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VARIATIONAL CONVERGENCE OVER p-UNIFORMLY CONVEX SPACES

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ABSTRACT. We establish a variational convergence over p-uniformly convex spaces for $p\geq 2$. Variational convergence for Cheeger type energy functionals over L^p-maps into p-uniformly convex space having NPC property of Busemann type and the existence of p-harmonic map for Cheeger type energy functionals with Dirichlet boundary condition are also presented.

1. INTRODUCTION AND MAIN RESULT

This article is a summary of a part of the paper [17] under preparation. We study a variational convergences over p-uniformly convex spaces having NPC property in the sense of Busemann, where a p-uniformly convex space is a natural generalization of p-uniformly convex Banach space. Typical examples of p-uniformly convex spaces are L^p-spaces with $p\geq 2$, CAT(0)-spaces, more concretely, Hadamard manifolds and trees, and so on. If the target space is a p-uniformly convex space having NPC property in the sense of Busemann, then the L^p-mapping space is also a p-uniformly convex geodesic spaces having NPC property in the sense of Busemann, and an energy functional defined in a suitable way becomes convex and lower semi-continuous. Thus, it is reasonable to consider that (H_i, d_{H_i}) and (H, d_H) are all p-uniformly convex geodesic spaces having the weak L-convexity of Busemann type instead of such L^p-mapping spaces (see Definition 3.1 below for the weak L-convexity), and $E_i : H_i\to[0,\infty]$ and $E:H\to[0,\infty]$ are convex lower semi-continuous functions with $E_i, E\not\equiv+\infty$. For any $\lambda\geq 0$ and $u\in H$, there exists a unique minimizer, say $J_\lambda^E(u)\in H$, of $v\mapsto\lambda^{p-1}E(v)+d_H^p(u,v)$. This defines a map $J_\lambda^E : H\to H$, called the *resolvent of* E (see Theorem 5.2 below and [9, 22, 20] for the case $p=2$). The minimum $E^\lambda(u):=\min_{v\in H}(\lambda^{p-1}E(v)+d_H^p(u,v))$ is called the *Moreau-Yosida approximation* or the *Hopf-Lax formula.* Note that if X is a Hilbert space and if E is a closed densely defined symmetric

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quadratic form on X, then we have $J_\lambda^E = (I + \lambda A)^{-1}$, where A is the infinitesimal generator associated with E. The one-parameter family $[0, +\infty[\ni \lambda \mapsto J_\lambda^E(u)$ gives a deformation of a given map $u \in H$ to a minimizer of E (or a harmonic map), $\lim_{\lambda\to+\infty} J_\lambda^E(u)$ (if any). Jost [13] studied convergence of resolvents and Moreau-Yosida approximations. Although his study is only on a fixed CAT(0)-space, we extend it for a sequence of p-uniformly convex geodesic spaces having the weak L-convexity of Busemann type with an asymptotic relation (Theorem 5.11 below). This is new even on a fixed p-uniformly convex geodesic spaces having the weak L-convexity of Busemann type.

We can apply our result in the following way. Let $(X_i, q_i) \to (X, q)$ and $(Y_i, o_i) \to (Y, o)$ $(i = 1, 2, 3, \dots)$ be two pointed Gromov-Hausdorff convergent sequences of proper metric spaces, where '*proper*' means that any bounded subset is relatively compact, and let us give a positive Radon measure m_i on X_i with full support which converge to a positive Radon measure m on X (see the definition for the convergence of measures in [20]). We are interested in the convergence and asymptotic behavior of maps $u_i : X_i \to Y_i$ and also energy functionals E_i defined on a family of maps from $X_i \to Y_i$. We set $L_i^p := L_{o_i}^p(X_i, Y_i, m_i)$ and $L^p := L_o^p(X, Y, m)$. For $u_i, v_i \in L_i^p$ (resp. $u, v \in L^p$), we set $d_{L_i^p}(u_i, v_i) := \|d_{Y_i}(u_i, v_i)\|_{L_i^p}$ (resp. $d_{L^p}(u, v) := \|d_Y(u, v)\|_{L^p}$), where $\|\cdot\|_{L_i^p}$ (resp. $\|\cdot\|_{L^p}$) is the L^p-norm with respect to the measure m_i (resp. m). Consider

$$\mathcal{L}^p := \bigsqcup_i L_i^p \sqcup L^p$$

and endowed the L^p-*topology* defined in [20] with $\mathcal{L}^p$. The L^p-topology on $\mathcal{L}^p$ has some nice properties involving the L^p-metric structure of L_i^p and L^p, such as, if $L_i^p \ni u_i, v_i \to u, v \in L^p$ respectively in L^p, then $d_{L_i^p}(u_i, v_i) \to d_{L^p}(u, v)$. By their properties we present a set of axioms for a topology on $\mathcal{L}^p$ for $(L_i^p, d_{L_i^p})$ and (L^p, d_{L^p}). We call such a topology satisfying the axioms the *asymptotic relation between* $\{L_i^p\}$ *and* L^p (see Definition 4.3). Since L_i^p and L^p are typically improper, the asymptotic relation can be thought as a non-uniform variant of Gromov-Hausdorff convergence.

We now assume that Y_i and Y are p-uniformly convex spaces with common parameter $k \in]0, 2]$ having NPC in the sense of Busemann and satisfying **(B)** and **(C)**. Then L_i^p and L^p are so. Let E_i (resp. E) be Cheeger type p-energy functional on $H^{1,p}(X_i, Y_i; m_i) (\subset L_i^p)$ (resp. $H^{1,p}(X, Y; m) (\subset L^p)$). Here $H^{1,p}(X_i, Y_i; m_i)$ (resp. $H^{1,p}(X, Y; m)$) is the Cheeger-type p-Sobolev space for L^p-maps with respect to m_i (resp. m) from X_i to Y_i (resp. X to Y) (see Section 6 below). Then E_i (resp. E) is a lower semi-continuous convex functional on L_i^p (resp. L^p). As a corollary of Theorem 5.11 below, we have the following:

Theorem 1.1. *If E_i converges to E in the Mosco sense, then for any $\lambda > 0$ we have the following* (1) *and* (2).

(1) E_i^λ *strongly converges to* E^λ.

(2) $J_\lambda^{E_i}$ *strongly converges to* J_λ^E.

Under a suitable condition like uniform Ricci lower bound condition for X_i, X, we can expect that the Mosco convergence of $\{E_i\}$ to E holds. At present, we are still in progress to deduce it.

As an addendum, we also show the existence of p-harmonic map for Cheeger type energy functionals over L^p-maps into p-uniformly convex space having NPC in the sense of Busemann with Dirichlet boundary condition (see Theorem 6.20 below).

2. p-UNIFORMLY CONVEX SPACES

Definition 2.1 (Geodesics). Let (Y, d) be a metric space. A map $\gamma : I \to Y$ is said to be a *curve* if it is continuous, where $I = [a, b] \subset \mathbb{R}$ is a closed interval. The length $L(\gamma)$ of a curve $\gamma : I \to Y$ is defined to be

$$L(\gamma) := \sup\left\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \;\middle|\; a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\right\}.$$

A curve $\gamma : I \to Y$ is said to be a *minimal geodesic* if $L(\gamma|_{[s,t]}) = d(\gamma_s, \gamma_t)$ holds for any $s, t \in I$, $s < t$, equivalently $d(\gamma_r, \gamma_t) = d(\gamma_r, \gamma_s) + d(\gamma_s, \gamma_t)$ for any $r < s < t$. A curve $\gamma : I \to Y$ is said to be a *geodesic* if for any $s, t \in I$, $s < t$ with sufficiently small $|t - s|$, $L(\gamma|_{[s,t]}) = d(\gamma_s, \gamma_t)$ holds. A metric space (Y, d) is called a *R-geodesic space* for $R \in]0, \infty]$ if any two points in Y whose distance is strictly less than R can be joined by a minimal geodesic. We simply say that (Y, d) is a *geodesic space* if it is an ∞-geodesic space. Throughout this paper, for given $x, y \in Y$, denote by $\gamma_{xy} : [0, 1] \to Y$ a minimal geodesics from $x =: \gamma_{xy}(0)$ to $y =: \gamma_{xy}(1)$ provided (Y, d) is an R-geodesic space and $d(x, y) < R$.

For $n \in \mathbb{N}$, we denote by $\mathbb{M}^n(\kappa)$ the n-dimensional space form of constant curvature $\kappa \in \mathbb{R}$. Let R_κ be the diameter of $\mathbb{M}^n(\kappa)$, that is, $R_\kappa := \infty$ if $\kappa \le 0$ and $R_\kappa := \pi/\sqrt{\kappa}$ if $\kappa > 0$.

Definition 2.2 (CAT(κ)-Inequality, see [2]). Let (Y, d) be a metric space and $\triangle$ a geodesic triangle in Y with perimeter strictly less than $2R_\kappa$. Let $\widetilde{\triangle}$ be a comparison triangle of $\triangle$ in $\mathbb{M}^2(\kappa)$. We say that $\triangle$ satisfies *CAT(κ)-inequality* if all $p, q \in \triangle$ and its corresponding points $\tilde{p}, \tilde{q} \in \widetilde{\triangle}$ satisfy

$$d(p, q) \le d(\tilde{p}, \tilde{q}).$$

Definition 2.3 (CAT(κ)-Space, see [2]). A metric space (Y, d) is said to be a *CAT(κ)-space* if (Y, d) is a R_κ-geodesic space and all geodesic triangles in Y with perimeter strictly less than $2R_\kappa$ satisfy CAT(κ)-inequality.

Definition 2.4 (p-Uniformly Convex Geodesic Space; cf. Naor-Silberman [25]). A metric space (Y,d) is called *p-uniformly convex with parameter* $k>0$ if (Y,d) is a geodesic space and for any three points $x,y,z\in Y$, any minimal geodesic $\gamma:=(\gamma_t)_{t\in[0,1]}$ in Y with $\gamma_0=x$, $\gamma_1=y$, and all $t\in[0,1]$,

$$d^p(z,\gamma_t)\le(1-t)d^p(z,x)+td^p(z,y)-\frac{k}{2}t(1-t)d^p(x,y). \tag{2.1}$$

By definition, putting $z=\gamma_t$, we see $k\in]0,2]$ and $p\in[2,\infty[$. The inequality (2.1) yields the (strict) convexity of $Y\ni x\mapsto d^p(z,x)$ for a fixed $z\in Y$. Any closed convex subset of a p-uniformly convex space is again a p-uniformly convex space with the same parameter. Any L^p space over a measurable space is p-uniformly convex with parameter $k=\frac{8}{4^pp^2}(\frac{p-1}{p})^{p-1}$ provided $p\ge 2$. Every CAT(0)-space is a p-uniformly convex space with parameter $k=\frac{8}{4^pp^2}(\frac{p-1}{p})^{p-1}$ for $p>2$ (we can take $k=2$ if $p=2$), because $\mathbb{R}^2$ is isometrically embedded into $L^p([0,1])$ for $p>1$ (see [5],[25]) and any L^p-space is p-uniformly convex for $p\ge 2$. Ohta [28] proved that for $\kappa>0$ any CAT(κ)-space Y with $\mathrm{diam}(Y)<R_\kappa/2$ is a 2-uniformly convex space with parameter $\{(\pi-2\sqrt{\kappa}\varepsilon)\tan\sqrt{\kappa}\varepsilon\}$ for any $\varepsilon\in]0,R_\kappa/2-\mathrm{diam}(Y)]$.

Remark 2.5. A Banach space $(Y,\|\cdot\|)$ is said to be *uniformly convex* if

$$\delta_Y(\varepsilon):=\inf\left\{1-\left\|\frac{x+y}{2}\right\|\;\middle|\;x,y\in Y,\|x\|=\|y\|=1,\|x-y\|\ge\varepsilon\right\},$$

the modulus of convexity of Y, satisfies $\delta_Y(\varepsilon)>0$ for $\varepsilon\in]0,2]$. For $p\ge 2$, $(Y,\|\cdot\|)$ is said to be *p-uniformly convex* if there exists $c>0$ such that $\delta_Y(\varepsilon)\ge c\varepsilon^p$ for $\varepsilon\in]0,2]$. It is known that for $p\ge 2$, $\delta_{L^p}(\varepsilon)=1-\left[1-\left(\frac{\varepsilon}{2}\right)^p\right]^{\frac{1}{p}}\ge\frac{1}{2^pp}\varepsilon^p$ for $\varepsilon\in]0,2]$. By Lemma 2.1 in [29], if a Banach space $(Y,\|\cdot\|)$ is p-uniformly convex for $p\ge 2$, then there exists $d=d(c,p)>0$ such that

$$\|(1-t)x+ty\|^p\le(1-t)\|x\|^p+t\|y\|^p-d\{t(1-t)^p+t^p(1-t)\}\|x-y\|^p$$

for all $x,y\in Y$ and $t\in]0,1[$. Actually, we can take $d=\frac{c}{p}\left(\frac{p-1}{p}\right)^{p-1}$ as an optimal value. Since $\frac{4}{2^p}\le(1-t)^{p-1}+t^{p-1}\le 1$ for all $t\in[0,1]$, p-uniform convexity of the Banach space implies the p-uniform convexity of geodesic space.

The following propositions can be proved in the same way as in [28]. So we omit its proof.

Proposition 2.6 (cf. Lemma 2.3 in [28]). *Let (Y,d) be a p-uniformly convex space. For $x,y,z\in Y$, any minimal geodesic $\gamma:=(\gamma_t)_{t\in[0,1]}$ in*

Y with $\gamma_0 = x$, $\gamma_1 = y$, and all $t \in [0,1]$, we have

$$(2.2) \qquad d^p(z,\gamma_t) \le \frac{2}{k}\cdot\frac{1}{t^{p-1}+(1-t)^{p-1}} \times \left((1-t)^{p-1}d^p(z,x)+t^{p-1}d^p(z,y)-(1-t)^{p-1}t^{p-1}d^p(x,y)\right).$$

Proposition 2.7 (cf. Lemma 2.2 and Proposition 2.4 in [28]). *Any two points in a p-uniformly convex space can be connected by a unique minimal geodesic and contractible.*

Lemma 2.8 (Projection Map to Convex Set). *Let (Y,d) be a complete p-uniformly convex space with parameter $k \in]0,2]$. The the following hold:*

(1) *Let F be a closed convex subset of (Y,d). Then for each $x \in Y$, there exists a unique element $\pi_F(x) \in F$ such that $d(x,F) = d(\pi_F(x),x)$ holds. We call $\pi_F : Y \to F$ the projection map to F.*
(2) *Let F be as above. Then π_F satisfies*

$$(2.3) \qquad d^p(z,\pi_F(z)) + \frac{k}{2}d^p(\pi_F(z),w) \le d^p(z,w), \quad \text{for } z \in Y, w \in F,$$

in particular, $\left(\frac{k}{2}\right)^{1/p} d(\pi_F(z),w) \le d(z,w)$ for $z \in Y, w \in F$.

Definition 2.9 (Vertical Geodesics). Let (Y,d) be a geodesic space. Take a geodesic η with a point p_0 on it and another geodesic γ through p_0. We say that γ *is vertical to η at p_0* (write $\gamma \perp_{p_0} \eta$ in short) if for any $x \in \gamma$ and $y \in \eta$,

$$d(x,p_0) \le d(x,y)$$

holds.

Let (Y,d) be a complete p-uniformly convex space with parameter $k \in]0,2]$. We consider the following conditions:

(A) For any closed convex set F in (Y,d), the projection map $\pi_F : Y \to Y$ satisfies $d(\pi_F(x),y) \le d(x,y)$ for $x \in Y$, $y \in F$.
(B) Let γ and η be minimal geodesics among two points such that γ intersects η at p_0. Then $\gamma \perp_{p_0} \eta$ imlies $\eta \perp_{p_0} \gamma$.
(C) Let σ and η be minimal geodesics among two points such that σ intersects η at p_0 and $\sigma \ne \{p_0\}$. Suppose that γ is a minimal geodesic among two points which contains σ. Then $\sigma \perp_{p_0} \eta$ implies $\gamma \perp_{p_0} \eta$.

Lemma 2.10. **(B)** *implies* **(A)**.

Remark 2.11. Theorem 2.13 below shows that the conditions **(A)**, **(B)**, **(C)** are satisfied for any complete CAT(κ)-space with diameter strictly less than $R_\kappa/2$. For any complete p-uniformly convex space (Y,d) with parameter $k \in]0,2]$ which is also a weakly L-convex space in the sense of Busemann for some (L_1, L_2) satisfying the conditions **(A)**, **(B)**, **(C)**, the space $L^p_h(X, Y; m)$ of L^p-maps from $(X, \mathfrak{X}, m)$ into Y with a map $h : X \to Y$ is also a complete p-uniformly convex space with the same parameter $k \in]0, 2]$ which is also a weakly L-convex space in the sense of Busemann for the same (L_1, L_2). and $L^p_h(X, Y; m)$ satisfies the conditions **(A)**, **(B)**, **(C)**.

Lemma 2.12. Take a geodesic triangle $\triangle ABC$ in $\mathbb{M}^n(\kappa)$ and set $a := d_{\mathbb{M}^n(\kappa)}(B, C)$, $b := d_{\mathbb{M}^n(\kappa)}(C, A)$, $c := d_{\mathbb{M}^n(\kappa)}(A, B)$. Assume $a, b, c < R_\kappa/2$ and $\angle BAC \geq \pi/2$. Then for any point P on AB, $d_{\mathbb{M}^n(\kappa)}(C, A) \leq d_{\mathbb{M}^n(\kappa)}(C, P) \leq d_{\mathbb{M}^n(\kappa)}(C, B)$ holds.

Theorem 2.13. Let $\kappa \in \mathbb{R}$. Any CAT(κ)-space (Y, d) with $\operatorname{diam}(Y) < R_\kappa/2$ is a 2-uniformly convex space with some parameter $k \in]0, 2]$ satisfying the conditions **(A)**, **(B)**, **(C)**.

3. L-convex spaces of Busemann type

Definition 3.1 (L-Convexity of Busemann Type, cf. Ohta [28]). Let $L_1, L_2 \geq 0$. A metric space (Y, d) is called an *L-convex space for (L_1, L_2) in the sense of Busemann* if (Y, d) is a geodesic space, and for any three points $x, y, z \in Y$ and any minimal geodesics $\gamma := \gamma_{xy} : [0, 1] \to Y$ and $\eta := \gamma_{xz} : [0, 1] \to Y$, and for all $t \in [0, 1]$,

$$d(\gamma_t, \eta_t) \leq \left(1 + L_1 \frac{\min\{d(x, y) + d(x, z), 2L_2\}}{2}\right) t d(y, z) \tag{3.1}$$

holds. A metric space (Y, d) is called a *weakly L-convex space for (L_1, L_2) in the sense of Busemann* if (Y, d) is a geodesic space, and for any three points $x, y, z \in Y$ and any minimal geodesics $\gamma := \gamma_{xy} : [0, 1] \to Y$ and $\eta := \gamma_{xz} : [0, 1] \to Y$, and for all $t \in [0, 1]$,

$$d(\gamma_t, \eta_t) \leq (1 + L_1 L_2) t d(y, z) \tag{3.2}$$

holds. A metric space (Y, d) is said to be *quasi-L-convex for (L_1, L_2) in the sense of Busemann* if (Y, d) is weakly L-convex for (L_1, L_2) in the sense of Busemann such that for any $x \in Y$, any two minimal geodesics γ and η emanating from x and $t, s \in [0, \infty[$, the limit

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(\gamma_{t\varepsilon}, \eta_{s\varepsilon}) \tag{3.3}$$

always exists.

Clearly, any complete separable CAT(0)-space is an L-convex space for (L_1, L_2) with $L_1 L_2 = 0$ in the sense of Busemann. Let (Y, d) be a CAT(1)-space with $\operatorname{diam}(Y) \leq \pi - \varepsilon$, $\varepsilon \in]0, \pi[$ in which no triangle has a perimeter greater than 2π. Then by Proposition 4.1 in [28], (Y, d) is an L-convex space for

$$(L_1, L_2) = \left(\frac{2\{(\pi - \varepsilon) - \sin \varepsilon\}}{(\pi - \varepsilon) \sin \varepsilon}, \pi - \varepsilon\right).$$

By Lemma 4.1 in [28], L-convexity of Busemann type implies the quasi-L-convexity of Busemann type.

Let (Y,d) be a quasi-L-convex space for some (L_1,L_2). For $x\in Y$, we define Σ'_x as the set of unit speed minimal geodesics emanating from $x\in Y$. Then $\gamma,\eta\in\Sigma'_x$ and $t,s\in[0,\infty[$, we can define the limit $\lim_{\varepsilon\to 0} d(\gamma_{t\varepsilon},\eta_{s\varepsilon})/\varepsilon$. Define the *space of directions* Σ_x at $x\in X$ by $\Sigma_x:=\Sigma'_x/\sim$, where $\gamma\sim\eta$ holds if $\lim_{\varepsilon\to 0} d(\gamma_\varepsilon,\eta_\varepsilon)/\varepsilon=0$. Put

$$K'_x := \Sigma_x \times [0,\infty[/\sim,$$

where $(\gamma,t)\sim(\eta,s)$ holds if $\lim_{\varepsilon\to 0} d(\gamma_{t\varepsilon},\eta_{s\varepsilon})/\varepsilon=0$. Then

$$d_{K'_x}((\gamma,t),(\eta,s)) := \lim_{\varepsilon\to 0}\frac{d(\gamma_{t\varepsilon},\eta_{s\varepsilon})}{\varepsilon}$$

gives a distance function on K'_x. Define the *tangent cone* (K_x,d_{K_x}) at $x\in X$ as the completion of $(K'_x,d_{K'_x})$.

The following proposition can be similarly proved as for Proposition 4.2 in [28].

Proposition 3.2 (cf. Proposition 4.2 in [28]). *For a p-uniformly convex space (Y,d) having the quasi-L-convexity of Busemann type for some (L_1,L_2) and $x\in Y$, the tangent cone (K_x,d_{K_x}) is a geodesic space. Moreover, it is weakly L-convex in the sense of Busemann with $L_1L_2=0$, that is, a Busemann's NPC space.*

4. Weak convergence over p-uniformly convex spaces

Throughout this section, we denote by i any element of a given directed set $\{i\}$. We need the following:

Proposition 4.1. *Let $\{(H_i,d_{H_i})\}$ be a net of complete p-uniformly convex spaces with common parameter $k\in]0,2]$ and all (H_i,d_{H_i}) have the weak L-convexity of Busemann type for some common (L_1,L_2). Let $x_i\in H_i$ be a net and $\gamma^i,\eta^i:[0,1]\to H_i$ a net of minimal segments. Set*

$$\alpha_0 := \overline{\lim_i}\, d_{H_i}(\gamma^i_0,\eta^i_0),\quad \alpha_1 := \overline{\lim_i}\, d_{H_i}(\gamma^i_1,\eta^i_1)$$

and $A:=(1+L_1L_2)(\alpha_0+\alpha_1)$. Then

$$\overline{\lim_i}\, d_{H_i}(\pi_{\gamma^i}(x_i),\pi_{\eta^i}(x_i)) \le A + \left(\frac{2p}{k}\right)^{1/p}\left(\sup_j d_j(x_j,y_j)+2A\right)^{\frac{p-1}{p}}\cdot(2A)^{\frac{1}{p}}$$

or

$$\overline{\lim_i}\, d_{H_i}(\pi_{\gamma^i}(x_i),\pi_{\eta^i}(x_i)) \le A + \left(\frac{2p}{k}\right)^{1/p}\left(\sup_j d_j(x_j,y_j)+2A\right)^{\frac{p-1}{p}}\cdot(2A)^{\frac{1}{p}}$$

holds.

Corollary 4.2. *Let $\{(H_i,d_{H_i})\}$ be a net of complete p-uniformly convex spaces with common parameter $k\in]0,2]$ and all (H_i,d_{H_i}) have the weak*

L-convexity of Busemann type for some common (L_1, L_2). *Let* $x_i \in H_i$ *be a net and* $\gamma^i, \eta^i : [0,1] \to H_i$ *a net of minimal segments. If*

$$\lim_i d_{H_i}(\gamma^i_0, \eta^i_0) = \lim_i d_{H_i}(\gamma^i_1, \eta^i_1) = 0$$

holds, then

$$\lim_i d_{H_i}(\pi_{\gamma^i}(x_i), \pi_{\eta^i}(x_i)) = 0.$$

Let $\{(H_i, d_{H_i})\}$ be a net of metric spaces and (H, d_H) a metric space. Define

$$\mathcal{H} := \left(\bigsqcup_i H_i \right) \sqcup H \quad \text{(disjoint union)}.$$

Definition 4.3 (Asymptotic Relation on $\mathcal{H}$). We call a topology on $\mathcal{H}$ satisfying the following (A1)–(A4) an *asymptotic relation between* $\{(H_i, d_{H_i})\}$ *and* (H, d_H).

(A1) H_i and H are all closed in $\mathcal{H}$ and the restricted topology of $\mathcal{H}$ on each of H_i and H coincides with its original topology.

(A2) For any $x \in H$ there exists a net $x_i \in H_i$ converging to x in $\mathcal{H}$.

(A3) If $H_i \ni x_i \to x \in H$ and $H_i \ni y_i \to y \in H$ in $\mathcal{H}$, then we have $d_{H_i}(x_i, y_i) \to d_H(x, y)$.

(A4) If $H_i \ni x_i \to x \in H$ in $\mathcal{H}$ and if $y_i \in H_i$ is a net with $d_{H_i}(x_i, y_i) \to 0$, then $y_i \to x$ in $\mathcal{H}$.

Definition 4.4 (Asymptotic Compactness of Asymptotic Relation). Assume that $\{(H_i, d_{H_i})\}$ and (H, d_H) have an asymptotic relation. We say that a net $x_i \in H_i$ is *bounded* if $d_{H_i}(x_i, o_i)$ is bounded for some convergent net $o_i \in H_i$. The asymptotic relation is said to be *asymptotically compact* if any bounded net $x_i \in H_i$ has a convergent subnet in $\mathcal{H}$ with respect to the asymptotic relation.

Hereafter, strong convergence on $\mathcal{H}$ means the convergence with respect to a given asymptotic relation over $\mathcal{H}$. Assume that an asymptotic relation between metric spaces $\{H_i\}$ and H given. Consider the following condition:

(G) If $\gamma^i : [0,1] \to H_i$ and $\gamma : [0,1] \to H$ are minimal geodesics such that $\gamma^i_0 \to \gamma_0$ and $\gamma^i_1 \to \gamma_1$, then $\gamma^i_t \to \gamma_t$ for any $t \in [0,1]$.

Proposition 4.5. (1) *If* **(G)** *is satisfied and if each* H_i *is a geodesic space, then* H *is so.*

(2) *If* **(G)** *is satisfied and if each* H_i *is p-uniformly convex with common parameter* $k \in]0,2]$, *then* H *is so.*

(3) *If each* H_i *is p-uniformly convex with common parameter* $k \in]0,2]$ *and* H *is a geodesic space, then* **(G)** *is satisfied and* H *is p-uniformly convex with parameter* $k \in]0,2]$.

In the proof of Proposition 4.5, we use Proposition 2.6.

We now define the *weak convergence* over $\mathcal{H}$, which generalize the notions introduced in [8, 6, 20].

Definition 4.6 (Weak Convergence on $\mathcal{H}$). Let $\{(H_i, d_{H_i})\}$ be a net of complete p-uniformly convex spaces with common parameter $k \in]0,2]$ and (H, d_H) a complete p-uniformly convex space with the same parameter k. We say that a net $x_i \in H_i$ *weakly converges to a point* $x \in H$ if for any net of geodesic segments γ^i in H_i strongly converging to a geodesic segment γ in H with $\gamma_0 = x$, $\pi_{\gamma^i}(x_i)$ strongly converges to x. Here the strong convergence of $\{\gamma^i\}$ to γ means that for any $t \in [0,1]$, γ^i_t strongly converges to γ_t. It is easy to prove that a strong convergence implies a weak convergence and that a weakly convergent net always has a unique weak limit.

The following proposition is omitted in [20]. We shall give it for completeness.

Proposition 4.7 (Weak Topology on $\mathcal{H}$). *The weak convergence over $\mathcal{H}$ of complete p-uniformly convex spaces with parameter $k \in]0,2]$ induces a Hausdorff topology on it. We call it* weak topology *of (H, d_H).*

Remark 4.8. The notion of weak convergence over a fixed CAT(0)-space is proposed by Jost [8]. In [20], we extend it over $\mathcal{H}$ of CAT(0)-spaces. In Kirk-Panyanak [14], they give a different approach on the weak convergence, so-called Δ-*convergence*, and Espínola and Fernández-León [6] proved the equivalence between the weak convergence and the Δ-convergence over a fixed CAT(0)-space or CAT(1)-space whose diameter strictly less than $\pi/2$ (see Proposition 5.2 in [6]). Such an equivalence is also valid for a fixed p-uniformly convex space in the same way as in the proof of Proposition 5.2 in [6].

Lemma 4.9. *Let $\{(H_i, d_{H_i})\}$ be a net of complete p-uniformly convex space with common parameter $k \in]0,2]$ and (H, d_H) a complete p-uniformly convex space with the same parameter k. Suppose that a net $x_i \in H_i$ is weakly convergent to $x \in H$ and a net $y_i \subset H_i$ is strongly convergent to $y \in H$. Then we have the following:*

1. *Under* **(A)** *for all (H_i, d_{H_i}), $d_H(x, y) \leq \underline{\lim}_i\, d_{H_i}(x_i, y_i)$.*
2. *Under* **(B)** *for all (H_i, d_{H_i}), $\lim_i d_{H_i}(x_i, y_i) = d_H(x, y)$ if and only if $x_i \in H_i$ strongly converges to $x \in H$.*

The main result of this section is the following theorem:

Theorem 4.10 (Banach-Alaoglu Type Theorem). *Let $\{(H_i, d_{H_i})\}$ be a net of complete p-uniformly convex spaces with common parameter $k \in]0,2]$ and (H, d_H) a complete p-uniformly convex space with the same parameter k and all (H_i, d_{H_i}) and (H, d_H) have the weak L-convexity of Busemann type for some common (L_1, L_2). Suppose one of the following:*

1. **(B)** *and* **(C)** *hold for (H, d_H) and $(H_i, d_{H_i}) = (H, d_H)$ holds for all i.*
2. *(H, d_H) is separable.*

Then every bounded net $\{x_i\} \subset \mathcal{H}$ has a weakly convergent subsequence.

Combining Theorems 2.13 and 4.10, we obtain the following:

Corollary 4.11 (Banach-Alaoglu Type Theorem over CAT(κ)-Spaces). *Let $\{(H_i, d_{H_i})\}$ be a net of complete CAT(κ)-spaces with* $\mathrm{diam}(H_i) < R_\kappa/2 - \varepsilon$ *with $\varepsilon \in]0, R_\kappa/2[$, and (H, d_H) a complete CAT(κ)-space with* $\mathrm{diam}(H) < R_\kappa/2 - \varepsilon$ *with $\varepsilon \in]0, R_\kappa/2[$. Assume that $(H_i, d_{H_i}) = (H, d_H)$ for all i or (H, d_H) is separable. Then every bounded net $\{x_i\} \subset \mathcal{H}$ has a weakly convergent subsequence.*

Remark 4.12. The assertion of Theorem 4.10 was proved by Theorem 2.1 in Jost [8] over a fixed complete CAT(0)-space without assuming the separability. In the framework of convergence over CAT(0)-spaces, Lemma 5.5 in [20] extends Theorem 2.1 in [8]. For a fixed CAT(κ)-space (H, d_H) with $\mathrm{diam}(H) < R_\kappa/2 - \varepsilon$ with $\varepsilon \in]0, R_\kappa/2[$, the assertion of Corollary 4.11 is essentially shown by combining Corollary 4.4 and Remark 5.3 of [6]. Corollary 4.11 also extends the result in [6].

5. Variational convergence over p-uniformly convex spaces

In this section we fix $p \geq 2$.

5.1. Resolvents.

Throughout this subsection, we fix a complete p-uniformly convex space (H, d_H) with parameter $k \in]0, 2]$. Consider a function $E : H \to [0, \infty]$ and set $D(E) := \{x \in H \mid E(x) < \infty\}$.

Definition 5.1 (Moreau-Yosida Approximation, [9]). For $E : H \to [0, +\infty]$ we define $E^\lambda : H \to [0, +\infty]$ by

$$E^\lambda(x) := \inf_{y \in H} (\lambda^{p-1} E(y) + d_H^p(y, x)), \qquad x \in H, \ \lambda > 0,$$

and call it the *Moreau-Yosida approximation* or the *Hopf-Lax formula* for E.

Theorem 5.2 (Existence of Resolvent). *If E is lower semi-continuous, convex and $E \not\equiv +\infty$, then for any $x \in H$ there exists a unique point, say $J_\lambda(x) \in H$, such that*

$$E^\lambda(x) = \lambda^{p-1} E(J_\lambda(x)) + d_H^p(x, J_\lambda(x)).$$

This defines a map $J_\lambda : H \to H$, called the resolvent of E.

Note that if H is a Hilbert space and $p = 2$, and if E is a closed densely defined non-negative quadratic form on H, then we have $J_\lambda = (I + \lambda A)^{-1} = \frac{1}{\lambda} G_{\frac{1}{\lambda}}$. Here, I is the identity operator, A the infinitesimal generator associated with E, i.e., the non-negative self-adjoint operator on H such that $D(E) = \sqrt{A}$ and $E(x) = (\sqrt{A}x, \sqrt{A}x)_H$ for any $x \in D(E)$, where $(\cdot, \cdot)_H$ is the Hilbert inner product on H, and $G_\alpha = (\alpha + A)^{-1}$, $\alpha > 0$ is the resolvent operator associated with A.

To the end of this subsection, we always assume the convexity of E. We have the following lemmas and theorems which are known for the case that (H, d_H) is a CAT(0)-space. The proofs are omitted.

Lemma 5.3. *For $\lambda, \mu > 0$, we have*

$$\frac{1}{\mu^{p-1}}\left(\frac{1}{\lambda^{p-1}}E^{\lambda}\right)^{\mu} = \frac{1}{(\lambda+\mu)^{p-1}}E^{\lambda+\mu}.$$

Lemma 5.4. *Let $E : H \to [0, \infty]$ be a lower semi-continuous function with $E \not\equiv \infty$. For $x \in H$ and $s \in [0, 1]$, we have*

$$J_\lambda(x) = J_{(1-s)\lambda}\left((1-s)x + sJ_\lambda(x)\right),$$

where $(1-s)x + sJ_\lambda(x)$ is the point on the geodesic joining x to $J_\lambda(x)$ such that $d_H(x, (1-s)x + sJ_\lambda(x)) = sd_H(x, J_\lambda(x))$.

Lemma 5.5. *Let $J_\lambda : H \to H$, $\lambda > 0$ be the resolvent associated with a lower semi-continuous convex function $E : H \to [0, \infty]$ with $E \not\equiv \infty$. For $x \in \overline{D(E)}$, then*

$$\lim_{\lambda \to 0} d_H(J_\lambda(x), x) = 0.$$

Theorem 5.6. *Let $E : H \to [0, \infty]$ be a lower semi-continuous convex function with $E \not\equiv \infty$. Take $x \in H$ and assume that $(J_{\lambda_n}(x))_{n \in \mathbb{N}}$ is bounded for some sequence $\lambda_n \to \infty$. Then $(J_\lambda(x))_{\lambda > 0}$ converges to a minimizer of E.*

5.2. **Variational Convergence.** Throughout this subsection, we fix a net $\{(H_i, d_{H_i})\}$ of complete p-uniformly convex spaces with common parameter $k \in]0, 2]$ and a complete p-uniformly convex space (H, d) with the same parameter $k \in]0, 2]$. Consider a net $\{E_i\}$ of functions $E_i : H_i \to [0, \infty]$ and a function $E : H \to [0, \infty]$.

Definition 5.7 (Asymptotic Compactness, [24],[20]). The net $\{E_i\}$ of functions is said to be *asymptotically compact* if for any bounded net $x_i \in H$ with $\overline{\lim}_i E_i(x_i) < +\infty$ there exists a convergent subnet of $\{x_i\}$.

Definition 5.8 (Γ-convergence). We say that E_i *Γ-converges to E* if the following (Γ1) and (Γ2) are satisfied:

(Γ1) For any $x \in H$ there exists a net $x_i \in H_i$ such that $x_i \to x$ and $E_i(x_i) \to E(x)$.

(Γ2) If $H_i \ni x_i \to x \in H$ then $E(x) \leq \underline{\lim}_i E_i(x_i)$.

Definition 5.9 (Mosco convergence). We say that E_i *converges to E in the Mosco sense* if both (Γ1) in Definition 5.8 and the following (Γ2') hold.

(Γ2') If $H_i \ni x_i \to x \in H$ weakly, then $E(x) \leq \underline{\lim}_i E_i(x_i)$.

Note that (Γ2') is a stronger condition than (Γ2), so that a Mosco convergence implies a Γ-convergence.

It is easy to prove the following proposition. The proof is omitted.

Proposition 5.10. *Assume that $\{E_i\}$ is asymptotically compact. Then the following* (1)–(3) *are all equivalent to each other.*

(1) *E_i converges to E in the Mosco sense.*
(2) *E_i Γ-converges to E.*
(3) *E_i compactly converges to E.*

In what follows, we assume that all H_i and H are p-uniformly convex spaces with a common parameter $k \in]0,2]$ having the weak L-convexity of Busemann type, and all functions $E_i : H_i \to [0,+\infty]$ and $E : H \to [0,+\infty]$ are all *lower semi-continuous, convex, and are not identically equal to* $+\infty$. Let J^i_λ and J_λ be the resolvents of E_i and E respectively.

Theorem 5.11. *Suppose that all (H_i, d_{H_i}) satisfy the condition* **(B)**. *Assume that $(H_i, d_{H_i}) = (H, d_H)$ for all i and (H, d_H) satisfies* **(C)**, *or (H, d_H) is separable. If E_i converges to E in the Mosco sense, then for any $\lambda > 0$ we have the following* (1) *and* (2).

(1) *E^λ_i strongly converges to E^λ.*
(2) *J^i_λ strongly converges to J_λ.*

Proposition 5.12. *If E^λ_i strongly converges to E^λ for any $\lambda > 0$, then E_i Γ-converges to E.*

Propositions 5.10, 5.12 and Theorem 5.11 together imply the following

Corollary 5.13. *Assume that $\{E_i\}$ is asymptotically compact and all (H_i, d_{H_i}) satisfies the condition* **(A)**. *Then, the following* (1) *and* (2) *are equivalent.*

(1) *E_i compactly converges to E.*
(2) *E^λ_i strongly converges to E^λ for any $\lambda > 0$.*

6. Cheeger type Sobolev space over L^p-maps

In this section, we prepare several notions for our main Theorem 1.1.

6.1. **The space of L^p-maps.** Let $(X, \mathfrak{X}, m)$ be a σ-finite measure space. Denote by $\mathfrak{X}^m$ the completion of $\mathfrak{X}$ with respect to m. In what follows, we simply say *measurable* (resp. *$\mathfrak{X}^m$-measurable*) for $\mathfrak{X}$-measurable (resp. $\mathfrak{X}^m$-measurable). A numerical function f on X is a map $f : X \to [-\infty, \infty]$. For a measurable numerical function f on X, we set $\|f\|_p := \left(\int_X |f(x)|^p m(dx)\right)^{1/p}$, $\|f\|_\infty := \inf\{\lambda > 0 \mid |f(x)| \leq \lambda \ m\text{-a.e. } x \in X\}$. For $p \in]0, \infty]$, denote by $L^p(X; m)$ the family of m-equivalence classes of $\mathfrak{X}^m$-measurable functions finite with respect to $\|\cdot\|_p$. Denote by $L^0(X; m)$ the family of m-equivalence classes of $\mathfrak{X}^m$-measurable numerical functions $f : X \to [-\infty, \infty]$ with $|f| < \infty$ m-a.e.

Let (Y,d) be a metric space. For $p \in]0,\infty]$ and measurable maps $f,g : X \to Y$, define a pseudo distance $d_p(f,g)$ by $d_p(f,g) := \|d(f,g)\|_p$. If $p < \infty$, then

$$d_p(f,g) := \left(\int_X d^p(f(x),g(x))m(dx)\right)^{1/p}.$$

If $p = \infty$, then $d_\infty(f,g)$ is the m-essentially supremum of $x \mapsto d(f(x),g(x))$. We say that f and g are m-equivalent if

$$f(x) = g(x) \ \ m\text{-a.e. } x \in X$$

and write $f \overset{m}{\sim} g$. For a fixed measurable map $h : X \to Y$, we set

$$L^p_h(X,Y;m) := \{f \in \mathfrak{X}/\mathfrak{B}(Y) \mid d(f,h) \in L^p(X;m)\}/\overset{m}{\sim}.$$

The map $h : X \to Y$ is called a *base map*. If $m(X) < \infty$ and $h : X \to Y$ is bounded, then $L^p_h(X,Y;m)$ is independent of the choice of such h.

Lemma 6.1. *Let (Y,d) be a metric space. For a fixed measurable map $h : X \to Y$ and $p \in [1,\infty]$, we have the following:*

(1) *If (Y,d) is complete (resp. separable), then $(L^p_h(X,Y;m),d_p)$ is so.*
(2) *Suppose that (Y,d) is a geodesic space and any two points can be connected by a unique minimal geodesic. For given $\gamma_0,\gamma_1 \in Y$ and each $t \in [0,1]$, let γ_t be the t-point in a unique minimal geodesic $\gamma : [0,1] \to Y$ joining γ_0 to γ_1. Assume that for each $t \in [0,1]$, γ_t is continuous with respect to (γ_0,γ_1). Then for given $f_0,f_1 \in L^p_h(X,Y;m)$, the map $f_t : X \to Y$ defined by $f_t(x) := (f_0(x)f_1(x))_t$ belongs to $L^p_h(X,Y;m)$ and forms a minimal geodesic joining f_0 to f_1 in $L^p_h(X,Y;m)$. In particular, $(L^p_h(X,Y;m),d_p)$ is a geodesic space.*

Theorem 6.2. *Let (Y,d) be a complete p-uniformly convex space having the weak L-convexity of Busemann type. Fix a measurable map $h : X \to Y$. Then we have the following:*

(1) *$(L^p_h(X,Y;m),d_p)$ is a complete p-uniformly convex space having the weak L-convexity of Busemann type.*
(2) *Let $\gamma : [0,\infty[\to L^p_h(X,Y;m)$ be a minimal geodesic. Then for each $x \in X$ and $L \in [0,\infty[$, there exists a minimal segment $\tilde{\gamma}^{(L)}(x) : [0,L] \to Y$ such that $d_p(\gamma_t,\tilde{\gamma}^{(L)}_t) = 0$ for all $t \in [0,L]$, where $\tilde{\gamma}^{(L)} : [0,L] \to L^p_h(X,Y;m)$ is a minimal segment defined by $\tilde{\gamma}^{(L)}(x)$.*
(3) *Assume that (Y,d) satisfies the quasi-L-convexity of Busemann type for some (L_1,L_2). Then $(L^p_h(X,Y;m),d_p)$ is so.*

Lemma 6.3. *Let (Y,d) be a complete p-uniformly convex space having the weak L-convexity of Busemann type such that (Y,d) satisfies* **(A)**. *Let F be a closed convex subset of $(L^p_h(X,Y;m),d_p)$. For each $x \in X$, set $F(x) := \{f(x) \mid f \in F\}$.*

(1) *For each* $x \in X$, $F(x)$ *is convex in* (Y, d).
(2) *Take an* $f \in L^p_h(X, Y; m)$. *Then* $\pi_F(f) = (\pi_{\overline{F(x)}}(f(x)))_{x\in X}$ *in* $L^p_h(X, Y; m)$.

Theorem 6.4. *Let* (Y, d) *be a complete p-uniformly convex space having the weak L-convexity of Busemann type. The following hold:*

(1) *If* (Y, d) *satisfies* **(A)**, *then* $(L^p_h(X, Y; m), d_p)$ *does so.*
(2) *If* (Y, d) *satisfies* **(B)**, *then* $(L^p_h(X, Y; m), d_p)$ *does so.*
(3) *If* (Y, d) *satisfies* **(C)**, *then* $(L^p_h(X, Y; m), d_p)$ *does so.*

Corollary 6.5. *For* $p \geq 2$, $L^p(X; m)$ *satisfies* **(A)**, **(B)**, **(C)**.

Corollary 6.6. *Let* (Y, d) *be a complete CAT*(κ)*-space with a diameter strictly less than* $R_\kappa/2$. *Then we have the following:*

(1) $(L^2_h(X, Y; m), d_2)$ *is a 2-uniformly convex space with the same parameter* $k \in]0, 2]$ *having the weak L-convexity of Busemann type.*
(2) $(L^2_h(X, Y; m), d_2)$ *satisfies* **(A)**, **(B)** *and* **(C)**.

Hereafter, we focus only on the case that X is a locally compact separable metric space and $h \equiv o$, where $o \in Y$ is a fixed base point. We write $L^r_o(X, Y; m)$ instead of $L^r_h(X, Y; m)$ in such a case.

Definition 6.7 (Lipschitz Maps with Compact Support). The *support* 'supp$[u]$ 'for a measurable map $u : X \to Y$ is defined to be the subset of X satisfying the condition that $x \in X \setminus \mathrm{supp}[u]$ if and only if there exists an open neighborhood U of x such that $u = o$ on U. Denote by $C^{\mathrm{Lip}}_o(X, Y)$ the set of Lipschitz continuous maps $u : X \to Y$ with compact support supp$[u]$.

Theorem 6.8. *Suppose that* (Y, d) *is a separable geodesic space. Let* $r \geq 1$. *Then* $C^{\mathrm{Lip}}_o(X, Y)$ *is a dense subset of* $(L^r_o(X, Y; m), d_r)$.

6.2. **Upper gradient and Cheeger's Sobolev spaces.** In what follows, let (X, d_X) be a metric space, and $U \subset X$ be an open set, and m be a Borel regular measure on X such that any ball with finite positive radius is of finite positive measure. Let (Y, d) be a complete geodesic space.

Definition 6.9 (Upper Gradient). A Borel function $g : U \to [0, \infty]$ is called an *upper gradient* for a map $u : U \to Y$ if, for any unit speed curve $c : [0, \ell] \to U$, we have

$$\Phi(u(c(0)), u(c(\ell))) \leq \int_0^\ell g(c(s))ds.$$

Definition 6.10 (Upper Pointwise Lipschitz Constant Function). For a map $u : U \to Y$ and a point $z \in U$, we define

$$Lip\,u(z) := \varliminf_{r \to 0} \sup_{d_X(z,w)=r} \frac{d(u(z),u(w))}{r},$$

$$\mathrm{Lip}\,u(z) := \lim_{r \to 0} \sup_{0<d_X(z,w)<r} \frac{d(u(z),u(w))}{d_X(z,w)}$$

and we put $Lip\,u(z) = \mathrm{Lip}\,u(z) = 0$ if z is an isolated point. Clearly $Lip\,u \leq \mathrm{Lip}\,u$ on X. We call $\mathrm{Lip}\,u$ the *upper pointwise Lipschitz constant function* for u.

Cheeger [4] proved that for a locally Lipschitz function $u : U \to \mathbb{R}$, then $Lip\,u$, hence $\mathrm{Lip}\,u$, is an upper gradient for u. We next define the Cheeger type Sobolev spaces. Fix a point $o \in Y$ as a base point and $p \in [1, \infty[$. Let $L^p_o(U,Y;m)$ be the space of L^p-maps as defined in the previous section. We write $L^p(U,Y;m)$ instead of $L^p_o(U,Y;m)$ for simplicity.

Definition 6.11 (Cheeger Type Sobolev Space). For $u \in L^p(U,Y;m)$, we define the Cheeger type p-energy of u as

$$E_p(u) := \inf_{\{(u_i,g_i)\}_{i=1}^{\infty}} \varliminf_{i \to \infty} \|g_i\|^p_{L^p(U;m)},$$

where the infimum is taken over all sequences $\{(u_i, g_i)\}_{i=1}^{\infty}$ such that $u_i \to u$ in $L^p(U,Y;m)$ as $i \to \infty$ and g_i is an upper gradient for u_i for each i. The *Cheeger type* $(1,p)$*-Sobolev space* is defined by

$$H^{1,p}(U,Y;m) := \{u \in L^p(U,Y;m) \mid E_p(u) < \infty\}.$$

By definition, if $u = v$ m-a.e. on U, then $E_p(u) = E_p(v)$.

The following is proved in [26].

Theorem 6.12 (Lower Semi Continuity of Energy, see Theorem 2.8 in [26]). *If a sequence* $\{u_i\}_{i=1}^{\infty}$ *converges to* u *in* $L^p(U,Y;m)$*, then* $E_p(u) \leq \varliminf_{i\to\infty} E_p(u_i)$.

Definition 6.13 (Generalized Upper Gradient). A function $g \in L^p(U;m)$ is called a *generalized upper gradient* for $u \in H^{1,p}(U,Y;m)$ if there exists a sequence $\{(u_i, g_i)\}_{i=1}^{\infty}$ such that g_i is an upper gradient for u_i and $u_i \to u$, $g_i \to g$ in $L^p(U,Y;m)$, $L^p(U;m)$ respectively as $i \to \infty$.

From the definition of the p-energy, $E_p(u) \leq \|g\|^p_{L^p(U;m)}$ for any generalized upper gradient $g \in L^p(U;m)$ for $u \in H^{1,p}(U,Y;m)$.

Definition 6.14 (Minimal Generalized Upper Gradient). A generalized upper gradient $g \in L^p(U;m)$ for a map $u \in H^{1,p}(U,Y;m)$ is said to be *minimal* if it satisfies $E_p(u) = \|g\|^p_{L^p(U;m)}$.

Hereafter, we assume that (Y,d) is weakly L-convex with $L_1L_2=0$, that is, (Y,d) is a Busemann's NPC space. Then the distance function $d: Y\times Y \to [0,\infty[$ is convex. We know the following results:

Lemma 6.15 (See, Lemma 3.1 in [28]). *Suppose that (Y,d) is weakly L-convex with $L_1L_2=0$. Let $u_1,\ u_2 : U\to Y$ be maps. For any upper gradient g_1, g_2 for u_1, u_2 respectively and $0\le\lambda\le 1$. The function $g:=(1-\lambda)g_1+\lambda g_2$ is an upper gradient for the map $v:=(1-\lambda)u_1+\lambda u_2$. In particular, for any $u_1,u_2\in H^{1,p}(U,Y;m)$ with $1\le p<\infty$ and for any $0\le\lambda\le1$, we have*

$$E_p((1-\lambda)u_1+\lambda u_2)^{1/p}\le(1-\lambda)E_p(u_1)^{1/p}+\lambda E_p(u_2)^{1/p}.$$

Theorem 6.16 (See, Theorem 3.2 in [26]). *Let $p\in]1,\infty[$. Suppose that (Y,d) is weakly L-convex with $L_1L_2=0$. Then for any $u\in H^{1,p}(U,Y;m)$, there exists a unique minimal generalized upper gradient g_u for u.*

For $p\in]1,\infty[$, we define a distance $d_{H^{1,p}}$ on $H^{1,p}(U,Y;m)$: for $u,v\in H^{1,p}(U,Y;m)$,

$$d_{H^{1,p}}(u,v):=d_p(u,v)+\|g_u-g_v\|_{L^p(U;m)}, \tag{6.1}$$

where g_u, g_v is the minimal generalized upper gradient for $u,v\in H^{1,p}(U,Y;m)$, respectively. Let $(\overline{H}^{1,p}(U,Y;m),d_{\overline{H}^{1,p}})$ be the completion of $(H^{1,p}(U,Y;m),d_{H^{1,p}})$.

The following assertion is not declared clearly in [26]. We provide its proof for completeness.

Theorem 6.17. *Let $p\in]1,\infty[$. We have $\overline{H}^{1,p}(U,Y;m)=H^{1,p}(U,Y;m)$.*

Remark 6.18. Theorem 6.17 does not necessarily imply the $d_{H^{1,p}}$-completeness of $H^{1,p}(U,Y;m)$, that is, $d_{\overline{H}^{1,p}}=d_{H^{1,p}}$ on $H^{1,p}(U,Y;m)$.

6.3. **p-harmonic maps.** In this subsection, we still assume that (Y,d) is weakly L-convex with $L_1L_2=0$.

Definition 6.19 (p-Harmonic Map). For $v\in H^{1,p}(U,Y;m)$, let $H^{1,p}_v(U,Y;m)$ be the $d_{H^{1,p}}$-closure of

$$\{u\in H^{1,p}(U,Y;m)\mid \operatorname{supp} d(u,v)\Subset U\}.$$

v is said to be p-harmonic if and only if $E_p(v)=\inf_{u\in H^{1,p}_v(U,Y;m)}E_p(u)$.

Theorem 6.20. *Suppose $p\ge2$. If there exists $C>0$ such that for any $f\in H^{1,p}_0(U)$,*

$$\int_U|f|^p dm\le C\int_U|g_f|^p dm,\quad \text{(Poincaré Inequality)}$$

then there exists a p-harmonic map in $H^{1,p}_v(U,Y;m)$ for given $v\in H^{1,p}(U,Y;m)$.

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