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Geometric Properties of Plane Quartics

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Abstract: It is well-known that smooth plane quartic curves in the two dimensional complex projective space are (non-hyperelliptic) curves of genus three. And the local property of singular plane quartic curves is also well known. We consider about the parameters of these defining equations.

1 Introduction

Let $\mathbb{P}^2$ be a 2-dimensional complex projective space with the coordinate $[x, y, z]$ and let $f_4(x, y, z)$ be a homogeneous polynomial of variables x, y, z with degree 4 in $\mathbb{P}^2$. We consider the set $V_4 := \{(x, y, z)|f_4(x, y, z) = 0\}$. We call V_4 complex projective plane quartics (in short: plane quartics). Nonhyperelliptic curves of genus 3 are non-singular plane quartics. Let M_g be the variety of moduli of curves of genus g. Then M_3 has dimension which is less than or equal to 6. Let $f(x, y, z)$ be a homogeneous polynomial of variables x, y, z with degree 4 in $\mathbb{C}^3$. The analytic set defined by $f(x, y, z) = 0$ has a singular point at the origin in $\mathbb{C}^3$. The analytic set is a smooth plane quartic in $\mathbb{P}^2$ if it has only isolated singular point at the origin in $\mathbb{C}^3$.

2 Normal forms of smooth plane quartics

For the defining equations of smooth plane quartics in $\mathbb{P}^2$, the following theorem holds.

Theorem 2.1. *For the defining equations of smooth quartics with the coordinate* $[x, y, z]$, *there exist the following two normal forms.*
Type I : $x^3z + (y^3 + pyz^2 + qz^3)x + ry^4 + sy^3z + ty^2z^2 + uyz^3 + vz^4 = 0,$
Type II : $x^3z + (py^2 + qyz + rz^2)xz + y^4 + sy^2z^2 + tyz^3 + uz^4 = 0,$
where $(p, q, r, s, t, u, v) \in \mathbb{C}^7$ *are parametric coefficients.*

We can rewrite any defining equation of smooth plane quartic into the one of above normal forms. This accords for a result of Shioda's Propositions 1 and 2 ([1]). And we got the restrictions of parameters for the above Type I, Type II ([2], [3]). As the result, the following theorem holds.

Theorem 2.2. Let C be a smooth plane quartic in a 2-dimensional complex projective space with the coordinate $[x, y, z]$ Then we can take the multiplicity of C and y-axis at [1,0,0] such that it is equal to 3 or 4 by replacing a suitable coordinate.

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3 Normal forms of singular plane quartics

By a same method as the rewriting of the smooth plane quartics, we can rewrite the defining equation of the singular quartic to one of the defining following equations.

Type I : $x^2yz + (y^3 + a_1y^2z + z^3)x + (a_2y^3 + a_3y^2z + a_4yz^2 + a_5z^3)z = 0,$
Type II : $x^2yz + a_1xz^3 + y^4 + a_2y^3z + a_3y^2z^2 + a_4yz^3 + a_5z^4 = 0,$
Type III : $z(x^2y + a_1xz^2 + y^3 + a_2y^2z + a_3yz^2 + a_4z^3) = 0,$
Type IV : $z(x^2y + a_1xz^2 + a_2y^2z + a_3yz^2 + a_4z^3) = 0,$
Type V : $x^2z^2 + (y^2 + 2a_1z^2)xy + (a_2y^3 + a_3y^2z + a_4yz^2 + a_5z^3)z = 0,$
Type VI : $x^2z^2 + 2a_1xy^2z + y^4 + a_2y^2z^2 + a_3yz^3 + a_4z^4 = 0,$
Type VII : $x^2z^2 + 2xy^2z + y^4 + y^3z + a_1yz^3 + a_2z^4 = 0,$
Type VIII : $x^2z^2 + 2xy^2z + y^4 + y^2z^2 + a_1z^4 = 0,$
Type IX : $x^2z^2 + 2xy^2z + (y^3 + z^3)y = 0,$
Type X : $x^2z^2 + 2xy^2z + y^4 + a_1z^4 = 0,$
Type XI : $z(x^2z + xy^2 + a_1y^2z + a_2yz^2 + a_3z^3) = 0,$
Type XII : $z(x^2z + y^3 + a_1yz^2 + a_2z^3) = 0,$
Type XIII : $xy^2z + xz^3 + y^4 + a_1yz^3 + a_2z^4 = 0,$
Type XIV : $z(xy^2 + xz^2 + a_1yz^2 + a_2z^3) = 0,$
Type XV : $xyz^2 + y^4 + a_1y^3z + a_2z^4 = 0,$
Type XVI : $z(xyz + y^3 + a_1z^3) = 0,$
Type XVII : $z^2(xy + a_1z^2) = 0,$
Type XVIII : $xz^3 + (y^2 + a_1z^2)y^2 = 0,$
Type XIX : $z(xz^2 + y^3) = 0,$
Type XX : $z^2(xz + y^2) = 0,$
Type XXI : $z(y^3 + a_1yz^2 + a_2z^3) = 0,$
Type XXII : $(y^2 + z^2)z^2 = 0,$
Type XXIII : $y^2z^2 = 0,$
Type XXIV : $yz^3 = 0,$
Type XXV : $z^4 = 0.$

We obtain a following table as a classification of singular point at [1,0,0] on irreducible plane quartics.

Type of singularity	Type of above normal form	Name
A_1	**I, II**	Node
A_2	**V**	Cusp ((2,3)-cusp)
A_3	**VI**	Tacnode
A_4	**VII**	Double cusp ((2,5)-cusp)
A_5	**VIII**	Osnode
A_6	**IX**	Ramphoid cusp ((2,7)-cusp)
D_4	**XIII**	Triple point
D_5	**XV**	Tacnode-cusp
E_6	**XVIII**	Cusp ((3,4)-cusp)

We can obtain the restrictions for the parameters of these defining equations by using the method of smooth cases. And we want to know those restrictions and their relations. It is our next purpose.

References

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