ a minimal geodesics from $x =: \gamma_{xy}(0)$ to $y =: \gamma_{xy}(1)$ provided $(Y, d)$ is an $R$-geodesic space and $d(x, y) < R$.

For $n \in \mathbb{N}$, we denote by $M^n(\kappa)$ the $n$-dimensional space form of constant curvature $\kappa \in \mathbb{R}$. Let $R_\kappa$ be the diameter of $M^n(\kappa)$, that is, $R_\kappa := \infty$ if $\kappa \leq 0$ and $R_\kappa := \pi/\sqrt{\kappa}$ if $\kappa > 0$.

Definition 2.2 (CAT($\kappa$)-Inequality, see [2]). Let $(Y, d)$ be a metric space and $\Delta$ a geodesic triangle in $Y$ with perimeter strictly less than $2R_\kappa$. Let $\hat{\Delta}$ be a comparison triangle of $\Delta$ in $M^2(\kappa)$. We say that $\Delta$ satisfies CAT($\kappa$)-inequality if all $p, q \in \Delta$ and its corresponding points $\hat{p}, \hat{q} \in \hat{\Delta}$ satisfy

$$d(p, q) \leq d(\hat{p}, \hat{q}).$$

Definition 2.3 (CAT($\kappa$)-Space, see [2]). A metric space $(Y, d)$ is said to be a CAT($\kappa$)-space if $(Y, d)$ is a $R_\kappa$-geodesic space and all geodesic triangles in $Y$ with perimeter strictly less than $2R_\kappa$ satisfy CAT($\kappa$)-inequality.
**Definition 2.4** ($p$-Uniformly Convex Geodesic Space; cf. Naor-Silberman [25]). A metric space $(Y, d)$ is called $p$-uniformly convex with parameter $k > 0$ if $(Y, d)$ is a geodesic space and for any three points $x, y, z \in Y$, any minimal geodesic $\gamma := (\gamma_t)_{t \in [0, 1]}$ in $Y$ with $\gamma_0 = x$, $\gamma_1 = y$, and all $t \in [0, 1]$,

$$
\delta_{p}(z, \gamma_{t}) \leq (1 - t)\delta_{p}(z, x) + t\delta_{p}(z, y) - \frac{k}{2} t(1 - t)\delta_{p}(x, y).
$$

By definition, putting $z = \gamma_t$, we see $k \in [0, 2]$ and $p \in [2, \infty[$. The inequality (2.1) yields the (strict) convexity of $Y$ if $x \mapsto \delta_{p}(z, x)$ for a fixed $z \in Y$. Any closed convex subset of a $p$-uniformly convex space is again a $p$-uniformly convex space with the same parameter. Any $L^p$ space over a measurable space is $p$-uniformly convex with parameter $k = \frac{8}{4p^2}(\frac{p-1}{p})^{p-1}$ provided $p \geq 2$. Every CAT(0)-space is a $p$-uniformly convex space with parameter $k = \frac{8}{4p^2}(\frac{p-1}{p})^{p-1}$ for $p > 2$ (we can take $k = 2$ if $p = 2$), because $\mathbb{R}^2$ is isometrically embedded into $L^p([0, 1])$ for $p > 1$ (see [5],[25]) and any $L^p$-space is $p$-uniformly convex for $p \geq 2$. Ohta [28] proved that for $\kappa > 0$ any CAT($\kappa$)-space $Y$ with $\text{diam}(Y) < \frac{\kappa}{2}$ is a 2-uniformly convex space with parameter $\{\langle \pi - 2\sqrt{\kappa\epsilon} \tan \sqrt{\kappa\epsilon} \rangle \}$ for any $\epsilon \in [0, \frac{\kappa}{2} - \text{diam}(Y)]$.

**Remark 2.5.** A Banach space $(Y, \| \cdot \|)$ is said to be uniformly convex if

$$
\delta_{Y}(\epsilon) := \inf \left\{ 1 - \frac{\| x + y \|}{2} \mid x, y \in Y, \| x \| = \| y \| = 1, \| x - y \| \geq \epsilon \right\},
$$

the modulus of convexity of $Y$, satisfies $\delta_{Y}(\epsilon) > 0$ for $\epsilon \in [0, 2]$. For $p \geq 2$, $(Y, \| \cdot \|)$ is said to be $p$-uniformly convex if there exists $c > 0$ such that $\delta_{Y}(\epsilon) \geq c \epsilon^p$ for $\epsilon \in [0, 2]$. It is known that for $p \geq 2$, $\delta_{1p}(\epsilon) = 1 - \left[ 1 - \left( \frac{\epsilon}{2} \right)^p \right]^{\frac{1}{p}} \geq \frac{1}{2p^2} \epsilon^p$ for $\epsilon \in [0, 2]$. By Lemma 2.1 in [29], if a Banach space $(Y, \| \cdot \|)$ is $p$-uniformly convex for $p \geq 2$, then there exists $d = d(c, p) > 0$ such that

$$
\| (1 - t)x + ty \| \leq (1 - t)\| x \| + t\| y \| - d\{ t(1 - t)^p + t^p(1 - t) \}\| x - y \|^{p}
$$

for all $x, y \in Y$ and $t \in [0, 1]$. Actually, we can take $d = \frac{c}{p} \left( \frac{p-1}{p} \right)^{p-1}$ as an optimal value. Since $\frac{\Delta}{p^2} \leq (1 - t)^{p-1} + t^{p-1} \leq 1$ for all $t \in [0, 1]$, $p$-uniform convexity of the Banach space implies the $p$-uniform convexity of geodesic space.

The following propositions can be proved in the same way as in [28]. So we omit its proof.

**Proposition 2.6** (cf. Lemma 2.3 in [28]). Let $(Y, d)$ be a $p$-uniformly convex space. For $x, y, z \in Y$, any minimal geodesic $\gamma := (\gamma_t)_{t \in [0, 1]}$ in...
$Y$ with $\gamma_0 = x$, $\gamma_1 = y$, and all $t \in [0, 1]$, we have
\begin{equation}
(2.2)
\begin{split}
    d^p(z, \gamma_t) & \leq \frac{2}{k} \cdot \frac{1}{t^{p-1} + (1-t)^{p-1}} \\
    & \times ((1-t)^{p-1}d^p(z, x) + t^{p-1}d^p(z, y) - (1-t)^{p-1}t^{p-1}d^p(x, y)).
\end{split}
\end{equation}

**Proposition 2.7** (cf. Lemma 2.2 and Proposition 2.4 in [28]). Any two points in a $p$-uniformly convex space can be connected by a unique minimal geodesic and contractible.

**Lemma 2.8** (Projection Map to Convex Set). Let $(Y, d)$ be a complete $p$-uniformly convex space with parameter $k \in ]0, 2]$. The following hold:

1. Let $F$ be a closed convex subset of $(Y, d)$. Then for each $x \in Y$, there exists a unique element $\pi_F(x) \in F$ such that $d(x, F) = d(\pi_F(x), x)$ holds. We call $\pi_F : Y \rightarrow F$ the projection map to $F$.

2. Let $F$ be as above. Then $\pi_F$ satisfies
\begin{equation}
(2.3)
    d^p(z, \pi_F(z)) + \frac{k}{2} d^p(\pi_F(z), w) \leq d^p(z, w), \quad \text{for } z \in Y, w \in F,
\end{equation}
in particular, $\left(\frac{k}{2}\right)^{1/p} d(\pi_F(z), w) \leq d(z, w)$ for $z \in Y, w \in F$.

**Definition 2.9** (Vertical Geodesics). Let $(Y, d)$ be a geodesic space. Take a geodesic $\eta$ with a point $p_0$ on it and another geodesic $\gamma$ through $p_0$. We say that $\gamma$ is vertical to $\eta$ at $p_0$ (write $\gamma \perp_{p_0} \eta$ in short) if for any $x \in \gamma$ and $y \in \eta$,
\[
d(x, p_0) \leq d(x, y)
\]
holds.

Let $(Y, d)$ be a complete $p$-uniformly convex space with parameter $k \in ]0, 2]$. We consider the following conditions:

(A) For any closed convex set $F$ in $(Y, d)$, the projection map $\pi_F : Y \rightarrow Y$ satisfies $d(\pi_F(x), y) \leq d(x, y)$ for $x \in Y$, $y \in F$.

(B) Let $\gamma$ and $\eta$ be minimal geodesics among two points such that $\gamma$ intersects $\eta$ at $p_0$. Then $\gamma \perp_{p_0} \eta$ implies $\eta \perp_{p_0} \gamma$.

(C) Let $\sigma$ and $\eta$ be minimal geodesics among two points such that $\sigma$ intersects $\eta$ at $p_0$ and $\sigma \neq \{p_0\}$. Suppose that $\gamma$ is a minimal geodesic among two points which contains $\sigma$. Then $\sigma \perp_{p_0} \eta$ implies $\gamma \perp_{p_0} \eta$.

**Lemma 2.10.** (B) implies (A).

**Remark 2.11.** Theorem 2.13 below shows that the conditions (A), (B), (C) are satisfied for any complete CAT($\kappa$)-space with diameter strictly less than $R_{\kappa}/2$. For any complete $p$-uniformly convex space $(Y, d)$ with parameter $k \in ]0, 2]$ which is also a weakly $L$-convex space in the sense
of Busemann for some \((L_1, L_2)\) satisfying the conditions (A), (B), (C), the space \(L^p_h(X, Y; m)\) of \(L^p\)-maps from \((X, X, m)\) into \(Y\) with a map \(h : X \to Y\) is also a complete \(p\)-uniformly convex space with the same parameter \(k \in [0, 2]\) which is also a weakly \(L\)-convex space in the sense of Busemann for the same \((L_1, L_2)\). and \(L^p_h(X, Y; m)\) satisfies the conditions (A), (B), (C).

**Lemma 2.12.** Take a geodesic triangle \(\triangle ABC\) in \(\mathbb{M}^n(\kappa)\) and set \(a := d_{\mathbb{M}^n(\kappa)}(B, C), b := d_{\mathbb{M}^n(\kappa)}(C, A), c := d_{\mathbb{M}^n(\kappa)}(A, B)\). Assume \(a, b, c < R_\kappa/2\) and \(\angle BAC \geq \pi/2\). Then for any point \(P\) on \(AB\), \(d_{\mathbb{M}^n(\kappa)}(C, A) \leq d_{\mathbb{M}^n(\kappa)}(C, P) \leq d_{\mathbb{M}^n(\kappa)}(C, B)\) holds.

**Theorem 2.13.** Let \(\kappa \in \mathbb{R}\). Any CAT(\(\kappa\))-space \((Y, d)\) with \(\text{diam}(Y) < R_\kappa/2\) is a 2-uniformly convex space with some parameter \(k \in [0, 2]\) satisfying the conditions (A), (B), (C).

3. \(L\)-convex spaces of Busemann type

**Definition 3.1** (\(L\)-Convexity of Busemann Type, cf. Ohta [28]). Let \(L_1, L_2 \geq 0\). A metric space \((Y, d)\) is called an \(L\)-convex space for \((L_1, L_2)\) in the sense of Busemann if \((Y, d)\) is a geodesic space, and for any three points \(x, y, z \in Y\) and any minimal geodesics \(\gamma := \gamma_{xy} : [0, 1] \to Y\) and \(\eta := \gamma_{xz} : [0, 1] \to Y\), and for all \(t \in [0, 1]\),

\[
(3.1) \quad d(\gamma_t, \eta_t) \leq \left(1 + L_1 \min\{d(x, y) + d(x, z), 2L_2\}\right) \frac{td(y, z)}{2}
\]

holds. A metric space \((Y, d)\) is called a weakly \(L\)-convex space for \((L_1, L_2)\) in the sense of Busemann if \((Y, d)\) is a geodesic space, and for any three points \(x, y, z \in Y\) and any minimal geodesics \(\gamma := \gamma_{xy} : [0, 1] \to Y\) and \(\eta := \gamma_{xz} : [0, 1] \to Y\), and for all \(t \in [0, 1]\),

\[
(3.2) \quad d(\gamma_t, \eta_t) \leq (1 + L_1L_2)td(y, z)
\]

holds. A metric space \((Y, d)\) is said to be quasi-\(L\)-convex for \((L_1, L_2)\) in the sense of Busemann if \((Y, d)\) is weakly \(L\)-convex for \((L_1, L_2)\) in the sense of Busemann such that for any \(x \in Y\), any two minimal geodesics \(\gamma\) and \(\eta\) emanating from \(x\) and \(t, s \in [0, \infty[\), the limit

\[
(3.3) \quad \lim_{\epsilon \to 0} \frac{1}{\epsilon} d(\gamma_{te}, \eta_{se})
\]

always exists.

Clearly, any complete separable CAT(0)-space is an \(L\)-convex space for \((L_1, L_2)\) with \(L_1L_2 = 0\) in the sense of Busemann. Let \((Y, d)\) be a CAT(1)-space with \(\text{diam}(Y) \leq \pi - \epsilon, \epsilon \in ]0, \pi[\) in which no triangle has a perimeter greater than \(2\pi\). Then by Proposition 4.1 in [28], \((Y, d)\) is an \(L\)-convex space for

\[
(L_1, L_2) = \left(\frac{2\{(\pi - \epsilon) - \sin \epsilon\}}{\pi - \epsilon} \sin \epsilon, \pi - \epsilon\right).
\]
By Lemma 4.1 in [28], $L$-convexity of Busemann type implies the quasi-$L$-convexity of Busemann type.

Let $(Y, d)$ be a quasi-$L$-convex space for some $(L_1, L_2)$. For $x \in Y$, we define $\Sigma_x$ as the set of unit speed minimal geodesics emanating from $x \in Y$. Then $\gamma, \eta \in \Sigma_x$ and $t, s \in [0, \infty[$, we can define the limit $\lim_{\epsilon \to 0} d(\gamma_{t\epsilon}, \eta_{s\epsilon})/\epsilon$. Define the space of directions $\Sigma_x$ at $x \in X$ by $\Sigma_x := \Sigma_x/\sim$, where $\gamma \sim \eta$ holds if $\lim_{\epsilon \to 0} d(\gamma_{\epsilon}, \eta_{\epsilon})/\epsilon = 0$.

Put $K_x' := \Sigma_x \times [0, \infty[ / \sim$, where $((\gamma, t), (\eta, s)) \sim (\gamma, \eta)$ holds if $\lim_{\epsilon \to 0} d(\gamma_{t\epsilon}, \eta_{s\epsilon})/\epsilon = 0$. Then $d_{K_x'}((\gamma, t), (\eta, s)) := \lim_{\epsilon \to 0} \frac{d(\gamma_{t\epsilon}, \eta_{s\epsilon})}{\epsilon}$ gives a distance function on $K_x'$. Define the space of directions $\Sigma_x$ at $x \in X$ by $\Sigma_x := \Sigma_x / \sim$.

Proposition 3.2 (cf. Proposition 4.2 in [28]). For a $p$-uniformly convex space $(Y, d)$ having the quasi-$L$-convexity of Busemann type for some $(L_1, L_2)$ and $x \in Y$, the tangent cone $(K_x, d_K)$ at $x \in X$ is a geodesic space. Moreover, it is weakly $L$-convex in the sense of Busemann with $L_1 L_2 = 0$, that is, a Busemann's NPC space.

4. Weak Convergence over $p$-Uniformly Convex Spaces

Throughout this section, we denote by $i$ any element of a given directed set $\{i\}$. We need the following:

Proposition 4.1. Let $\{(H_i, d_{H_i})\}$ be a net of complete $p$-uniformly convex spaces with common parameter $k \in [0, 2]$ and all $(H_i, d_{H_i})$ have the weak $L$-convexity of Busemann type for some common $(L_1, L_2)$. Let $x_i \in H_i$ be a net and $\gamma^i, \eta^i : [0, 1] \to H_i$ a net of minimal segments. Set

$$\alpha_0 := \lim_{i} d_{H_i}(\gamma^i_0, \eta^i_0), \quad \alpha_1 := \lim_{i} d_{H_i}(\gamma^i_1, \eta^i_1)$$

and $A := (1 + L_1 L_2)(\alpha_0 + \alpha_1)$. Then

$$\lim_{i} d_{H_i}(\pi_{\gamma^i}(x_i), \pi_{\eta^i}(x_i)) \leq A + \left(\frac{2p}{k}\right)^{1/p} \left(\sup_j d_j(x_j, y_j) + 2A\right)^{\frac{p-1}{p}} \cdot (2A)^{\frac{1}{p}}$$

or

$$\lim_{i} d_{H_i}(\pi_{\gamma^i}(x_i), \pi_{\eta^i}(x_i)) \leq A + \left(\frac{2p}{k}\right)^{1/p} \left(\sup_j d_j(x_j, y_j) + 2A\right)^{\frac{p-1}{p}} \cdot (2A)^{\frac{1}{p}}$$

holds.

Corollary 4.2. Let $\{(H_i, d_{H_i})\}$ be a net of complete $p$-uniformly convex spaces with common parameter $k \in [0, 2]$ and all $(H_i, d_{H_i})$ have the weak
L-convexity of Busemann type for some common \((L_1, L_2)\). Let \(x_i \in H_i\) be a net and \(\gamma^i, \eta^i : [0, 1] \to H_i\) a net of minimal segments. If
\[
\lim_i d_{H_i}(\gamma^i_0, \eta^i_0) = \lim_i d_{H_i}(\gamma^i_1, \eta^i_1) = 0
\]
holds, then
\[
\lim_i d_{H_i}(\pi_{\gamma^i}(x_i), \pi_{\eta^i}(x_i)) = 0.
\]

Let \(\{(H_i, d_{H_i})\}\) be a net of metric spaces and \((H, d_H)\) a metric space. Define
\[
\mathcal{H} := \left( \bigcup_i H_i \right) \sqcup H \quad \text{(disjoint union)}.
\]

**Definition 4.3** (Asymptotic Relation on \(\mathcal{H}\)). We call a topology on \(\mathcal{H}\) satisfying the following (A1)–(A4) an asymptotic relation between \(\{(H_i, d_{H_i})\}\) and \((H, d_H)\).

A1) \(H_i\) and \(H\) are all closed in \(\mathcal{H}\) and the restricted topology of \(\mathcal{H}\) on each of \(H_i\) and \(H\) coincides with its original topology.

A2) For any \(x \in H\) there exists a net \(x_i \in H_i\) converging to \(x\) in \(\mathcal{H}\).

A3) If \(H_i \ni x \to x \in H\) and \(H_i \ni y \to y \in H\) in \(\mathcal{H}\), then we have
\[
d_{H_i}(x_i, y_i) \to d_H(x, y).
\]

A4) If \(H_i \ni x_i \to x \in H\) in \(\mathcal{H}\) and if \(y_i \in H_i\) is a net with \(d_{H_i}(x_i, y_i) \to 0\), then \(y_i \to x\) in \(\mathcal{H}\).

**Definition 4.4** (Asymptotic Compactness of Asymptotic Relation). Assume that \(\{(H_i, d_{H_i})\}\) and \((H, d_H)\) have an asymptotic relation. We say that a net \(x_i \in H_i\) is bounded if \(d_{H_i}(x_i, o_i)\) is bounded for some convergent net \(o_i \in H_i\). The asymptotic relation is said to be asymptotically compact if any bounded net \(x_i \in H_i\) has a convergent subnet in \(\mathcal{H}\) with respect to the asymptotic relation.

Hereafter, strong convergence on \(\mathcal{H}\) means the convergence with respect to a given asymptotic relation over \(\mathcal{H}\). Assume that an asymptotic relation between metric spaces \(\{H_i\}\) and \(H\) given. Consider the following condition:

\(\text{G}\) If \(\gamma^i : [0, 1] \to H_i\) and \(\gamma : [0, 1] \to H\) are minimal geodesics such that \(\gamma^i_0 \to \gamma_0\) and \(\gamma^i_1 \to \gamma_1\), then \(\gamma^i_t \to \gamma_t\) for any \(t \in [0, 1]\).

**Proposition 4.5.** (1) If \(\text{G}\) is satisfied and if each \(H_i\) is a geodesic space, then \(H\) is so.

(2) If \(\text{G}\) is satisfied and if each \(H_i\) is \(p\)-uniformly convex with common parameter \(k \in [0, 2]\), then \(H\) is so.

(3) If each \(H_i\) is \(p\)-uniformly convex with common parameter \(k \in [0, 2]\) and \(H\) is a geodesic space, then \(\text{G}\) is satisfied and \(H\) is \(p\)-uniformly convex with parameter \(k \in [0, 2]\).

In the proof of Proposition 4.5, we use Proposition 2.6.

We now define the weak convergence over \(\mathcal{H}\), which generalize the notions introduced in \([8, 6, 20]\).
Definition 4.6 (Weak Convergence on $\mathcal{H}$). Let $\{(H_i, d_{H_i})\}$ be a net of complete $p$-uniformly convex spaces with common parameter $k \in [0, 2]$ and $(H, d_H)$ a complete $p$-uniformly convex space with the same parameter $k$. We say that a net $x_i \in H_i$ weakly converges to a point $x \in H$ if for any net of geodesic segments $\gamma^i$ in $H_i$ strongly converging to a geodesic segment $\gamma$ in $H$ with $\gamma_0 = x$, $\pi_{\gamma}(x_i)$ strongly converges to $x$. Here the strong convergence of $\{\gamma^i\}$ to $\gamma$ means that for any $t \in [0, 1]$, $\gamma^i_t$ strongly converges to $\gamma_t$. It is easy to prove that a strong convergence implies a weak convergence and that a weakly convergent net always has a unique weak limit.

The following proposition is omitted in [20]. We shall give it for completeness.

Proposition 4.7 (Weak Topology on $\mathcal{H}$). The weak convergence over $\mathcal{H}$ of complete $p$-uniformly convex spaces with parameter $k \in [0, 2]$ induces a Hausdorff topology on it. We call it weak topology of $(H, d_H)$.

Remark 4.8. The notion of weak convergence over a fixed CAT(0)-space is proposed by Jost [8]. In [20], we extend it over $\mathcal{H}$ of CAT(0)-spaces. In Kirk-Panyanak [14], they give a different approach on the weak convergence, so-called $\Delta$-convergence, and Espínola and Fernández-León [6] proved the equivalence between the weak convergence and the $\Delta$-convergence over a fixed CAT(0)-space or CAT(1)-space whose diameter strictly less than $\pi/2$ (see Proposition 5.2 in [6]). Such an equivalence is also valid for a fixed $p$-uniformly convex space in the same way as in the proof of Proposition 5.2 in [6].

Lemma 4.9. Let $\{(H_i, d_{H_i})\}$ be a net of complete $p$-uniformly convex space with common parameter $k \in [0, 2]$ and $(H, d_H)$ a complete $p$-uniformly convex space with the same parameter $k$. Suppose that a net $x_i \in H_i$ is weakly convergent to $x \in H$ and a net $y_i \in H_i$ is strongly convergent to $y \in H$. Then we have the following:

1. Under (A) for all $(H_i, d_{H_i})$, $d_H(x, y) \leq \lim d_{H_i}(x_i, y_i)$.
2. Under (B) for all $(H_i, d_{H_i})$, $\lim d_H(x_i, y_i) = d_H(x, y)$ if and only if $x_i \in H_i$ strongly converges to $x \in H$.

The main result of this section is the following theorem:

Theorem 4.10 (Banach-Alaoglu Type Theorem). Let $\{(H_i, d_{H_i})\}$ be a net of complete $p$-uniformly convex spaces with common parameter $k \in [0, 2]$ and $(H, d_H)$ a complete $p$-uniformly convex space with the same parameter $k$ and all $(H_i, d_{H_i})$ and $(H, d_H)$ have the weak $L$-convexity of Busemann type for some common $(L_1, L_2)$. Suppose one of the following:

1. (B) and (C) hold for $(H, d_H)$ and $(H_i, d_{H_i}) = (H, d_H)$ holds for all $i$.
2. $(H, d_H)$ is separable.
Then every bounded net \( \{x_i\} \subset \mathcal{H} \) has a weakly convergent subsequence.

Combining Theorems 2.13 and 4.10, we obtain the following:

**Corollary 4.11** (Banach-Alaoglu Type Theorem over CAT(\(\kappa\))-Spaces). Let \( \{(H_i, d_{H_i})\} \) be a net of complete CAT(\(\kappa\))-spaces with \( \text{diam}(H_i) < R_{\kappa}/2 - \epsilon \) with \( \epsilon \in [0, R_{\kappa}/2] \), and \( (H, d_H) \) a complete CAT(\(\kappa\))-space with \( \text{diam}(H) < R_{\kappa}/2 - \epsilon \) with \( \epsilon \in [0, R_{\kappa}/2] \). Assume that \( (H_i, d_{H_i}) = (H, d_H) \) for all \( i \) or \( (H, d_H) \) is separable. Then every bounded net \( \{x_i\} \subset \mathcal{H} \) has a weakly convergent subsequence.

**Remark 4.12.** The assertion of Theorem 4.10 was proved by Theorem 2.1 in Jost [8] over a fixed complete CAT(0)-space without assuming the separability. In the framework of convergence over CAT(0)-spaces, Lemma 5.5 in [20] extends Theorem 2.1 in [8]. For a fixed CAT(\(\kappa\))-space \( (H, d_H) \) with \( \text{diam}(H) < R_{\kappa}/2 - \epsilon \) with \( \epsilon \in [0, R_{\kappa}/2] \), the assertion of Corollary 4.11 is essentially shown by combining Corollary 4.4 and Remark 5.3 of [6]. Corollary 4.11 also extends the result in [6].

5. VARIATIONAL CONVERGENCE OVER \( p \)-UNIFORMLY CONVEX SPACES

In this section we fix \( p \geq 2 \).

5.1. Resolvents. Throughout this subsection, we fix a complete \( p \)-uniformly convex space \( (H, d_H) \) with parameter \( k \in [0, 2] \). Consider a function \( E : H \to [0, \infty] \) and set \( D(E) := \{x \in H \mid E(x) < \infty\} \).

**Definition 5.1** (Moreau-Yosida Approximation, [9]). For \( E : H \to [0, +\infty] \) we define \( E^\lambda : H \to [0, +\infty] \) by

\[
E^\lambda(x) := \inf_{y \in H} (\lambda^{p-1}E(y) + d_H^p(y, x)), \quad x \in H, \lambda > 0,
\]

and call it the Moreau-Yosida approximation or the Hopf-Lax formula for \( E \).

**Theorem 5.2** (Existence of Resolvent). If \( E \) is lower semi-continuous, convex and \( E \not\equiv +\infty \), then for any \( x \in H \) there exists a unique point, say \( J_\lambda(x) \in H \), such that

\[
E^\lambda(x) = \lambda^{p-1}E(J_\lambda(x)) + d_H^p(x, J_\lambda(x)).
\]

This defines a map \( J_\lambda : H \to H \), called the resolvent of \( E \).

Note that if \( H \) is a Hilbert space and \( p = 2 \), and if \( E \) is a closed densely defined non-negative quadratic form on \( H \), then we have \( J_\lambda = (I + \lambda A)^{-1} = \frac{1}{\lambda} G_\lambda \). Here, \( I \) is the identity operator, \( A \) the infinitesimal generator associated with \( E \), i.e., the non-negative self-adjoint operator on \( H \) such that \( D(E) = \sqrt{A} \) and \( E(x) = (\sqrt{A}x, \sqrt{A}x)_H \) for any \( x \in D(E) \), where \( (\cdot, \cdot)_H \) is the Hilbert inner product on \( H \), and \( G_\alpha = (\alpha + A)^{-1}, \alpha > 0 \) is the resolvent operator associated with \( A \).
To the end of this subsection, we always assume the convexity of $E$. We have the following lemmas and theorems which are known for the case that $(H,d_H)$ is a CAT(0)-space. The proofs are omitted.

Lemma 5.3. For $\lambda,\mu > 0$, we have
\[
\frac{1}{\mu^{p-1}} \left( \frac{1}{\lambda^{p-1}} E^\lambda \right)^\mu = \frac{1}{(\lambda + \mu)^{p-1}} E^{\lambda+\mu}.
\]

Lemma 5.4. Let $E : H \to [0,\infty]$ be a lower semi-continuous function with $E \not\equiv \infty$. For $x \in H$ and $s \in [0,1]$, we have
\[
J_\lambda(x) = J_{(1-s)\lambda}((1-s)x + sJ_\lambda(x)),
\]
where $(1-s)x + sJ_\lambda(x)$ is the point on the geodesic joining $x$ to $J_\lambda(x)$ such that $d_H(x,(1-s)x + sJ_\lambda(x)) = sd_H(x,J_\lambda(x))$.

Lemma 5.5. Let $J_\lambda : H \to H$, $\lambda > 0$ be the resolvent associated with a lower semi-continuous convex function $E : H \to [0,\infty]$ with $E \not\equiv \infty$. For $x \in D(E)$, then
\[
\lim_{\lambda \to 0} d_H(J_\lambda(x),x) = 0.
\]

Theorem 5.6. Let $E : H \to [0,\infty]$ be a lower semi-continuous convex function with $E \not\equiv \infty$. Take $x \in H$ and assume that $(J_{\lambda_n}(x))_{n\in\mathbb{N}}$ is bounded for some sequence $\lambda_n \to \infty$. Then $(J_{\lambda}(x))_{\lambda>0}$ converges to a minimizer of $E$.

5.2. Variational Convergence. Throughout this subsection, we fix a net $\{(H_i,d_{H_i})\}$ of complete $p$-uniformly convex spaces with common parameter $k \in [0,2]$ and a complete $p$-uniformly convex space $(H,d)$ with the same parameter $k \in [0,2]$. Consider a net $\{E_i\}$ of functions $E_i : H_i \to [0,\infty]$ and a function $E : H \to [0,\infty]$.

Definition 5.7 (Asymptotic Compactness, [24],[20]). The net $\{E_i\}$ of functions is said to be asymptotically compact if for any bounded net $x_i \in H$ with $\lim_i E_i(x_i) < +\infty$ there exists a convergent subnet of $\{x_i\}$.

Definition 5.8 ($\Gamma$-convergence). We say that $E_i$ $\Gamma$-converges to $E$ if the following (T1) and (T2) are satisfied:

(T1) For any $x \in H$ there exists a net $x_i \in H_i$ such that $x_i \to x$ and $E_i(x_i) \to E(x)$.

(T2) If $H_i \ni x_i \to x \in H$ then $E(x) \leq \lim_i E_i(x_i)$.

Definition 5.9 (Mosco convergence). We say that $E_i$ converges to $E$ in the Mosco sense if both (T1) in Definition 5.8 and the following (T2') hold.

(T2') If $H_i \ni x_i \to x \in H$ weakly, then $E(x) \leq \lim_i E_i(x_i)$.

Note that (T2') is a stronger condition than (T2), so that a Mosco convergence implies a $\Gamma$-convergence.

It is easy to prove the following proposition. The proof is omitted.
Proposition 5.10. Assume that $\{E_i\}$ is asymptotically compact. Then the following (1)–(3) are all equivalent to each other.

1. $E_i$ converges to $E$ in the Mosco sense.
2. $E_i$ $\Gamma$-converges to $E$.
3. $E_i$ compactly converges to $E$.

In what follows, we assume that all $H_i$ and $H$ are $p$-uniformly convex spaces with a common parameter $k \in [0, 2]$ having the weak $L$-convexity of Busemann type, and all functions $E_i : H_i \to [0, +\infty]$ and $E : H \to [0, +\infty]$ are all lower semi-continuous, convex, and are not identically equal to $+\infty$. Let $J^I_\lambda$ and $J_\lambda$ be the resolvents of $E_i$ and $E$ respectively.

Theorem 5.11. Suppose that all $(H_i, d_{H_i})$ satisfy the condition (B). Assume that $(H_i, d_{H_i}) = (H, d_H)$ for all $i$ and $(H, d_H)$ satisfies (C), or $(H, d_H)$ is separable. If $E_i$ converges to $E$ in the Mosco sense, then for any $\lambda > 0$ we have the following (1) and (2).

1. $E^\lambda_i$ strongly converges to $E^\lambda$.
2. $J^\lambda_i$ strongly converges to $J_\lambda$.

Proposition 5.12. If $E^\lambda_i$ strongly converges to $E^\lambda$ for any $\lambda > 0$, then $E_i \Gamma$-converges to $E$.

Propositions 5.10, 5.12 and Theorem 5.11 together imply the following.

Corollary 5.13. Assume that $\{E_i\}$ is asymptotically compact and all $(H_i, d_{H_i})$ satisfies the condition (A). Then, the following (1) and (2) are equivalent.

1. $E_i$ compactly converges to $E$.
2. $E^\lambda_i$ strongly converges to $E^\lambda$ for any $\lambda > 0$.

6. Cheeger Type Sobolev Space over $L^p$-Maps

In this section, we prepare several notions for our main Theorem 1.1.

6.1. The space of $L^p$-maps. Let $(X, \mathcal{X}, m)$ be a $\sigma$-finite measure space. Denote by $\mathcal{X}^m$ the completion of $\mathcal{X}$ with respect to $m$. In what follows, we simply say measurable (resp. $\mathcal{X}^m$-measurable) for $\mathcal{X}$-measurable (resp. $\mathcal{X}^m$-measurable). A numerical function $f$ on $X$ is a map $f : X \to [-\infty, \infty]$. For a measurable numerical function $f$ on $X$, we set $\|f\| : = (\int_X |f(x)|^p m(dx))^{1/p}$, $\|f\|_{\infty} : = \inf\{\lambda > 0 \mid |f(x)| \leq \lambda \text{ m-a.e. } x \in X\}$. For $p \in [0, \infty]$, denote by $L^p(X; m)$ the family of $m$-equivalence classes of $\mathcal{X}^m$-measurable functions finite with respect to $\|\cdot\|_p$. Denote by $L^0(X; m)$ the family of $m$-equivalence classes of $\mathcal{X}^m$-measurable numerical functions $f : X \to [-\infty, \infty]$ with $|f| < \infty$ m-a.e.
Let \((Y, d)\) be a metric space. For \(p \in [0, \infty]\) and measurable maps \(f, g : X \to Y\), define a pseudo distance \(d_p(f, g)\) by \(d_p(f, g) := \|d(f, g)\|_p\). If \(p < \infty\), then
\[
d_p(f, g) := \left( \int_X d^p(f(x), g(x))m(dx) \right)^{1/p}.
\]
If \(p = \infty\), then \(d_\infty(f, g)\) is the \(m\)-essentially supremum of \(x \mapsto d(f(x), g(x))\).
We say that \(f\) and \(g\) are \(m\)-equivalent if
\[
f(x) = g(x) \text{ m-a.e. } x \in X
\]
and write \(f \sim m g\). For a fixed measurable map \(h : X \to Y\), we set
\[
L_h^p(X, Y; m) := \{f \in \mathcal{X}/\mathcal{B}(Y) \mid d(f, h) \in L^p(X; m)\}/\sim m.
\]
The map \(h : X \to Y\) is called a base map. If \(m(X) < \infty\) and \(h : X \to Y\) is bounded, then \(L_h^p(X, Y; m)\) is independent of the choice of such \(h\).

**Lemma 6.1.** Let \((Y, d)\) be a metric space. For a fixed measurable map \(h : X \to Y\) and \(p \in [1, \infty]\), we have the following:

1. If \((Y, d)\) is complete (resp. separable), then \((L_h^p(X, Y; m), d_p)\) is so.
2. Suppose that \((Y, d)\) is a geodesic space and any two points can be connected by a unique minimal geodesic. For given \(\gamma_0, \gamma_1 \in Y\) and each \(t \in [0, 1]\), let \(\gamma_t\) be the \(t\)-point in a unique minimal geodesic \(\gamma : [0, 1] \to Y\) joining \(\gamma_0\) to \(\gamma_1\). Assume that for each \(t \in [0, 1]\), \(\gamma_t\) is continuous with respect to \((\gamma_0, \gamma_1)\). Then for given \(f_0, f_1 \in L_h^p(X, Y; m)\), the map \(f_t : X \to Y\) defined by \(f_t(x) := (f_0(x)f_1(x))_t\) belongs to \(L_h^p(X, Y; m)\) and forms a minimal geodesic joining \(f_0\) to \(f_1\) in \(L_h^p(X, Y; m)\). In particular, \((L_h^p(X, Y; m), d_p)\) is a geodesic space.

**Theorem 6.2.** Let \((Y, d)\) be a complete \(p\)-uniformly convex space having the weak \(L\)-convexity of Busemann type. Fix a measurable map \(h : X \to Y\). Then we have the following:

1. \((L_h^p(X, Y; m), d_p)\) is a complete \(p\)-uniformly convex space having the weak \(L\)-convexity of Busemann type.
2. Let \(\gamma : [0, \infty[ \to L_h^p(X, Y; m)\) be a minimal geodesic. Then for each \(x \in X\) and \(L \in [0, \infty[\), there exists a minimal segment \(\tilde{\gamma}^{(L)}(x) : [0, L] \to Y\) such that \(d_p(\gamma_t, \tilde{\gamma}^{(L)}(x)) = 0\) for all \(t \in [0, L]\), where \(\tilde{\gamma}^{(L)} : [0, L] \to L_h^p(X, Y; m)\) is a minimal segment defined by \(\tilde{\gamma}^{(L)}(x)\).
3. Assume that \((Y, d)\) satisfies the quasi-\(L\)-convexity of Busemann type for some \((L_1, L_2)\). Then \((L_h^p(X, Y; m), d_p)\) is so.

**Lemma 6.3.** Let \((Y, d)\) be a complete \(p\)-uniformly convex space having the weak \(L\)-convexity of Busemann type such that \((Y, d)\) satisfies (A). Let \(F\) be a closed convex subset of \((L_h^p(X, Y; m), d_p)\). For each \(x \in X\), set \(F(x) := \{f(x) \mid f \in F\}\).
For each $x \in X$, $F(x)$ is convex in $(Y, d)$.

Take an $f \in L_h^p(X, Y;m)$. Then $\pi_F(f) = (\pi_{F(x)}(f(x)))_{x \in X}$ in $L_h^p(X, Y;m)$.

Theorem 6.4. Let $(Y, d)$ be a complete $p$-uniformly convex space having the weak $L$-convexity of Busemann type. The following hold:

1. If $(Y, d)$ satisfies (A), then $(L_h^p(X, Y;m), d_p)$ does so.
2. If $(Y, d)$ satisfies (B), then $(L_h^p(X, Y;m), d_p)$ does so.
3. If $(Y, d)$ satisfies (C), then $(L_h^p(X, Y;m), d_p)$ does so.

Corollary 6.5. For $p \geq 2$, $L^p(X;m)$ satisfies (A), (B), (C).

Corollary 6.6. Let $(Y, d)$ be a complete $CAT(\kappa)$-space with a diameter strictly less than $R_\kappa/2$. Then we have the following:

1. $(L_h^2(X, Y;m), d_2)$ is a 2-uniformly convex space with the same parameter $k \in [0, 2]$ having the weak $L$-convexity of Busemann type.
2. $(L_h^2(X, Y;m), d_2)$ satisfies (A), (B) and (C).

Hereafter, we focus only on the case that $X$ is a locally compact separable metric space and $h \equiv 0$, where $o \in Y$ is a fixed base point. We write $L^r_o(X, Y;m)$ instead of $L_h^r(X, Y;m)$ in such a case.

Definition 6.7 (Lipschitz Maps with Compact Support). The support 'supp$[u]$' for a measurable map $u : X \rightarrow Y$ is defined to be the subset of $X$ satisfying the condition that $x \in X \setminus$ supp$[u]$ if and only if there exists an open neighborhood $U$ of $x$ such that $u = o$ on $U$. Denote by $C_o^{Lip}(X, Y)$ the set of Lipschitz continuous maps $u : X \rightarrow Y$ with compact support supp$[u]$.

Theorem 6.8. Suppose that $(Y, d)$ is a separable geodesic space. Let $r \geq 1$. Then $C_o^{Lip}(X, Y)$ is a dense subset of $(L^r_o(X, Y;m), d_r)$.

6.2. Upper gradient and Cheeger’s Sobolev spaces. In what follows, let $(X, d_X)$ be a metric space, and $U \subset X$ be an open set, and $m$ be a Borel regular measure on $X$ such that any ball with finite positive radius is of finite positive measure. Let $(Y, d)$ be a complete geodesic space.

Definition 6.9 (Upper Gradient). A Borel function $g : U \rightarrow [0, \infty]$ is called an upper gradient for a map $u : U \rightarrow Y$ if, for any unit speed curve $c : [0, \ell] \rightarrow U$, we have

$$\Phi(u(c(0)), u(c(\ell))) \leq \int_0^\ell g(c(s))ds.$$
Definition 6.10 (Upper Pointwise Lipschitz Constant Function). For a map $u : U \to Y$ and a point $z \in U$, we define

$$\text{Lip } u(z) := \lim_{r \to 0} \sup_{d_X(z,w) = r} \frac{d(u(z), u(w))}{r},$$

and we put $\text{Lip } u(z) = \text{Lip } u(z) = 0$ if $z$ is an isolated point. Clearly $\text{Lip } u \leq \text{Lip } u$ on $X$. We call $\text{Lip } u$ the upper pointwise Lipschitz constant function for $u$.

Cheeger [4] proved that for a locally Lipschitz function $u : U \to \mathbb{R}$, then $\text{Lip } u$, hence $\text{Lip } u$, is an upper gradient for $u$. We next define the Cheeger type Sobolev spaces. Fix a point $o \in Y$ as a base point and $p \in [1, \infty]$. Let $L^p_o(U,Y;m)$ be the space of $L^p$-maps as defined in the previous section. We write $L^p(U,Y;m)$ instead of $L^p_o(U,Y;m)$ for simplicity.

Definition 6.11 (Cheeger Type Sobolev Space). For $u \in L^p(U,Y;m)$, we define the Cheeger type $p$-energy of $u$ as

$$E_p(u) := \inf_{\{(u_i, g_i)\}_{i=1}^\infty} \lim_{i \to \infty} \|g_i\|_{L^p(U;m)}^p,$$

where the infimum is taken over all sequences $\{(u_i, g_i)\}_{i=1}^\infty$ such that $u_i \to u$ in $L^p(U,Y;m)$ as $i \to \infty$ and $g_i$ is an upper gradient for $u_i$ for each $i$. The Cheeger type $(1, p)$-Sobolev space is defined by

$$H^{1,p}(U,Y;m) := \{u \in L^p(U,Y;m) | E_p(u) < \infty\}.$$ 

By definition, if $u = v$ $m$-a.e. on $U$, then $E_p(u) = E_p(v)$.

The following is proved in [26].

Theorem 6.12 (Lower Semi Continuity of Energy, see Theorem 2.8 in [26]). If a sequence $\{u_i\}_{i=1}^\infty$ converges to $u$ in $L^p(U,Y;m)$, then $E_p(u) \leq \lim_{i \to \infty} E_p(u_i)$.

Definition 6.13 (Generalized Upper Gradient). A function $g \in L^p(U;m)$ is called a generalized upper gradient for $u \in H^{1,p}(U,Y;m)$ if there exists a sequence $\{(u_i, g_i)\}_{i=1}^\infty$ such that $g_i$ is an upper gradient for $u_i$ and $u_i \to u$, $g_i \to g$ in $L^p(U,Y;m)$, $L^p(U;m)$ respectively as $i \to \infty$.

From the definition of the $p$-energy, $E_p(u) \leq \|g\|_{L^p(U;m)}^p$ for any generalized upper gradient $g \in L^p(U;m)$ for $u \in H^{1,p}(U,Y;m)$.

Definition 6.14 (Minimal Generalized Upper Gradient). A generalized upper gradient $g \in L^p(U;m)$ for a map $u \in H^{1,p}(U,Y;m)$ is said to be minimal if it satisfies $E_p(u) = \|g\|_{L^p(U;m)}^p$. 
Hereafter, we assume that \((Y,d)\) is weakly \(L\)-convex with \(L_1L_2 = 0\), that is, \((Y,d)\) is a Busemann’s NPC space. Then the distance function \(d: Y \times Y \to [0,\infty]\) is convex. We know the following results:

**Lemma 6.15** (See, Lemma 3.1 in [28]). Suppose that \((Y,d)\) is weakly \(L\)-convex with \(L_1L_2 = 0\). Let \(u_1, u_2 : U \to Y\) be maps. For any upper gradient \(g_1, g_2\) for \(u_1, u_2\) respectively and \(0 \leq \lambda \leq 1\). The function \(g := (1-\lambda)g_1 + \lambda g_2\) is an upper gradient for the map \(v := (1-\lambda)u_1 + \lambda u_2\). In particular, for any \(u_1, u_2 \in H^{1,p}(U,Y;m)\) with \(1 \leq p < \infty\) and for any \(0 \leq \lambda \leq 1\), we have

\[
E_p((1-\lambda)u_1 + \lambda u_2)^{1/p} \leq (1-\lambda)E_p(u_1)^{1/p} + \lambda E_p(u_2)^{1/p}.
\]

**Theorem 6.16** (See, Theorem 3.2 in [26]). Let \(p \in ]1,\infty[\). Suppose that \((Y,d)\) is weakly \(L\)-convex with \(L_1L_2 = 0\). Then for any \(u \in H^{1,p}(U,Y;m)\), there exists a unique minimal generalized upper gradient \(g_u\) for \(u\).

For \(p \in ]1,\infty[\), we define a distance \(d_{H^{1,p}}\) on \(H^{1,p}(U,Y;m)\): for \(u, v \in H^{1,p}(U,Y;m)\),

\[
d_{H^{1,p}}(u,v) := d_p(u,v) + \|g_u - g_v\|_{L^{p}(U;m)},
\]

where \(g_u, g_v\) is the minimal generalized upper gradient for \(u, v \in H^{1,p}(U,Y;m)\), respectively. Let \((\overline{H}^{1,p}(U,Y;m), d_{\overline{H}^{1,p}})\) be the completion of \((H^{1,p}(U,Y;m), d_{H^{1,p}})\).

The following assertion is not declared clearly in [26]. We provide its proof for completeness.

**Theorem 6.17.** Let \(p \in ]1,\infty[\). We have \(\overline{H}^{1,p}(U,Y;m) = H^{1,p}(U,Y;m)\).

**Remark 6.18.** Theorem 6.17 does not necessarily imply the \(d_{H^{1,p}}\)-completeness of \(H^{1,p}(U,Y;m)\), that is, \(d_{\overline{H}^{1,p}} = d_{H^{1,p}}\) on \(H^{1,p}(U,Y;m)\).

**6.3. \(p\)-harmonic maps.** In this subsection, we still assume that \((Y,d)\) is weakly \(L\)-convex with \(L_1L_2 = 0\).

**Definition 6.19** \((p\text{-Harmonic Map})\). For \(v \in H^{1,p}(U,Y;m)\), let \(H^{1,p}_0(U,Y;m)\) be the \(d_{H^{1,p}}\)-closure of

\[
\{ u \in H^{1,p}(U,Y;m) \mid \text{supp} \, d(u,v) \subseteq U \}.
\]

\(v\) is said to be \(p\)-harmonic if and only if \(E_p(v) = \inf_{u \in H^{1,p}_0(U,Y;m)} E_p(u)\).

**Theorem 6.20.** Suppose \(p \geq 2\). If there exists \(C > 0\) such that for any \(f \in H^{1,p}_0(U)\),

\[
\int_U |f|^p dm \leq C \int_U |g_f|^p dm, \quad \text{(Poincaré Inequality)}
\]

then there exists a \(p\)-harmonic map in \(H^{1,p}_0(U,Y;m)\) for given \(v \in H^{1,p}(U,Y;m)\).
REFERENCES


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