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Lower bound of the lifespan of solutions to nonlinear elastic wave equation

Hideo Kubo
Graduate School of Information Sciences, Tohoku University

1. INTRODUCTION

In this paper we consider the Cauchy problem for homogeneous, isotropic, hyperelastic wave equations:

\[ (\partial_{t}^{2} - L)u(t, x) = F(\nabla u, \nabla^2 u), \quad (t, x) \in (0, T) \times \mathbb{R}^3, \]
\[ u(0, x) = \epsilon f(x), \quad (\partial_{t}u)(0, x) = \epsilon g(x), \quad x \in \mathbb{R}^3, \]

where \( u(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \\ u_3(t, x) \end{pmatrix} \) is the displacement vector from the configuration, \( \nabla u = (\partial_1 u, \partial_2 u, \partial_3 u), \) \( \partial_j = \partial/\partial x_j (j = 1, 2, 3), \) and

\[ L = c_2^2 \Delta + (c_1^2 - c_2^2) \text{grad div}, \quad \Delta = \text{div grad} \]

with material constants \( c_1, c_2 \) satisfying \( 0 < c_2 < c_1. \) Here grad and div stand for the spatial gradient and divergence, respectively. Besides, \( f, g \) are smooth functions with compact support and \( \epsilon \) is a positive parameter. In addition, the nonlinearity is expressed as

\[ F(\nabla u, \nabla^2 u) = A_1 \text{grad (divu)}^2 + A_2 \text{grad |rotu|}^2 
+ A_3 \text{rot ((divu)(rotu))} + N(u). \]

Here, \( A_1, A_2 \) and \( A_3 \) are real constants and each components of \( N(u) \) is a linear combination of the so-called null-forms. (for the detail, see Appendix below; also [1]).

We denote the lifespan of the problem (1.1)-(1.2) by \( T_\epsilon \) which is the supremum of all \( T > 0 \) such that the problem admits a unique smooth solution in \( [0, T) \times \mathbb{R}^3. \) In John [10] the lower bound for the lifespan \( T_\epsilon \geq e^{C/\epsilon} \) with a positive number \( C \) was obtained for sufficiently small \( \epsilon \) (see also [13]). Moreover, if \( A_1 = 0, \) then the global solvability of the problem for sufficiently small initial data was proved by Agemi [1] and Sideris [14], independently.

On the other hand, concerning the Cauchy problem for scalar wave equations:

\[ (\partial_{t}^{2} - \Delta)v(t, x) = \sum_{j,k,l=0}^{3} g_{jkl}(\partial_{j}v)(\partial_{k}\partial_{l}v), \quad (t, x) \in (0, T) \times \mathbb{R}^3, \]
\[ v(0, x) = \epsilon \phi(x), \quad (\partial_{t}v)(0, x) = \epsilon \psi(x), \quad x \in \mathbb{R}^3, \]
not only the estimate of the lifespan $\tilde{T}_\varepsilon$ of this problem from below but also much precise information of $\tilde{T}_\varepsilon$ are known (here, $g_{jkl}$ are real constants and $\phi, \psi \in C_0^\infty(\mathbb{R}^3)$). More explicitly, it was independently shown by Hörmander [5] and John [9] that

\[
(1.6) \quad \liminf_{\varepsilon \to +0} \varepsilon \log \tilde{T}_\varepsilon \geq \left( \max \{ -2^{-1}G(\theta) \partial_s^2 \tilde{R}[\phi, \psi](s, \theta); s \in \mathbb{R}, \theta \in S^2 \} \right)^{-1},
\]

provided the right-hand side is a finite number. Here, the functions $G$ and $\tilde{R}[\phi, \psi]$ are defined by

\[
G(\theta) = \sum_{j,k,l=0}^{3} g_{jkl} \theta_j \theta_k \theta_l \quad \text{with} \quad \theta_0 = -1, \ (\theta_1, \theta_2, \theta_3) \in S^2,
\]

\[
\tilde{R}[\phi, \psi](s, \theta) = \frac{1}{4\pi} (\mathcal{R}[\psi](s, \theta) - \partial_s \mathcal{R}[\phi](s, \theta)), \quad (s, \theta) \in \mathbb{R} \times S^2,
\]

where $\mathcal{R}[\phi]$ is the Radon transform of $\phi$, that is,

\[
(1.7) \quad \mathcal{R}[\phi](s, \theta) = \int_{\theta \cdot y = s} \phi(y) \, dS_y, \quad (s, \theta) \in \mathbb{R} \times S^2.
\]

The counter part of the estimate (1.6) has been studied by Alinhac [2]. We remark that $G \equiv 0$ on $S^2$ is equivalent to the null condition introduced by Klainerman [12], and the condition implies $\tilde{T}_\varepsilon = +\infty$ (see also [3]). While, $\tilde{R}[\phi, \psi] \equiv 0$ on $\mathbb{R} \times S^2$ is equivalent to $\phi \equiv \psi \equiv 0$ on $\mathbb{R}^3$.

Therefore, a natural question is if it is possible to derive an analogous estimate to (1.6) for the lifespan $T_\varepsilon$ of the problem (1.1)-(1.2) or not. The difficulty for dealing with the elastic wave equation (1.1) comes from the fact that the equation has two distinct propagation speeds. For this, the hyperbolic boosts $x_j \partial_t + t \partial_j$ do not work well, and construction of a nonlinear approximate solution is not straightforward as in the case of the wave equation. Nevertheless, by using a higher order approximation (see (5.36) below) together with careful treatments of the decay factor $(1 + |c_\eta t - |x||)^{-1}$, we are able to overcome the difficulty.

In order to state our result, we define

\[
(1.8) \quad \tilde{R}_i[f, g](s, \theta) = \frac{1}{4\pi} \left( c_i^{-1} \mathcal{R}[g](s, \theta) - \partial_s \mathcal{R}[f](s, \theta) \right) \quad (i = 1, 2)
\]

for $(s, \theta) \in \mathbb{R} \times S^2$, where the Radon transform $\mathcal{R}[f]$ of $f = t(f_1, f_2, f_3) \in (C_0^\infty(\mathbb{R}^3))^3$ is given by $\mathcal{R}[f] = t(\mathcal{R}[f_1], \mathcal{R}[f_2], \mathcal{R}[f_3])$. We note that $\tilde{R}_i[f, g]$ is bounded on $\mathbb{R} \times S^2$ and compactly supported in $s$ for $f,$
$g \in (C_0^\infty(R^3))^3$. In particular, if
\begin{equation}
(1.9) \quad p_0(s, \theta) := \theta \cdot \mathcal{R}_1[f, g](s, \theta)
\end{equation}
is not identically zero on $R \times S^2$, then $\partial_s^2 p_0(s, \theta)$ takes both positive and negative values. Therefore, one can define a positive number
\begin{equation}
(1.10) \quad \tau_* = (\max\{-c_1^{-2} A_1 \partial_s^2 p_0(s, \theta) ; s \in R, \theta \in S^2 \})^{-1},
\end{equation}
provided $A_1 \neq 0$ and $p_0 \not\equiv 0$ on $R \times S^2$.

Then, our main result is the following:

**Theorem 1.1.** Let $f, g \in (C_0^\infty(R^3))^3$. If $A_1 \neq 0$ and $p_0 \not\equiv 0$ on $R \times S^2$, then we have
\begin{equation}
(1.11) \quad \liminf_{\epsilon \to +0} \epsilon \log T_\epsilon \geq \tau_*.
\end{equation}

**Remark 1.2.** (i) Unfortunately, we do not have the estimate in the opposite direction to (1.11), that is to say
\begin{equation}
(1.12) \quad \limsup_{\epsilon \to +0} \epsilon \log T_\epsilon \leq \tau_*
\end{equation}
in general. But, when the initial data take the following form:
\begin{equation}
(1.12) \quad f(x) = \phi(r)x, \quad g(x) = \psi(r)x, \quad x \in R^3,
\end{equation}
was shown by John [8], provided $A_1 \neq 0$ and the corresponding $p_0$ does not identically vanish on $R \times S^2$. Hence, the lower bound (1.11) seems to be optimal.

(ii) The number $\tau_*$ is related to the lifespan of the following Cauchy problem for $p = p(s, \theta, \tau)$:
\begin{align}
2c_1^2 \partial_\tau p + A_1(\partial_s p)^2 &= 0 \quad \text{in } R \times S^2 \times [0, \tau_*), \quad (1.13) \\
p(s, \theta, 0) &= p_0(s, \theta) \quad \text{for } (s, \theta) \in R \times S^2. \quad (1.14)
\end{align}
Indeed, it is known that the solution to the above problem uniquely exists in $R \times S^2 \times [0, \tau_*)$ (for the proof, see Lemma 6.5.4 with $G(\omega) \equiv 2A_1/c_1^2$ in [6]).

This paper is organized as follows. In the next section we gather notation. In Section 3 we give some preliminaries. Basic results on the linear elastic wave equation are introduced in Section 4. An approximate solution is constructed in Section 5, and useful estimates for the approximation are established in Proposition 5.5. Outline of the proof of Theorem 1.1 is given in Section 6. In Appendix a way to deduce (1.1) is discussed.
2. Notation

In this section, we introduce notation which will be used throughout this paper. We denote \( r = |x| \) and \( \omega = x/r \). We set \( \partial_r = \sum_{j=1}^{3}(x_j/r)\partial_j \) and \( O = t(O_1, O_2, O_3) = x \wedge t(\partial_1, \partial_2, \partial_3) \), where \( \wedge \) stands for the outer product in \( \mathbb{R}^3 \). Then we have

\[
(t(\partial_1, \partial_2, \partial_3) = \omega \partial_r - r^{-1} \omega \wedge O. \tag{2.1}
\]

We denote \( Z = \{Z_0, Z_1, \ldots, Z_6\} = \{\partial_t, \partial_1, \partial_2, \partial_3, O_1, O_2, O_3\} \). We write \( Z^\alpha \) for \( Z_0^{\alpha_0} \cdots Z_6^{\alpha_6} \) with a multi-index \( \alpha = (\alpha_0, \ldots, \alpha_6) \). Note that we have \( [Z_a, \partial_t^2 - \Delta] = 0 \) (\( a = 0, \ldots, 6 \)), where we have set \( [A, B] = AB - BA \).

We also use \( \tilde{Z} = \{	ilde{Z}_0, \tilde{Z}_1, \ldots, \tilde{Z}_6\} = \{\partial_t I, \partial_1 I, \partial_2 I, \partial_3 I, \tilde{O}_1, \tilde{O}_2, \tilde{O}_3\} \) for \( \mathbb{R}^3 \)-valued functions, where \( I \) is the \( 3 \times 3 \) identity matrix and

\[
\tilde{O}_j = O_j I + U_j \quad (j = 1, 2, 3) \tag{2.2}
\]

with

\[
U_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The vector fields \( \tilde{O}_j \) is closely related to the fact that if \( u(t, x) \) solves (1.1), then so does \( A^{-1}u(t, Ax) \) for any orthogonal matrix \( A \). This observation leads to the good algebraic relations \( [\tilde{Z}_j, L] = 0 \) for \( a = 0, \ldots, 6 \). We write \( \tilde{Z}^\alpha \) for \( \tilde{Z}_0^{\alpha_0} \cdots \tilde{Z}_6^{\alpha_6} \) with a multi-index \( \alpha = (\alpha_0, \ldots, \alpha_6) \).

For functions of \((s, \theta, \tau) \in \mathbb{R} \times S^2 \times [0, \infty)\), we denote the differentiation with respect to \( s \), \( \theta \) and \( \tau \) by

\[
\Lambda_0 = \partial_s, \quad \Lambda_1 = \partial_1, \quad \Lambda_2 = \partial_2, \quad \Lambda_3 = \partial_3, \quad \Lambda_4 = \partial_\tau, \tag{2.3}
\]

where differential operators \( o_i \) on \( S^2 \) are (formally) defined by \( t(o_1, o_2, o_3) = \theta \wedge t(\partial_1, \partial_2, \partial_3) \). We write \( \Lambda^\beta \) for \( \Lambda_0^{\alpha_0} \cdots \Lambda_4^{\alpha_4} \) and \( \Lambda^\gamma = \Lambda_0^{\gamma_0} \cdots \Lambda_3^{\gamma_3} \) with multi-indices \( \beta = (\beta_0, \ldots, \beta_4) \) and \( \gamma = (\gamma_0, \ldots, \gamma_3) \).

For a non-negative integer \( k \), and a real-valued smooth function \( \varphi(t, x) \), we define

\[
|\varphi(t, x)|_k = \sum_{|\alpha| \leq k} |(Z^\alpha \varphi)(t, x)|, \quad |\partial \varphi(t, x)|_k = \sum_{|\alpha| \leq k} \sum_{a=0}^{3} |(Z^\alpha \partial_a \varphi)(t, x)|.
\]

For a \( \mathbb{R}^3 \)-valued function \( u(t, x) \), we use the same notation \( |u(t, x)|_k \) and \( |\partial u(t, x)|_k \) with \( Z \) replaced by \( \tilde{Z} \).
For $\nu \geq 0$, a non-negative integer $k$, and $\phi \in \mathcal{S}(\mathbb{R}^3)$, we define

$$
\Vert \phi \Vert_{k,\nu} = \left( \sup_{x \in \mathbb{R}^3} \sum_{|\alpha| \leq k} (1 + |x|^2)^\nu |\partial_x^\alpha \phi(x)|^2 \right)^{1/2}.
$$

Here, $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz class, the set of rapidly decreasing real-valued functions. Besides, for $f, g \in (\mathcal{S}(\mathbb{R}^3))^3$, we set

$$
(2.4) \quad \mathcal{A}_{k,\nu}[f, g] = \sum_{j=1}^3 (\Vert f_j \Vert_{k+1,\nu} + \Vert g_j \Vert_{k,\nu}).
$$

As usual, various positive constants which may change line by line are denoted just by the same letter $C$ throughout this paper.

3. Preliminaries

First we recall basic properties of the Radon transform discussed in the section 4 of [11] for the case of $n = 3$ and $\chi \equiv 1$ (note that when $\chi \equiv 1$, $\mathcal{S}_\chi(\mathbb{R}^3)$ and $\Vert \phi \Vert_{\chi, k, \nu}$ in [11] become to $\mathcal{S}(\mathbb{R}^3)$ and $\Vert \phi \Vert_{k, \nu}$, respectively). It holds that

$$
(3.1) \quad \partial_s \mathcal{R}[\varphi](s, \theta) = \mathcal{R}[(\theta \cdot \text{grad})\varphi](s, \theta),
$$

$$
(3.2) \quad \partial_i \mathcal{R}[\varphi](s, \theta) = \mathcal{R}[O_i \varphi](s, \theta), \quad i = 1, 2, 3,
$$

$$
(3.3) \quad \mathcal{R}[\partial_i \varphi](s, \theta) = \theta_i \partial_s \mathcal{R}[\varphi](s, \theta), \quad i = 1, 2, 3
$$

for a real-valued function $\varphi \in \mathcal{S}(\mathbb{R}^3)$. Moreover, for $\nu \geq 0$, a nonnegative integer $k$, and a multi-index $\alpha$, we have

$$
(3.4) \quad |\partial_s^\alpha \mathcal{R}[\varphi](s, \theta)| \leq C \Vert \varphi \Vert_{k+|\alpha|, \nu+3+|\alpha|}(1 + s^2)^{-\frac{\nu}{2}}
$$

for $(s, \theta) \in \mathbb{R} \times S^2$. Here $C = C(k, \nu, \alpha)$ is a positive constant.

Next we define

$$
(3.5) \quad Q_\gamma[\varphi](t, x) = \frac{1}{4\pi} \int_{\theta \in S^2} \theta^\gamma \varphi(x + t\theta) dS'_\theta, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3
$$

for a multi-index $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, a real-valued function $\varphi \in \mathcal{S}(\mathbb{R}^3)$. Here, $dS'_\theta$ is the area element on $S^2$. Note that $Q_0[\varphi]$ is the spherical mean of $\varphi$. We shall derive decay property of $Q_\gamma[\varphi]$.

**Proposition 3.1.** Let $k$ be a nonnegative integer, $\nu > 0$, and $\gamma$ be a multi-index. Then there exists a positive constant $C$ such that we have

$$
(3.6) \quad |\partial_t^\alpha Q_\gamma[\varphi](t, x)| \leq C \Vert \varphi \Vert_{k, \nu+2}(1 + t + r)^{-2}(1 + |r - t|)^{-\nu}
$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^3$ with $r = |x|$, provided that $\varphi \in \mathcal{S}(\mathbb{R}^3)$.
Proof. It follows that

\begin{equation}
\partial_t^k Q_\gamma[\varphi](t, x) = \sum_{|\alpha|=k} \frac{1}{4\pi} \int_{\theta \in S^2} c_\alpha \theta^{\gamma+\alpha}(\partial_x^\alpha \varphi)(x + t\theta) dS'_\theta
\end{equation}

with some appropriate constants $c_\alpha$. Therefore, we get

\begin{equation}
|\partial_t^k Q_\gamma[\varphi](t, x)| \leq C \Vert \varphi \Vert_{k+1, N_0} (1+t+r)^{-2}(1+|r-t|)^{-\nu}
\end{equation}

Hence, the desired estimate follows from

\begin{equation}
\frac{1}{tr} \int_{|t-r|}^{t+r} \lambda(1+\lambda)^{-\nu-2}d\lambda \leq C(1+t+r)^{-2}(1+|r-t|)^{-\nu}
\end{equation}

for $t, r > 0$. By symmetry, it suffices to show (3.8) for $0 < r \leq t$.

First suppose $0 < r \leq t < 1$. Then the desire estimate follows from

\begin{equation}
\frac{1}{tr} \int_{|t-r|}^{t+r} \lambda(1+\lambda)^{-\nu-2}d\lambda \leq \frac{1}{tr} \int_{|t-r|}^{t+r} \lambda d\lambda = 2.
\end{equation}

Next suppose $t \geq 1$ and $0 < r \leq t$. Since $t \geq (t+r+1)/3$, we get

\begin{equation}
\frac{1}{tr} \int_{|t-r|}^{t+r} \lambda(1+\lambda)^{-\nu-2}d\lambda \leq \frac{3}{(1+t+r)r} \int_{|t-r|}^{t+r} (1+\lambda)^{-\nu-1}d\lambda.
\end{equation}

Observing that $t-r \geq (t+r)/3$ for $t \geq 2r$ and that $r \geq (t+r)/3$ for $t \leq 2r$, we obtain (3.8). This completes the proof. \hfill \square

The following proposition shows that the leading term of $Q_\gamma[\varphi]$ is described by the Radon transform. Since the proof of the proposition is similar to that of Lemma 4.3 in [11], we omit it.

**Proposition 3.2.** Let $k$ be a nonnegative integer, $\nu \geq 0$, $\gamma$ be a multi-index, and $c_* \geq 1$. Then there exist a positive constant $C$ and an integer $N_0 (\geq \nu+4)$ such that we have

\begin{equation}
|t \partial_t^k Q_\gamma[\varphi](t, x) - (4\pi r)^{-1}(-\omega)^\gamma ((-\partial_s)^k \mathcal{R}[\varphi])(r-t, \omega)| \leq C \Vert \varphi \Vert_{k+1, N_0} (1+t+r)^{-2}(1+|r-t|)^{-\nu}
\end{equation}

for $(t, x) \in (0, \infty) \times \mathbb{R}^3$ satisfying $r \geq t/(2c_*) \geq 1$ with $r = |x|$ and $\omega = x|x|^{-1}$, provided that $\varphi \in S(\mathbb{R}^3)$.

Next we derive a couple of estimates of the following integral operator for the latter sake:

\begin{equation}
T_\gamma[\varphi](t, x) = \int_{c_2 t}^{c_1 t} \tau^{-1} Q_\gamma[\varphi](\tau, x) d\tau, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3.
\end{equation}
Proposition 3.3. Let $k$ be a nonnegative integer, $\nu > 0$, $\gamma$ be a multi-index, and $\varphi \in \mathcal{S}(\mathbb{R}^3)$. When $(t, x) \in (0, \infty) \times \mathbb{R}^3$ satisfies one of

$r > 2c_1 t$, $r < c_2 t/2$ or $0 < t + r \leq 1$, we have

\begin{equation}
|T_\gamma[\varphi](t, x)| \leq C \|\varphi\|_{0, \nu+2} (1 + t + r)^{-2-\nu}.
\end{equation}

While, when $(t, x) \in (0, \infty) \times \mathbb{R}^3$ satisfies $c_2 t/2 < r < 2c_1 t$ and $t+r \geq 1$, we have

\begin{equation}
|T_\gamma[\varphi](t, x)| \leq C \|\varphi\|_{0, \nu+2} (1 + t + r)^{-3},
\end{equation}

provided $\nu > 1$. Moreover, if $k \geq 1$, then we have

\begin{equation}
|\partial_t^k T_\gamma[\varphi](t, x)| \leq C \|\varphi\|_{k, \nu+2} (1+t)^{-1} (1+t+r)^{-2} \max_{i=1,2} \{(1+|r-c_i t|)^{-\nu}\}
\end{equation}

for $(t, x) \in (0, \infty) \times \mathbb{R}^3$. Furthermore, we have

\begin{equation}
|\partial_j T_\gamma[\varphi](t, x)| \leq C \|\varphi\|_{2, N_0} (1+t+r)^{-3} \max_{i=1,2} \{(1+|r-c_i t|)^{-1}\},
\end{equation}

where $N_0$ is the number from Lemma 3.2.

Proof. First we prove (3.11). By (3.6) we have

\begin{equation}
|T_\gamma[\varphi](t, x)| \leq C \|\varphi\|_{0, \nu+2} \int_{c_2 t}^{c_1 t} \tau^{-1} (1+\tau+r)^{-2} (1+|\tau-r|)^{-\nu} d\tau
\end{equation}

(3.15)

\begin{align*}
&\leq C \|\varphi\|_{0, \nu+2} (1 + c_2 t + r)^{-2} \int_{c_2 t}^{c_1 t} \tau^{-1} (1+|\tau-r|)^{-\nu} d\tau.
\end{align*}

Observe that if $r \leq c_2 t/2$ and $\tau \geq c_2 t$ then $|\tau - r| \geq (c_2 t + r)/3$, and that if $r \geq 2c_1 t$ and $\tau \leq c_1 t$, then $|\tau - r| \geq (c_1 t + r)/3$. Thus we get (3.11) for $r \leq c_2 t/2$ or $r \geq 2c_1 t$. On the one hand, from (3.15) we have

\begin{align*}
|T_\gamma[\varphi](t, x)| &\leq C \|\varphi\|_{0, \nu+2} \int_{c_2 t}^{c_1 t} \tau^{-1} d\tau
\leq C \|\varphi\|_{0, \nu+2},
\end{align*}

which yields (3.11) for $0 < t + r \leq 1$.

Next we prove (3.12). Since $\tau > C(1 + t + r)$ for $\tau > c_2 t$, $c_2 t/2 < r < 2c_1 t$, and $t+r \geq 1$, we get (3.12) from (3.15) by $\nu > 1$.

Next we prove (3.13). It follows from (3.10) that

\begin{equation}
\partial_t T_\gamma[\varphi](t, x) = t^{-1} (Q_\gamma[\varphi](c_1 t, x) - Q_\gamma[\varphi](c_2 t, x)).
\end{equation}

When $t \geq 1$, we easily have (3.13) by (3.6). While, when $0 < t < 1$, we rewrite the right-hand side of (3.16) as

\begin{align*}
(c_1 - c_2) \int_0^1 (\partial_t Q_\gamma[\varphi]) (c_1 t \sigma + c_2 t (1-\sigma), x) d\sigma.
\end{align*}
Since $0 \leq c_1 t \sigma + c_2 t (1 - \sigma) \leq C$ for $0 < \sigma, t < 1$, we get from (3.6)

$$|\partial^k T_\gamma [\varphi](t, x)| \leq C \|\varphi\|_{k, \nu + 2} (1 + r)^{-2-\nu},$$

which yields (3.13) for $0 < t \leq 1$.

Finally, we prove (3.14). When one of $r > 2 c_1 t, r < c_2 t/2$ or $0 < t + r \leq 1$ holds, (3.11) with $\nu = 2$ yields (3.14). Therefore, we have only to consider the case where $c_2 t/2 \leq r \leq 2 c_1 t$ and $t + r \geq 1$. We rewrite

$$T_\gamma [\partial_j \varphi](t, x) = (4 \pi r)^{-1} \int_{c_2 t}^{c_1 t} \tau^{-2} (-\omega)^\gamma R[\partial_j \varphi](r - \tau, \omega) d\tau$$

$$+ \int_{c_2 t}^{c_1 t} \tau^{-2} (\tau Q_\gamma [\partial_j \varphi](\tau, x) - (4 \pi r)^{-1} (-\omega)^\gamma R[\partial_j \varphi](r - \tau, \omega)) d\tau.$$

Let $\nu > 1$ in the following. Then, by (3.9) with $k = 0$ the second term on the right-hand side is estimated by

$$C \|\varphi\|_{2, N_0} \int_{c_2 t}^{c_1 t} \tau^{-2} (1 + \tau + r)^{-2} (1 + |r - \tau|)^{-\nu} d\tau$$

$$\leq C \|\varphi\|_{2, N_0} (1 + t + r)^{-4},$$

because $\tau \geq C (1 + t + r)$ in this case. Using (3.3), we can make integration by parts in $\tau$ in the first term. Then it is rewritten as

$$(4 \pi r)^{-1} \int_{c_2 t}^{c_1 t} (-2 \tau^{-3}) \omega_j (-\omega)^\gamma R[\varphi](r - \tau, \omega) d\tau$$

$$- (4 \pi r)^{-1} (c_1 t)^{-2} \omega_j (-\omega)^\gamma R[\varphi](r - c_1 t, \omega)$$

$$- (c_2 t)^{-2} \omega_j (-\omega)^\gamma R[\varphi](r - c_2 t, \omega)).$$

By (3.4) we have $|R[\varphi](s, \omega)| \leq C \|\varphi\|_{0, \nu + 3} (1 + s)^{-\nu}$. Since $\nu > 1$, we thus find (3.14) in this case. This completes the proof. \(\square\)

### 4. LINEAR ELASTIC WAVE EQUATIONS

First of all, we consider the Cauchy problem:

(4.1) \( (\partial^2_t - L) u_0(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3, \)

(4.2) \( u_0(0, x) = f(x), \ (\partial_t u_0)(0, x) = g(x), \quad x \in \mathbb{R}^3, \)

where $f, g \in (S(\mathbb{R}^3))^3$. We recall the explicit representation of the solution $u_0$. We define

(4.3) \( E[g](t, x) = E_1[g](t, x) + E_2[g](t, x) + E_3[g](t, x), \)
\begin{align*}
E_1[g](t, x) &= \frac{t}{4\pi} \int_{\theta \in S^2} \Pi(\theta) g(x + c_1 t \theta) dS_{\theta}, \\
E_2[g](t, x) &= \frac{t}{4\pi} \int_{\theta \in S^2} (I - \Pi(\theta)) g(x + c_2 t \theta) dS_{\theta}, \\
E_3[g](t, x) &= -\frac{t}{4\pi} \int_{c_2 t}^{c_1 t} \tau^{-1} d\tau \\
&\quad \times \int_{\theta \in S^2} (g(x + \tau \theta) - 3(\theta \cdot g(x + \tau \theta))\theta) dS_{\theta}.
\end{align*}

Here, for each fixed $\theta \in S^2$, $\Pi(\theta) : \mathbb{R}^3 \to \mathbb{R}^3$ is the projection defined by $\Pi(\theta)v = (\theta \cdot v)\theta$ for $v \in \mathbb{R}^3$. Then it is known that

\begin{equation}
\dot{u}_0(t, x) = \partial_t E[f](t, x) + E[g](t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^3
\end{equation}

holds (see, e.g., John [10]). By virtue of Propositions 3.1 and 3.3, we can prove the following estimates which are refinement of those in Theorem 1 in [10] in the sense that we can replace the decaying factor $1+r$ by $1+t+r$ and that the derivatives enjoy better decay property with respect to $1+|r-c_i t|$ with $i = 1, 2$.

**Proposition 4.1.** Let $k$ be a nonnegative integer, $f, g \in (S(\mathbb{R}^3))^3$, $\nu > 1$, and $N_0$ be the number from Proposition 3.2. Then, for $(t, x) \in (0, \infty) \times \mathbb{R}^3$, we have

\begin{equation}
|u_0(t, x)|_k \leq C A_{k, \nu+2}[f, g] (1+t+r)^{-1} W_{-1}(t, r)
\end{equation}

and

\begin{equation}
|\partial u_0(t, x)|_k \leq C A_{k+2, N_0}[f, g] (1+t+r)^{-1} W_{-2}(t, r),
\end{equation}

where $A_{k, \nu}[f, g]$ is defined by (2.4), and for $\nu \in \mathbb{R}$ we put

\begin{equation}
W_{\nu}(t, r) = \max_{i=1,2} \{(1+|r-c_i t|)^\nu\}.
\end{equation}

Next we consider the radiation field for the free elastic wave (for the case of the scalar wave equation, see Friedlander [4], and also [11]). Having Proposition 3.2 in mind, we define the radiation field $\mathcal{F}_i[f, g]$ $(i = 1, 2)$ for $u_0$ associated with the propagation speed $c_i$ by

\begin{align*}
\mathcal{F}_1[f, g](s, \theta) &= \Pi(\theta) \tilde{\mathcal{R}}_1[f, g](s, \theta), \\
\mathcal{F}_2[f, g](s, \theta) &= (I - \Pi(\theta)) \tilde{\mathcal{R}}_2[f, g](s, \theta)
\end{align*}

for $(s, \theta) \in \mathbb{R} \times S^2$, and $f, g \in (S(\mathbb{R}^3))^3$. Here, $\tilde{\mathcal{R}}_i[f, g](s, \theta)$ is defined by (1.8). We remark that (3.4) implies

\begin{equation}
|\partial^k s^\alpha \mathcal{F}_i[f, g](s, \theta)| \leq C (1+s)^{-\nu}, \quad (s, \theta) \in \mathbb{R} \times S^2.
\end{equation}
for any $\nu > 0$, nonnegative integer $k$, multi-index $\alpha$, and $f, g \in (S(R^3))^3$. Then we have the following.

**Proposition 4.2.** Let $f, g \in (S(R^3))^3$ and let $u_0$ be the solution to the problem (4.1)-(4.2). Then for any non-negative integer $k$ and any multi-index $\alpha$ with $|\alpha| \geq 1$, there exists a positive constant $C$ such that

\[
(4.14) \quad \left| u_0(t, x) - \sum_{m=1}^{2} r^{-1} \mathcal{F}_m[f, g](r - c_m t, \omega) \right|_k \leq C(1 + t + r)^{-2},
\]

and

\[
(4.15) \quad \left| \partial_t u_0(t, x) - \sum_{m=1}^{2} (-c_m) r^{-1} \partial_s (\mathcal{F}_m[f, g])(r - c_m t, \omega) \right|_k + \left| \partial_x^\alpha u_0(t, x) - \sum_{m=1}^{2} \omega^\alpha r^{-1} \partial_s^{|\alpha|} (\mathcal{F}_m[f, g])(r - c_m t, \omega) \right|_k \leq C(1 + t + r)^{-2} W_{-1}(t, r)
\]

for $(t, x) \in (0, \infty) \times R^3$ with $r \geq c_2 t/2 \geq 1$. Here, $\omega = (\omega_1, \omega_2, \omega_3) = r^{-1}x$.

Next we consider the inhomogeneous elastic wave equation with zero initial data:

\[
(4.16) \quad \begin{cases} 
(\partial_t^2 - L)u(t, x) = h(t, x) & \text{for } (t, x) \in (0, T) \times R^3, \\
u(0, x) = 0, (\partial_t u)(0, x) = 0 & \text{for } x \in R^3.
\end{cases}
\]

The following estimate is an improvement of the corresponding estimate given by [1, Proposition 5.1] in the sense that the exponent of the weight in the right hand side $1 + \mu$ is replaced by $1 - \mu$. This kind of modification was well studied in the case of the scalar wave equation, and the detail of the proof of (4.17) will appear elsewhere.

**Proposition 4.3.** Let $u$ be the solution to (4.16) and let $\mu > 0$, $c_0 = 0$. Then we have

\[
(4.17) \quad |\partial u(t, x)| \leq C(1 + r)^{-1} W_{-1}(t, r) \sup_{(s, x) \in [0, t] \times R^3} (1 + |x|) \times (1 + s + |x|)^{1+\mu} \left( \max_{i=0,1,2} \{1 + |c_i s - |x||\} \right)^{1-\mu} |h(s, x)|_1
\]

for $(t, x) \in [0, T) \times R^3$.

On the other hand, the following estimate was proved by [10, Theorem 3].
**Proposition 4.4.** Let $u$ be the solution to (4.16). Then we have

\[ (4.18) \quad |\partial u(t, x)| \leq C(1 + r)^{-1} W_{-1}(t, r) \times \log(2 + t + r) \sup_{s \in [0, t]} \int_{\mathbb{R}^3} \min_{i=1,2} \{1 + |c_i s - |x||\} |h(s, x)|_{7} dy \]

for $(t, x) \in [0, T) \times \mathbb{R}^3$, and

\[ (4.19) \quad \int_{\mathbb{R}^3} |\partial u(t, x)| \frac{dx}{|x|} \leq C \log(2 + t) \times \sup_{s \in [0, t]} \left( \int_{\mathbb{R}^3} \min_{i=1,2} (1 + |y|) |h(s, x)|_{1}^2 dy \right)^{1/2} \]

for $t \in [0, T)$.

---

**5. Approximate solutions**

This section is the core of the present paper. We shall construct an approximate solution and derive important estimates given in Proposition 5.5 below in proving Theorem 1.1. Throughout this section we assume that $f, g \in (C_0^\infty(\mathbb{R}^3))^3$ satisfy

\[ (5.1) \quad f(x) = g(x) = 0 \text{ for } |x| \geq R \]

with some $R > 1$, and that $A_1 \neq 0$ and $p_0 \neq 0$ on $\mathbb{R} \times S^2$, where $p_0$ is defined by (1.9).

**Lemma 5.1.** Let $p(s, \theta, \tau)$ be the solution to (1.13)-(1.14) vanishing for $|s| \geq R$. Let $0 < \tau_0 < \tau_* \text{ with } \tau_*$ being defined by (1.10). Then for any $N > 0$, and for any multi-indices $\beta = (\beta_0, \ldots, \beta_4)$ and $\gamma = (\gamma_0, \ldots, \gamma_3)$, there exists a positive constant $C = C(\tau_0, \beta, \gamma, N)$ such that

\[ (5.2) \quad |\Lambda^\beta p(s, \theta, \tau)| \leq C, \]

\[ (5.3) \quad |\Lambda^\beta \partial_s p(s, \theta, \tau)| \leq C(1 + s)^{-N}, \]

\[ (5.4) \quad |\Lambda_\tau^\gamma \{p(s, \theta, \tau) - p_0(s, \theta)\}| \leq C \tau, \]

\[ (5.5) \quad |\Lambda_\tau^\gamma \partial_s \{p(s, \theta, \tau) - p_0(s, \theta)\}| \leq C \tau (1 + s)^{-N} \]

for all $(s, \theta, \tau) \in \mathbb{R} \times S^2 \times [0, \tau_0]$.

**Proof.** First of all, we note that (5.2) and (5.4) follows from (5.3) and (5.5) with $N > 1$ respectively, because both $p(s, \theta, \tau)$ and $p_0(s, \theta)$ vanish for $|s| \geq R$. 

Next we prove (5.3). If we set $P = \partial_s p$, then it satisfies

\begin{align}
(5.6) & \quad c_1^2 \partial_\tau P + A_1 \partial_s P = 0 \quad \text{in } \mathbb{R} \times S^2 \times [0, \tau_*), \\
(5.7) & \quad P(s, \theta, 0) = \partial_s p_0(s, \theta) \quad \text{for } (s, \theta) \in \mathbb{R} \times S^2.
\end{align}

Observe that for $(s, s_0, \theta, \tau) \in \mathbb{R} \times \mathbb{R} \times S^2 \times [0, \tau_0)$, the equation

\begin{align}
F(s, s_0, \theta, \tau) := c_1^2 (s_0 - s) + \partial_s p_0(s_0, \theta) A_1 \tau = 0
\end{align}

determines the implicit function $s_0 = s_0(s, \theta, \tau)$, because

\[ \partial_{s_0} F(s, s_0, \theta, \tau) = c_1^2 + \partial_{S}^2 p_0(s_0, \theta) A_1 \tau \geq c_1^2 (1 - \tau/\tau_*) > 0. \]

Therefore, the solution to (5.6)-(5.7) is given by $P(s, \theta, \tau) = (\partial_s p_0)(s_0(s, \theta, \tau), \theta)$, and hence for $(s, \theta, \tau) \in \mathbb{R} \times S^2 \times [0, \tau_0)$, we have

\begin{align}
\partial_s p(s, \theta, \tau) = (\partial_s p_0)(s_0(s, \theta, \tau), \theta).
\end{align}

Since (3.4) implies $|\Lambda^\beta p_0(s, \theta)| \leq C(1 + s)^{-N}$ for any $(s, \theta, \tau) \in \mathbb{R} \times S^2$ and $N > 0$, we see that $\Lambda^\beta s_0(s, \theta, \tau)$ is bounded for any $(s, \theta, \tau) \in \mathbb{R} \times S^2 \times [0, \tau_0)$, because we have

\begin{align*}
\partial_s s_0(s, \theta, \tau) &= \frac{c_1^2}{c_1^2 + (\partial_s^2 p_0)(s_0(s, \theta, \tau), \theta) A_1 \tau}, \\
\partial_\tau s_0(s, \theta, \tau) &= \frac{-A_1 (\partial_s p_0)(s_0(s, \theta, \tau), \theta)}{c_1^2 + (\partial_s^2 p_0)(s_0(s, \theta, \tau), \theta) A_1 \tau}, \\
o_i s_0(s, \theta, \tau) &= \frac{-A_1 \tau (o_i \partial_s p_0)(s_0(s, \theta, \tau), \theta)}{c_1^2 + (\partial_s^2 p_0)(s_0(s, \theta, \tau), \theta) A_1 \tau}.
\end{align*}

Therefore, we get (5.3) by using (5.9).

Next we prove (5.5). Since $s_0(s, \theta, 0) = s$, we get

\[ \partial_s p(s, \theta, \tau) - \partial_s p_0(s, \theta) = \tau \int_0^1 (\partial_s^2 p_0)(s_0(s, \theta, \sigma \tau), \theta) \partial_\tau s_0(s, \theta, \sigma \tau) d\sigma. \]

In view of (5.8), we see that $(1 + s_0(s, \theta, \tau))^{-N}$ is equivalent to $(1 + s)^{-N}$ for $(s, \theta, \tau) \in \mathbb{R} \times S^2 \times [0, \tau_0)$, because $|\Lambda^\beta p_0(s_0(s, \theta, \tau), \theta) A_1 \tau|$ is bounded. Thus we find (5.5) holds. This completes the proof.

For a real-valued function $\varphi = \varphi(s, \theta, \tau)$, we shall write

\[ \tilde{\varphi}(t, x) := \varphi(r - c_1 t, \omega, \epsilon \log(\epsilon t)) \]

with $r = |x|$ and $\omega = r^{-1} x$. Then we have

\begin{align}
(5.10) & \quad \partial_t \tilde{\varphi} = -c_1 \partial_s \tilde{\varphi} + \epsilon t^{-1} \partial_\tau \tilde{\varphi}, \\
(5.11) & \quad \text{grad } \tilde{\varphi} = \omega \partial_s \tilde{\varphi} - r^{-1} \omega \wedge o \tilde{\varphi},
\end{align}

where we have used (2.1) to get (5.11).
Let $p(s, \theta, \tau)$ be the solution to (1.13)-(1.14) vanishing for $|s| \geq R$. Using the above notation, we define

$$w_1(t, x) = \varepsilon r^{-1}(\tilde{p}(t, x)\omega + \mathcal{F}_2[f, g](r - c_2 t, \omega))$$

for $(t, x) \in \left[1/\varepsilon, \exp(\tau_*/\varepsilon)\right) \times (\mathbb{R}^3 \setminus \{0\})$. Note that

$$w_1(t, x) = 0 \quad \text{for} \quad |x| \geq c_1 t + R.$$

The following estimates, which shows that $w_1$ is a good approximation of $u_0$ near the characteristic cones $r = c_i t \ (i = 1, 2)$, are reduced from Lemma 5.1.

**Corollary 5.2.** Let $0 < \tau_0 < \tau_*$ and let $0 < \varepsilon \leq 1$. Then for any nonnegative integer $k$, there exists a positive constant $C = C(\tau_0, k)$ such that

1. $|w_1(t, x)|_k \leq C\varepsilon(1 + t + r)^{-1},$ 
2. $|\partial w_1(t, x)|_k \leq C\varepsilon(1 + t + r)^{-1}W_{-1}(t, r),$ 
3. $|dvi w_1(t, x)|_k \leq C\varepsilon(1 + t + r)^{-1}(1 + |r - c_1 t|)^{-1},$ 
4. $|rot w_1(t, x)|_k \leq C\varepsilon(1 + t + r)^{-1}(1 + |r - c_2 t|)^{-1}$

for $c_2 t/2 \leq |x| \leq c_1 t + R$ and $1 \leq t \leq \exp(\tau_0/\varepsilon)$. Moreover, we have

1. $|w_1(t, x) - \varepsilon u_0(t, x)|_k \leq C\varepsilon(1 + t + r)^{-2},$ 
2. $|\partial\{w_1(t, x) - \varepsilon u_0(t, x)\}|_k \leq C\varepsilon(1 + t + r)^{-2}W_{-1}(t, r),$ 

for $c_2 t/2 \leq |x| \leq c_1 t + R$ and $1/\varepsilon \leq t \leq 2/\varepsilon$. Here, $u_0$ is the solution of the Cauchy problem (4.1)-(4.2).

**Proof.** We suppose that $c_2 t/2 \leq r \leq c_1 t + R$ and $1 \leq t \leq \exp(\tau_0/\varepsilon)$ in what follows. Then we have

$$|t^{-1}|_k + |r^{-1}|_k + |(1 + t + r)^{-1}|_k \leq C(1 + t + r)^{-1}.$$

First we prove (5.14) and (5.15). It follows from (4.13), (2.1), and (5.20) that

$$|\mathcal{F}_2[f, g](r - c_2 t, \omega)|_k \leq C(1 + |r - c_2 t|)^{-1}.$$

While, from (5.2), (5.3) with $N = 1$, (5.10), (5.11), and (5.20), we get

$$|\tilde{p}(t, x)|_k \leq C \sum_{|\beta| \leq k} |\overline{\Lambda^\beta p}(t, x)| \leq C,$$

(5.21)

$$|\partial\tilde{p}(t, x)|_k \leq C \sum_{|\beta| \leq k} |\overline{\Lambda^\beta \partial_s p}(t, x)| + C(1 + t + r)^{-1} \sum_{|\beta| \leq k+1} |\overline{\Lambda^\beta p}(t, x)| \leq C(1 + |r - c_1 t|)^{-1}.$$
Thus we obtain (5.14) and (5.15) from (5.12).

Next we prove (5.16). A direct computation shows that

\begin{align}
(5.23) \quad \text{dvi} (r^{-1}\tilde{p}(t, x)\omega) &= r^{-1}\tilde{\omega}_p(t, x) + r^{-2}\tilde{p}(t, x), \\
(5.24) \quad \text{dvi} (r^{-1}\mathcal{F}_2[f, g](r - c_2 t, \omega)) &= -r^{-2}(2\omega \cdot \tilde{R}_2[f, g](r - c_2 t, \omega) + \Omega \cdot \tilde{R}_2[f, g](r - c_2 t, \omega)),
\end{align}

where we put \(\Omega \cdot f(x) = \sum_{j=1}^{3} \Omega_j f_j(x)\) with \(\Omega = \omega \wedge O\) (recall also (4.12)). Therefore, by (5.2), (5.3) with \(N = 1\), and (3.4), we get (5.16).

Next we prove (5.17). A direct computation shows that

\begin{align}
(5.25) \quad \text{rot} (r^{-1}\tilde{p}(t, x)\omega) &= -r^{-2}\Omega \wedge (\tilde{p}(t, x)\omega), \\
(5.26) \quad \text{rot} (r^{-1}\mathcal{F}_2[f, g](r - c_2 t, \omega)) &= r^{-1}\text{rot}\tilde{\mathcal{R}}_2[f, g](r - c_2 t, \omega) \\
&- r^{-2}(\omega \wedge \tilde{\mathcal{R}}_2[f, g](r - c_2 t, \omega) - \Omega \wedge \Pi(\omega)\tilde{\mathcal{R}}_2[f, g](r - c_2 t, \omega)).
\end{align}

Thus (5.2) and (3.4) yields (5.17).

Next we prove (5.19). Suppose that we also have \(1/\epsilon \leq t \leq 2/\epsilon\) from now on. In view of (4.15), it suffices to show

\[ |\partial_t\{w(t, x) - \sum_{m=1}^{2} \epsilon r^{-1}\mathcal{F}_m[f, g](r - c_m t, \omega)\}|_k \leq C\epsilon(1+t+r)^{-2}W_{-1}(t, r),\]

or

\[ |\partial_t\{\tilde{p}(t, x)\omega - \mathcal{F}_1[f, g](r - c_1 t, \omega)\}|_k \leq C(1+t+r)^{-1}W_{-1}(t, r),\]

because of (5.12) and (5.20). We see from (4.11) and (1.9) that the above estimate follows from

\begin{equation}
(5.27) \quad |\partial_t\{\tilde{p}(t, x) - p_0(r - c_1 t, \omega)\}|_k \leq C(1+t+r)^{-1}W_{-1}(t, r).
\end{equation}

It follows from (5.10), (5.11), (5.2), and (5.5) with \(N = 1\) that the left hand side of (5.27) is bounded by

\[ C \sum_{|\gamma| \leq k} |\tilde{\Lambda}_\gamma \partial_t p(t, x) - (\Lambda_\gamma \partial_t p_0)(r - c_1 t, \omega)| \\
+ C\epsilon(1+t+r)^{-1}\sum_{|\beta| \leq k} |\tilde{\Lambda}_\beta \partial_t p(t, x)|
\leq C\epsilon ((\log(\epsilon t))(1 + |r - c_1|)^{-1} + (1+t+r)^{-1}),
\]

which yields (5.27), because \(t \leq 2/\epsilon\) implies \(\epsilon \leq C(1+t+r)^{-1}\).

Similarly, one can show (5.18) by using (4.14), (5.4) instead of (4.15), (5.5), respectively. This completes the proof.
Next we examine how well \( w_1(t, x) \) satisfies the original equation (1.1) near the characteristic cones \( r = c_i t \) \((i = 1, 2)\). We set

(5.28) \[ E[u](t, x) = (\partial_t^2 - L)u(t, x) - F(\nabla u(t, x), \nabla^2 u(t, x)). \]

**Lemma 5.3.** Let \( 0 < \tau_0 < \tau_* \) and let \( 0 < \epsilon \leq 1 \). Then for any nonnegative integer \( k \), there exists a positive constant \( C = C(\tau_0, k) \) such that

(5.29) \[
|E[w_1](t, x) - (c_1^2 - c_2^2)\epsilon r^{-2}\{\omega \wedge \partial_s^2 p(t, x) + (\partial_s Y)(r - c_2 t, \omega)\omega + A_2 \text{grad} |\text{rot} w_1(t, x)|^2\}|_k \leq C\epsilon(1 + t + r)^{-3},
\]

for \( c_2 t/2 \leq |x| \leq c_1 t + R \) and \( 1 \leq t \leq \exp(\tau_0/\epsilon) \). Here we have set \( Y(s, \omega) = 2\omega \cdot \tilde{\mathcal{R}}_2[f, g](s, \omega) + \Omega \cdot \tilde{\mathcal{R}}_2[f, g](s, \omega) \).

**Proof.** Let \( c_2 t/2 \leq |x| \leq c_1 t + R \) and \( 1 \leq t \leq \exp(\tau_0/\epsilon) \). Then, \( t \) and \( r \) are equivalent to \( 1 + t + r \).

It holds that

\[
\partial_t^2 \tilde{p}(t, x) = c_1^2 \partial_s^2 p(t, x) - 2c_1 \epsilon t^{-1} \partial_r \partial_s p + t^{-2}(\epsilon^2 \partial_t^2 \tilde{p} - \epsilon \partial_r \tilde{p}),
\]

\[
\Delta(r^{-1} \tilde{p}(t, x) \omega) = r^{-1} \partial_s^2 p(t, x) \omega + r^{-3} \Delta_\omega(\tilde{p}(t, x) \omega),
\]

\[
\text{grad div}(r^{-1} \tilde{p}(t, x) \omega) = r^{-1} \partial_s^2 p(t, x) \omega - r^{-2} \omega \wedge \partial_s p(t, x)
\]  
\[-2r^{-3} \tilde{p}(t, x) \omega - r^{-3} \omega \wedge \tilde{p}(t, x),
\]

where \( \Delta_\omega = \sum_{j=1}^{3} O_j^2 \). Therefore, we have

(5.30) \[
| (\partial^2_t - L)(\epsilon r^{-1} \tilde{p}(t, x) \omega) + 2c_1 \epsilon^2 (tr)^{-1} \partial_r \partial_s p(t, x) \omega
\]  
\[-(c_1^2 - c_2^2)\epsilon r^{-2} \omega \wedge \partial_s p(t, x) |_k \leq C\epsilon(1 + r + t)^{-3}.
\]

While, we have

(\( \partial_t^2 - c_2^2 \Delta \))(r^{-1} \mathcal{F}_2[f, g](r - c_2 t, \omega)) = -c_2^2 r^{-3} \Delta_\omega \mathcal{F}_2[f, g](r - c_2 t, \omega),
\]

Hence, recalling (5.24), (5.10), and (5.11), we obtain

(5.31) \[
| (\partial^2_t - L)(\epsilon r^{-1} \mathcal{F}_2[f, g](r, \omega))
\]  
\[-(c_1^2 - c_2^2)\epsilon r^{-2} (\partial_s Y)(r - c_2 t, \omega) \omega |_k \leq C\epsilon(1 + r + t)^{-3}.
\]

Next we consider the nonlinear term. It follows from (5.16), (5.17) that

\[
|\text{rot}(\text{div} w_1(t, x))\text{rot} w_1(t, x))|_k \leq C\epsilon^2(1 + t + r)^{-3}.
\]

By using (2.1), we get from (5.14)

\[
|N(w_1(t, x))|_k \leq C\epsilon^2(1 + t + r)^{-3}.
\]
We see from (5.23), (5.24) that
\[ |\text{grad} (\text{div} w_1)^2 - \text{grad} (\varepsilon r^{-1} \tilde{\partial}_s p(t, x))^2|_k \leq C\varepsilon^2 (1 + t + r)^{-3}, \]
and hence
\[ |\text{grad} (\text{div} w_1)^2 - 2\varepsilon^2 r^{-2} \tilde{\partial}_s p(t, x) \tilde{\partial}_s^2 p(t, x) \omega|_k \leq C\varepsilon^2 (1 + t + r)^{-3}. \]
Thus we obtain
\[(5.32)\]
\[ |F(\nabla w_1, \nabla^2 w_1) - A_2 \text{grad} \text{rot} w_1|^2 - 2A_1 \varepsilon^2 r^{-2} \tilde{\partial}_s p(t, x) \tilde{\partial}_s^2 p(t, x) \omega|_k \leq C\varepsilon^2 (1 + t + r)^{-3}. \]
Observe that (5.2) and (5.3) (with $N = 1/2$) yield
\[(5.33)\]
\[ |\tilde{\partial}_s p(t, x)|_k \leq C(1 + |c_1 t - r|)^{-1/2}. \]
By (5.6) with $P = \partial_s p$, and (5.33), we obtain
\[(5.34)\]
\[ |2c_1 \varepsilon^2 (tr)^{-1} \tilde{\partial}_t \tilde{\partial}_s p + 2A_1 \varepsilon^2 r^{-2} \tilde{\partial}_s p \tilde{\partial}_s^2 p|_k \leq C\varepsilon^2 (1 + t + r)^{-3}. \]
Now (5.30), (5.31), (5.32), and (5.34) imply (5.29). This completes the proof. \(\square\)

In order to eliminate $r^{-2}\{\omega \wedge \tilde{\partial}_s p(t, x) + (\partial_s Y)(r - c_2 t, \omega)\omega\}$ in the estimate (5.29), we need to construct a more precise approximation. For this reason, we set
\[(5.35)\]
\[ q_1(s, \theta, \tau) = \int_s^\infty \theta \wedge (o p)(s', \theta, \tau) ds', \quad q_2(s, \theta) = \int_s^\infty Y(s', \theta) \theta ds', \]
and define
\[(5.36)\]
\[ w(t, x) = w_1(t, x) + \varepsilon r^{-2} (\tilde{q}_1(t, x) + q_2(r - c_2 t, \omega)) \]
for $(t, x) \in [1/\varepsilon, \exp(\tau_*/\varepsilon)) \times (\mathbb{R}^3 \setminus \{0\})$. Then, $w$ enjoys the same estimates as in Corollary 5.2 together with a suitable estimates for $E[w]$ as follows.

**Lemma 5.4.** Let $0 < \tau_0 < \tau_*$. We assume that $0 < \varepsilon \leq 1$. Then for any nonnegative integer $k$, there exists a positive constant $C = C(\tau_0, k)$
such that

\[(5.37)\quad |w(t, x)|_k \leq C\epsilon(1 + t + r)^{-1},\]

\[(5.38)\quad |\partial w(t, x)|_k \leq C\epsilon(1 + t + r)^{-1}W_{-1}(t, r),\]

\[(5.39)\quad |\text{div} w(t, x)|_k \leq C\epsilon(1 + t + r)^{-1}(1 + |r - c_1 t|)^{-1},\]

\[(5.40)\quad |\text{rot} w(t, x)|_k \leq C\epsilon(1 + t + r)^{-1}(1 + |r - c_2 t|)^{-1},\]

\[(5.41)\quad |E[w](t, x) + A_2 \text{grad}|\text{rot} w_1(t, x)|^2|_k \leq C\epsilon(1 + t + r)^{-3}\]

for \(c_2 t/2 \leq |x| \leq c_1 t + R\) and \(1 \leq t \leq \exp(\tau_0/\epsilon)\). Moreover, we have

\[(5.42)\quad |w(t, x) - \epsilon u_0(t, x)|_k \leq C\epsilon(1 + t + r)^{-2},\]

\[(5.43)\quad |\partial_t \{w(t, x) - \epsilon u_0(t, x)\}|_k \leq C\epsilon(1 + t + r)^{-2}W_{-1}(t, r),\]

for \(c_2 t/2 \leq |x| \leq c_1 t + R\) and \(1/\epsilon \leq t \leq 2/\epsilon\). Here, \(u_0\) is the solution of the Cauchy problem \((4.1)-(4.2)\).

**Proof.** Since \(p(s, \theta, \tau) = 0\) for \(|s| \geq R\), we see from \((5.2), (5.3),\) and \((3.4)\) that

\[(5.44)\quad |\Lambda^\beta q_1(s, \theta, \tau)| \leq C, \quad |\Lambda^\beta \partial_s q_1(s, \theta, \tau)| \leq C(1 + s)^{-1},\]

\[(5.45)\quad |\Lambda^\beta q_2(s, \theta)| \leq C, \quad |\Lambda^\beta \partial_s q_2(s, \theta)| \leq C(1 + s)^{-1},\]

for multi-indices \(\beta\) and \((s, \theta, \tau) \in \mathbb{R} \times S^2 \times [0, \tau_0]\). Therefore, if we set

\[w_2(t, x) = \epsilon r^{-2} (\tilde{q}_1(t, x) + q_2(r - c_2 t, \omega)),\]

then we get

\[(5.46)\quad |w_2(t, x)|_k \leq C\epsilon(1 + t + r)^{-2},\]

\[(5.47)\quad |\partial w_2(t, x)|_k \leq C\epsilon(1 + t + r)^{-2}W_{-1}(t, r),\]

so that the estimates in Lemma 5.4 except for \((5.41)\) immediately follow from Corollary 5.2.

In order to show \((5.41)\), we write

\[(5.48)\quad E[w] + A_2 \text{grad} |\text{rot} w_1|^2\]

\[= (E[w_1] + (c_1^2 - c_2^2)\epsilon r^{-2}\{\partial^2_s q_1(t, x) + (\partial^2_s q_2)(r - c_2 t, \omega)\})\]

\[+ A_2 \text{grad} |\text{rot} w_1|^2\]

\[+ \left((\partial_t^2 - L)w_2 - (c_1^2 - c_2^2)\epsilon r^{-2}\{\partial^2_s q_1(t, x) + (\partial^2_s q_2)(r - c_2 t, \omega)\}\right)\]

\[+ (F(\nabla w_1, \nabla^2 w_1) - F(\nabla w, \nabla^2 w))\]
By (5.15), (5.47) we get

\begin{equation}
|F(\nabla w_1, \nabla^2 w_1) - F(\nabla w, \nabla^2 w)|_k \leq C\varepsilon^2 (1 + t + r)^{-3}.
\end{equation}

Using (5.44), we find

\begin{equation}
|\partial_t^2 - \partial_s^2 q_1(t, x)| - (c_1^2 - c_2^2)\partial_s^2 q_1(t, x)|_k \leq C(1 + t + r)^{-3}.
\end{equation}

Since \( \theta \cdot q_1(s, \theta) = 0 \), we have dvi \( (r^{-2}q_1(t, x)) = -r^{-3}\Omega \cdot q_1(t, x) \) by (2.1). Therefore, we get

\begin{equation}
|\partial_t^2 - \partial_s^2 q_1(t, x)| - (c_1^2 - c_2^2)\partial_s^2 q_1(t, x)|_k \leq C(1 + t + r)^{-3}.
\end{equation}

While, we have from (5.45)

\begin{equation}
|\partial_t^2 - \partial_s^2 q_2(r - c_2t, \omega)|_k \leq C(1 + t + r)^{-3}.
\end{equation}

Since dvi \( (r^{-2}q_2(r - c_2t, \omega)) = -r^{-2}Y(r - c_2t, \omega) \), we obtain

\begin{equation}
|\partial_t^2 - \partial_s^2 q_2(r - c_2t, \omega)| - (c_1^2 - c_2^2)r^{-2}\partial_s^2 q_2(r - c_2t, \omega)|_k \leq C(1 + t + r)^{-3}.
\end{equation}

Now, in view of (5.48), we see from (5.49), (5.50), (5.51), and (5.29) that (5.41) holds, because \( \partial_s^2 q_1(t, x) + (c_1^2 - c_2^2)\partial_s^2 q_2(t, x) = \omega \wedge \overline{\partial_s p}(t, x) + (\partial_s Y)(r - c_2t, \omega) \). This completes the proof. \( \square \)

Now we are in a position to construct an approximate solution \( u_1 \) for all \((t, x) \in [0, \exp(\tau_*/\varepsilon)) \times (\mathbb{R}^3 \setminus \{0\})\): Let \( \chi \) and \( \xi \) be smooth and nonnegative functions on \([0, \infty)\) such that

\[
\chi(s) = \begin{cases} 
1, & s \leq 1, \\
0, & s \geq 2,
\end{cases}
\xi(s) = \begin{cases} 
0, & s \leq c_2/2, \\
1, & s \geq 3c_2/4.
\end{cases}
\]

Let \( 0 < \varepsilon \leq 1 \) in the following. We put \( \chi_\varepsilon(t) = \chi(\varepsilon t) \) and \( \eta(t, x) = \xi(|x|/t) \). Since

\begin{equation}
\varepsilon \leq C(1 + t)^{-1} \quad \text{if } 0 \leq \varepsilon t \leq 2,
\end{equation}

we get

\begin{equation}
\left| \frac{d^m \chi_\varepsilon(t)}{dt^m} \right| = \varepsilon^m \left| \frac{d^m \chi(t)}{dt^m} \right| \leq C(1 + t)^{-m} \quad \text{for } t \geq 0,
\end{equation}

where \( m \) is a nonnegative integer. While, we easily have \( O_j \eta(t, x) = 0 \) for \( 1 \leq j \leq 3 \). Since \( c_2 t/2 \leq r \leq 3c_2 t/4 \) for \((t, x) \in \text{supp} \partial \eta\), we have

\begin{equation}
\sum_{|\alpha| = m} |\partial^\alpha \eta(t, x)| \leq C(1 + t + r)^{-m} \quad \text{for } (t, x) \in [1, \infty) \times \mathbb{R}^3,
\end{equation}

where \( m \) is a nonnegative integer.
where $m$ is a nonnegative integer, $\partial = (\partial_t, \nabla_x)$, and $\alpha$ is a multi-index. Besides, we get

$$W_{-1}(t, r) \leq C(1 + t + r)^{-1} \quad \text{if } 0 \leq r \leq 3c_2 t/4.$$  \hspace{1cm} (5.55)

Let $u_0$ be the solution of the Cauchy problem (4.1)-(4.2), and let $w$ be given by (5.36). We define

$$u_1(t, x) = \chi_\epsilon(t) \epsilon u_0(t, x) + (1 - \chi_\epsilon(t)) \eta(t, x) w(t, x)$$  \hspace{1cm} (5.56)

for $(t, x) \in [0, \exp(\tau_*/\epsilon)) \times \mathbb{R}^3$. By (5.1) and the property of finite propagation, we have $|x| \leq c_1 t + R$ in $\text{supp} u_0$. Hence, recalling (5.13), we find that

$$u_0(t, x) = w(t, x) = u_1(t, x) = 0 \quad \text{for } |x| \geq c_1 t + R.$$  \hspace{1cm} (5.57)

Then we have the following:

**Proposition 5.5.** Let $0 < \tau_0 < \tau_*$, $k$ be a nonnegative integer, $0 \leq \lambda \leq 1/2$, $0 < \mu \leq 1/4$, and $0 < \epsilon \leq 1$. Then there exists a positive constant $C = C(\tau_0, k, \lambda, \mu)$ such that

$$|u_1(t, x)|_k \leq C\epsilon(1 + t)W_{-1}(t, r)^{-1},$$  \hspace{1cm} (5.58)

$$|\partial u_1(t, x)|_k \leq C\epsilon(1 + t + r)^{-1}W_{-1}(t, r),$$  \hspace{1cm} (5.59)

$$|E[u_1](t, x)|_k \leq C\epsilon^{1+\lambda}(1 + t + r)^{-2+\lambda-\mu}W_{-1+\mu}(t, r)$$  \hspace{1cm} (5.60)

for $(t, x) \in [0, \exp(\tau_0/\epsilon)] \times \mathbb{R}^3$.

For $t \in [0, \exp(\tau_0/\epsilon)]$, and

$$\|\|E[u_1](t, \cdot)|_k\|_{L^2} \leq C\epsilon^{1+\lambda}(1 + t)^{-(3/2)+\lambda}$$  \hspace{1cm} (5.61)

Next we consider (5.62) and (5.63). If we set

$$v(t, x) = \eta(t, x) w(t, x) - \epsilon u_0(t, x),$$  \hspace{1cm} (5.63)

then we have $u_1 = \epsilon u_0 + (1 - \chi_\epsilon) v$ by (5.56). Therefore, it follows that

$$E[u_1] = I_0 + I_1 + I_2 + I_3$$  \hspace{1cm} (5.64)
where we put
\[ I_0 = -\chi_{\epsilon}(t)F(\nabla u_1, \nabla^2 u_1), \]
\[ I_1 = -\chi''_{\epsilon}(t)u(t, x), \]
\[ I_2 = -2\chi'_{\epsilon}(t)\partial_t v(t, x), \]
\[ I_3 = (1 - \chi_{\epsilon}(t)) \{(\partial_{t}^{2} - L)(\eta(t, x)w(t, x)) - F(\nabla u_1, \nabla^2 u_1)\}. \]

We will estimate \( I_j \) for \( 0 \leq j \leq 3 \). Let \( 0 \leq \lambda \leq 1/2 \) and \( 0 < \mu \leq 1/4 \) in the following.

By (5.52) and (5.57), we have
\[ \epsilon \leq C(1 + t + r)^{-1} \quad \text{for} \quad (t, x) \in \text{supp} \, I_0 \cup \text{supp} \, I_1 \cup \text{supp} \, I_2. \]

From (5.59) and (5.65) we get
\[ |I_0|_k \leq C\epsilon^2(1 + t + r)^{-2}W_{-2}(t, r), \]
\[ \leq C\epsilon^{1+\lambda}(1 + t + r)^{-3+\lambda}W_{-2}(t, r), \]
which yields
\[ |||I_0|||_k \leq C\epsilon^{1+\lambda}(1 + t)^{-2+\lambda}. \]

Next we estimate \( I_1 \). We may assume \( t \geq 1 \), because \( \epsilon t \geq 1 \) in \( \text{supp} \chi''_{\epsilon} \). Therefore, (5.54), (5.55) and (5.62) yield
\[ |(1 - \eta(t, x))u_0(t, x)|_k \leq C(1 + t + r)^{-2}. \]

Observe that we have \( 1/\epsilon \leq t \leq 2/\epsilon \) and \( c_2 t/2 \leq r \) in \( \text{supp}(\chi''_{\epsilon} \eta) \). Thus, writing \( I_1 = -\epsilon^2 \chi''(\epsilon t)(\eta(w - \epsilon u_0) - \epsilon(1 - \eta)u_0) \), by (5.42), (5.68), and (5.65), we get
\[ |I_1|_k \leq C\epsilon^3(1 + t + r)^{-2} \leq C\epsilon^2(1 + t + r)^{-3}. \]

In order to evaluate \( I_2 \), we use
\[ |(1 - \eta(t, x))\partial_t u_0(t, x)|_k \leq C(1 + t + r)^{-3}, \]
which follows from (5.54), (5.55) and (4.9). Then, writing
\[ I_2 = -2\epsilon\chi'(\epsilon t)((\partial_t \eta)w - \epsilon u_0) + (\partial_t \eta)\epsilon u_0 \]
\[ + \eta(\partial_t w - \epsilon \partial_t u_0) - (1 - \eta)\epsilon \partial_t u_0, \]
by (5.42), (5.43), (5.54), (5.62), and (5.70) that
\[ |I_2|_k \leq C\epsilon^2(1 + t + r)^{-2}W_{-1}(t, r). \]

By (5.69), (5.71), and (5.65) we get
\[ |I_1 + I_2|_k \leq C\epsilon^{1+\lambda}(1 + t + r)^{-3+\lambda}W_{-1}(t, r), \]
\[ |||I_1 + I_2|||_k \leq C\epsilon^{1+\lambda}(1 + t)^{-2+\lambda}. \]

Next we consider \( I_3 \) by rewriting it as
\[ I_3 = (1 - \chi_{\epsilon}(t))(I_{31} + I_{32} + I_{33}), \]
where we have set
\[
I_{31} = - F(\nabla u_1, \nabla^2 u_1) + \eta F(\nabla w, \nabla^2 w),
\]
\[
I_{32} = [\partial_t^2 - L, \eta] w
\]
\[
I_{33} = \eta((\partial_t^2 - L)w - F(\nabla w, \nabla^2 w)).
\]

In the following, we assume \( t \geq 1 \), because \( \epsilon t \geq 1 \) in \( \text{supp}(1 - \chi_\epsilon) \).

We first estimate \( I_{31} \). We may assume \( \epsilon t \leq 2 \) or \( r \leq 3c_2 t/4 \), because \( I_{31} = 0 \) otherwise. If \( 0 \leq \epsilon t \leq 2 \), then we have (5.65) in \( \text{supp} u_1 \cup \text{supp} w \). Therefore, by (5.38) and (5.59), we get
\[
|(1 - \chi_\epsilon)I_{31}|_k \leq C\epsilon^{1+\lambda}(1+t+r)^{-3+\lambda}W_{-2}(t, r),
\]
similarly to (5.66). While, if \( r \leq 3c_2 t/4 \), then (5.38), (5.59), and (5.55) yield
\[
|(1 - \chi_\epsilon)I_{31}|_k \leq C\epsilon^2(1+t+r)^{-4}.
\]

Summing up, we have proved
\[
|(1 - \chi_\epsilon)I_{31}|_k \leq C\epsilon^{1+\lambda}(1+t+r)^{-3+\lambda}W_{-2}(t, r) + C\epsilon^2(1+t+r)^{-4}.
\]

By (5.37), (5.38), (5.54) with \( m = 1, 2 \), and (5.55), we get
\[
|(1 - \chi_\epsilon)I_{32}|_k \leq C\epsilon(1+t+r)^{-3}.
\]

From (5.41), we have
\[
|(1 - \chi_\epsilon)I_{33}|_k \leq C\epsilon(1+t+r)^{-3}.
\]
Thus, (5.75), (5.76), and (5.77) lead to
\[
|I_3|_k \leq C\epsilon(1+t+r)^{-3} + C\epsilon^{1+\lambda}(1+t+r)^{-3+\lambda}W_{-2}(t, r).
\]

Since \( \epsilon t \geq 1 \) in \( \text{supp} I_3 \), we have \( \epsilon \geq t^{-1} \geq (1+t+r)^{-1} \). Hence, we get
\[
|I_3|_k \leq C\epsilon^{1+\lambda}(1+t+r)^{-3+\lambda},
\]
which yields
\[
||I_3||_{L^2} \leq C\epsilon^{1+\lambda}(1+t)^{-(3/2)+\lambda}.
\]

Finally (5.60) follows from (5.66), (5.72), and (5.79). We also obtain (5.61) from (5.67), (5.73), and (5.80). This completes the proof. \( \square \)
6. OUTLINE OF THE PROOF OF THEOREM 1.1

We assume that $0 < \epsilon \leq 1$ and that (5.1) holds for some $R > 1$. Let $u_1(t, x)$ be the approximation defined by (5.56) for $(t, x) \in [0, \exp(\tau_* / \epsilon)) \times \mathbb{R}^3$. If we set

$$u_2(t, x) = u(t, x) - u_1(t, x),$$

then (1.1)-(1.2) is reduced to

\begin{align*}
(6.1) & \quad (\partial_t^2 - L)u_2 = H(u_1, u_2) - E[u_1] \quad \text{in} \quad [0, \exp(\tau_* / \epsilon)) \times \mathbb{R}^3, \\
(6.2) & \quad u_2(0, x) = (\partial_t u_2)(0, x) = 0 \quad \text{for} \quad x \in \mathbb{R}^3,
\end{align*}

where $E[u]$ is defined by (5.28), and $H(u_1, u_2)$ is given by

$$H(u_1, u_2) = F(\nabla(u_1 + u_2), \nabla^2(u_1 + u_2)) - F(\nabla u_1, \nabla^2 u_1).$$

Observe that for any nonnegative integer $k$, there exists a constant $C_k$ such that

$$\sup_{x \in \mathbb{R}^3} |u_2(0, x)|_k \leq C_k \epsilon^2,$$

because for $0 \leq t \leq \epsilon^{-1}$ and $x \in \mathbb{R}^3$, we have

$$(\partial_t^2 - L)u_2 = F(\nabla(u_1 + u_2), \nabla^2(u_1 + u_2)).$$

$u_2(0, x) = \partial_t u_2(0, x) = 0$, and $u_1(t, x) = \epsilon u_0(t, x)$ by (5.56). Therefore, by the local existence theorem (see [7]), what we need for proving Theorem 1.1 is to establish a suitable a-priori estimate. More explicitly, for $0 < T < \max\{T_\epsilon, \exp(\tau_0 / \epsilon)\}$ with $\tau_0 \in (0, \tau_*)$, we wish to evaluate the following quantity:

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \left\{ (1 + r)(W_{-1}^1(t, r))^{-1} |\partial u_2(t, x)|_K + (1 + r)(1 + |c_1 t - r|) |\text{div} u_2(t, x)|_K + (1 + r)(1 + |c_2 t - r|) |\text{rot} u_2(t, x)|_K \right\},$$

provided $K$ is an integer large enough and $\epsilon$ is small enough.

In order to carry out this purpose, we employ (4.17) for estimating $E[u_1]$ and (4.18) for evaluating $H(u_1, u_2)$, respectively. Note that (5.60), (5.61) enable us to regard $E[u_1]$ as a harmless term. In addition, when $T < \exp(\tau_0 / \epsilon)$, we see that $0 \leq t \leq T$ implies $\epsilon \log(2 + t) \leq C(1 + \tau_0)$. Hence, one can develop the argument as in [1], and find that Theorem 1.1 is valid.
APPENDIX: DERIVATION OF (1.1)

In this appendix we derive the quadratically perturbed wave equation (1.1) as the Euler-Lagrange equation of the following lagrangian:

(A.1) \[ I(u) = \iint_{\mathbb{R}^{1+3}} \left\{ \frac{1}{2} |\partial_t u|^2 - W(\epsilon(u)) \right\} dx dt, \]

where \( u = u(t, x) \) is the displacement vector, \( W(\epsilon(u)) \) is the strain energy, and

(A.2) \[ \epsilon(u) = \frac{1}{2} \left( (\nabla \otimes u) + \epsilon(\nabla \otimes u) \right) = \epsilon_{ij}(u) \]

with \( \epsilon_{ij}(u) = (\partial_i u_j + \partial_j u_i)/2 \) for \( i, j = 1, 2, 3 \). We underline that one can obtain the same equation as in Agemi [1]. Sideris [14], although our choice of the strain tensor \( \epsilon(u) \) is just the linear approximation of

\[ \tilde{\epsilon}(u) = \{I + \nabla u)(I + \nabla u\}^{1/2} - I, \]

used in [1]. [14].

Since we assumed that the elastic body is isotropic, the strain energy \( W(\epsilon(u)) \) is a function of the principal invariants \( \alpha(u), \beta(u), \) and \( \gamma(u) \) which are explicitly given by

(A.3) \[ \alpha(u) = \epsilon_{11}(u) + \epsilon_{22}(u) + \epsilon_{33}(u) = \text{div } u, \]

(A.4) \[ \beta(u) = \epsilon_{11}(u)\epsilon_{22}(u) + \epsilon_{22}(u)\epsilon_{33}(u) + \epsilon_{33}(u)\epsilon_{11}(u) \]
\[ - ( (\epsilon_{13}(u))^2 + (\epsilon_{32}(u))^2 + (\epsilon_{21}(u))^2 ) \]
\[ = - \frac{1}{4} |\text{rot } u|^2 + Q_{13}(u_1, u_3) + Q_{32}(u_3, u_2) + Q_{21}(u_2, u_1) \]

(A.5) \[ \gamma(u) = \det \epsilon(u) \]

where for scalar functions \( \phi \) and \( \psi \), we put

(A.6) \[ Q_{ij}(\phi, \psi) = (\partial_i \phi)(\partial_j \psi) - (\partial_j \phi)(\partial_i \psi) \quad (i, j = 1, 2, 3). \]

If we assume that \( W(\epsilon(u)) \) is of cubic order with respect to \( u \), then it is expressed as

(A.7) \[ W(\epsilon(u)) = W_0(\epsilon(u)) + a(\alpha(u))^3 + b(\alpha(u))^2\beta(u) + c\gamma(u) \]

where \( a, b, \) and \( c \) are constants, while \( W_0(\epsilon(u)) \) is the quadratic part of \( W(\epsilon(u)) \) defined by

(A.8) \[ W_0(\epsilon(u)) = \frac{1}{2}(\lambda + 2\mu)(\alpha(u))^2 - 2\mu\beta(u) \]

with the Lamé constants \( \lambda \) and \( \mu \).
The variational principle tells us that if \( u \) describes the phenomenon associated with the lagrangian \( I(u) \), then it must satisfy
\[
(A.9) \quad \lim_{\eta \to 0} \eta^{-1}\{I(u + \eta \varphi) - I(u)\} = 0
\]
for any \( \varphi = t(\varphi_1, \varphi_2, \varphi_3) \in C_0^\infty(\mathbb{R}^{1+3}) \). We shall show that (A.9) implies
\[
(A.10) \quad \partial_t^2 u_i - (\mu \Delta u_i + (\lambda + \mu) \partial_i \text{div} u) - A_1 \partial_i (\text{div} u)^2
\]
\[
- A_2 \partial_i \|\text{rot} u\|^2 - A_3 \left\{ \left( \text{div} u \right) \left( \partial_i \text{div} u - \Delta u_i \right)
\right.
\]
\[
+ \left( \partial_2 \text{div} u \right) \left( \partial_2 u_1 - \partial_1 u_2 \right) - \left( \partial_3 \text{div} u \right) \left( \partial_3 u_1 - \partial_1 u_3 \right)
\]
\[
+ N_i(u) = 0 \quad (i = 1, 2, 3),
\]
where \( N_i(u) \) is a linear combination of null-forms \( Q_{kl} \) defined by (A.6).

Since
\[
(A.11) \quad \text{rot} \left( \left( \text{div} u \right) (\text{rot} u) \right) = \left( \text{div} u \right) \text{rot} (\text{rot} u) + (\text{grad} \text{div} u) \wedge \text{rot} u,
\]
\[
(A.12) \quad \text{rot} (\text{rot} u) = \text{grad} \text{div} u - \Delta u,
\]
we find (1.1) from (A.10) by setting \( c_1^2 = \lambda + 2\mu, \ c_2^2 = \mu, A_1 = 3a, A_2 = -b/4, \) and \( A_3 = b/2 \).

For simplicity, we shall write \( f \prec g \) if there exist \( h_i \ (i = 1, 2, 3) \) such that \( f(x) - g(x) = \sum_{i=1}^3 \partial_i h_i(x) \). In order to prove (A.10) for \( i = 1 \), we take \( \varphi = t(\varphi_1, 0, 0) \) in the following.

Since \( \epsilon_{ij}(u) \) is linear in \( u \), we see from (A.3) that \( \alpha(u) \) is also linear functional, and hence we get
\[
(A.13) \quad \lim_{\eta \to 0} \eta^{-1}\{(\alpha(u + \eta \varphi))^2 - (\alpha(u))^2\} = 2\alpha(u)\alpha(\varphi)
\]
\[
= 2(\partial_1 \varphi_1)(\text{div} u) \prec -2\varphi_1 \partial_1 (\text{div} u).
\]

From (A.4) we get
\[
\beta(u + \eta \varphi) = -\frac{1}{4} \text{rot} u + \eta \text{rot} \varphi|^2 + Q_{13}(u_1 + \eta \varphi_1, u_3)
\]
\[
+ Q_{32}(u_3, u_2) + Q_{21}(u_2, u_1 + \eta \varphi_1).
\]

Therefore, we obtain
\[
(A.14) \quad \lim_{\eta \to 0} \eta^{-1}\{\beta(u + \eta \varphi) - \beta(u)\}
\]
\[
= -\frac{1}{2} \left( (\partial_3 u_1)(\partial_3 \varphi_1) + (\partial_2 u_1)(\partial_2 \varphi_1) \right)
\]
\[
+ (\partial_2 u_2 + \partial_3 u_3)(\partial_1 \varphi_1) - \frac{1}{2}(\partial_1 u_3)(\partial_3 \varphi_1) - \frac{1}{2}(\partial_1 u_2)(\partial_2 \varphi_1)
\]
\[
\prec -\frac{1}{2} \varphi_1 (\partial_1 \text{div} u - \Delta u_1).
\]
Thus we find from (A.13) and (A.14) that

(A.15) \[ \lim_{\eta \to 0} \eta^{-1} \{ W_0(u + \eta \varphi) - W_0(u) \} \]
\[ \cong - \varphi_1 (\mu \Delta u_1 + (\lambda + \mu) \partial_1 \text{div} u). \]

In particular, when \( a = b = c = 0 \), we obtain the homogeneous elastic wave equation (4.1) from (A.9).

Next we consider the higher order terms in \( W(\varepsilon(u)) \). It is easy to see that

(A.16) \[ \lim_{\eta \to 0} \eta^{-1} \{ (\alpha(u + \eta \varphi))^3 - (\alpha(u))^3 \} \cong -3 \varphi_1 \partial_1 (\text{div} u)^2. \]

It follows that

\[ \lim_{\eta \to 0} \eta^{-1} \{ \alpha(u + \eta \varphi)\beta(u + \eta \varphi) - \alpha(u)\beta(u) \} \]
\[ = \alpha(u) \lim_{\eta \to 0} \eta^{-1} \{ \beta(u + \eta \varphi) - \beta(u) \} + \alpha(\varphi) \lim_{\eta \to 0} \beta(u + \eta \varphi). \]
\[ \cong - \frac{1}{2} \alpha(u)(\partial_1 \text{div} u - \Delta u_1)\varphi_1 + \frac{1}{2} (\partial_3 \alpha(u)(\partial_3 u_1) + \partial_2 \alpha(u)(\partial_2 u_1))\varphi_1 \]
\[ - \partial_1 \alpha(u)(\partial_2 u_2 + \partial_3 u_3)\varphi_1 + \frac{1}{2} (\partial_3 \alpha(u)(\partial_1 u_3) + \partial_2 \alpha(u)(\partial_1 u_2))\varphi_1 \]
\[ - \varphi_1 \partial_1 \beta(u), \]

in view of (A.14). Rearranging the terms in the last expression, we get

(A.17) \[ \lim_{\eta \to 0} \eta^{-1} \{ \alpha(u + \eta \varphi)\beta(u + \eta \varphi) - \alpha(u)\beta(u) \} \]
\[ \cong - \frac{1}{2} \varphi_1 \{ \alpha(u)(\partial_1 \text{div} u - \Delta u_1) \]
\[ + (-\partial_3 \alpha(u)(\partial_3 u_1 - \partial_1 u_3) + \partial_2 \alpha(u)(\partial_2 u_1 - \partial_1 u_2)) \}
\[ + \varphi_1 \{ Q_{12}(u_2, \alpha(u)) + Q_{13}(u_3, \alpha(u)) \} \]
\[ + \varphi_1 \partial_1 (\frac{1}{4} |\text{rot} u|^2 - Q_{13}(u_1, u_3) - Q_{32}(u_3, u_2) - Q_{21}(u_2, u_1)). \]

A direct computation shows that

\[ \lim_{\eta \to 0} \eta^{-1} \{ \gamma(u + \eta \varphi) - \gamma(u) \} = \begin{vmatrix} \partial_1 \varphi_1 & \partial_2 \varphi_1 & \partial_3 \varphi_1 \\ \varepsilon_{12}(u) & \varepsilon_{22}(u) & \varepsilon_{23}(u) \\ \varepsilon_{13}(u) & \varepsilon_{23}(u) & \varepsilon_{33}(u) \end{vmatrix} \]
\[ \cong - \varphi_1 \{ \partial_1 (\varepsilon_{22}(u)\varepsilon_{33}(u) - (\varepsilon_{23}(u))^2) \]
\[ + \partial_2 (\varepsilon_{23}(u)\varepsilon_{13}(u) - \varepsilon_{12}(u)\varepsilon_{33}(u)) \]
\[ + \partial_3 (\varepsilon_{23}(u)\varepsilon_{12}(u) - \varepsilon_{22}(u)\varepsilon_{13}(u)) \}, \]
which implies
\begin{equation}
\lim_{\eta \to 0} \eta^{-1} \{ \gamma(u + \eta \varphi) - \gamma(u) \}
= - \varphi_1 \left\{ \partial_1 Q_{23}(u_2, u_3) + \frac{1}{4} \partial_2 (Q_{23}(u_3, u_1) + Q_{31}(u_2, u_3)) 
+ \frac{1}{4} \partial_3 (Q_{12}(u_2, u_3) + Q_{23}(u_1, u_2)) 
+ \frac{1}{4} (Q_{12}(u_3, \partial_2 u_3 - \partial_3 u_2) + Q_{13}(u_2, \partial_3 u_2 - \partial_2 u_3) 
+ Q_{23}(u_2, \partial_1 u_3 + \partial_3 u_1) + Q_{23}(u_3, \partial_2 u_1 + \partial_1 u_2)) \right\}.
\end{equation}

Finally, one can conclude from (A.15), (A.16), (A.17), and (A.18) that (A.9) yields (A.10), and hence (1.1).

REFERENCES