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Singular Casimir Elements of the Euler Equation

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Abstract

Casting the Euler equation of incompressible inviscid fluid into a Hamiltonian formalism, we encounter a singularity in the Poisson bracket; if the symplectic operator defining the bracket has a kernel, the system is said non-canonical; the center of the Lie-Poisson algebra, then, consists of nontrivial members which are called Casimir elements. The nonlinearity of the Euler equation makes the symplectic operator inhomogeneous on the phase space (a Hilbert space of the state variables), and creates a singularity where the nullity of the symplectic operator changes. We have unearthed "singular Casimir elements" stemming in the singularity; the functional derivative of the singular Casimir element is a generalized gradient of "ragged" functional on a Hilbert space.

1 Introduction

We start by reviewing the standard Hamiltonian mechanics, and generalizing it to non-canonical systems, we formulate our problem. A canonical Hamiltonian system of a finite dimension n (here we assume that n is an even number) is endowed with an $n \times n$ antisymmetric regular matrix $J(\boldsymbol{z})$ (assumed to be a holomolphic function on $\boldsymbol{z} \in \mathbb{R}^n$) defining a Poisson bracket $\{\boldsymbol{a}, \boldsymbol{b}\} := (J\boldsymbol{a}, \boldsymbol{b})$, where $(\ ,\ )$ the inner product of the phase space. We assume that the Poisson bracket satisfies Jacobi's identity. Provided by a Hamiltonian $H(\boldsymbol{z})$, the equation of motion is written as

$$\frac{d}{dt}\boldsymbol{z} = J(\boldsymbol{z})\partial_{\boldsymbol{z}} H(\boldsymbol{z}). \tag{1}$$

The fixed point (stationary point) of the dynamics is $\boldsymbol{z}$ such that

$$\partial_{\boldsymbol{z}} H(\boldsymbol{z}) = 0. \tag{2}$$

A *non-canonical* Hamiltonian system [1] allows $J(\boldsymbol{z})$ to have a kernel, i.e. Rank($J(\boldsymbol{z})$) may be less than n (and may vary as a function of $\boldsymbol{z}$). Then, the fixed points may not be the only critical points of the Hamiltonian $H(\boldsymbol{z})$. A *Casimir element* $C(\boldsymbol{z})$ is a non-trivial (non-constant) solution to the differential equation

$$J(\boldsymbol{z})\partial_{\boldsymbol{z}} C(\boldsymbol{z}) = 0. \tag{3}$$

Given such $C(z)$, a transformation $H(z) \mapsto H_\mu(z) = H(z) - \mu C(z)$ (μ is a constant) does not change the dynamics. Thus, a critical point of

$$\partial_z H_\mu(z) \equiv \partial_z[H(z) - \mu C(z)] = 0 \tag{4}$$

will also give a fixed point.

If $\mathrm{Rank}(J(z)) = n$, (3) has only trivial solution ($C(z) =$ constant). If $\mathrm{Rank}(J(z)) = n - 2m$ ($m > 0$ is a constant), (3) has $2m$ independent solutions (Lie-Darboux theorem). The problem becomes more interesting if there is a *singularity* where $\mathrm{Rank}(J(z))$ changes: In this case, have a singular (hyper-function) Casimir element. For example, let us consider one-dimensional system with $J = ix$ ($x \in \mathbb{R}$). At $x = 0$, $\mathrm{Rank}(J)$ drops to 0, which is a singular point of the differential equation $J(x)\partial_x C = 0$. The singular Casimir element is, then, $C(x) = Y(x)$ (Heaviside's step function).

We generalize (1) further to an infinite-dimension space: Let $u \in X$ be the state vector (X is some function space), $\mathcal{J}(u)$ be a linear antisymmetric operator in X (generally depending on u; for a fixed u, $\mathcal{J}(u)$ may be regarded as a linear operator $X \to X$; see Remark 1), and $H(u)$ be a functional $X \to \mathbb{R}$. Introducing an appropriate functional derivative (gradient) $\partial_u H(u)$, we consider an evolution equation of the form

$$\frac{d}{dt}u = \mathcal{J}(u)\partial_u H(u). \tag{5}$$

A Casimir element $C(u)$ (a functional $X \to \mathbb{R}$) is a non-trivial solution to

$$\mathcal{J}(u)\partial_u C(u) = 0. \tag{6}$$

We may solve (6) by two steps:

1. Find the kernel of $\mathcal{J}(u)$, i.e., solve a "linear equation" (cf. Remark 1)

$$\mathcal{J}(u)v = 0 \tag{7}$$

   to determine v for a given u. Let us write the solution as $v(u)$.

2. "Integrate" $v(u)$ with respect to u to find a functional $C(u)$ such that $v(u) = \partial_u C(u)$.

As noticed in the forgoing finite-dimension practice, *step-1* should involve "singular solutions" if $\mathcal{J}(u)$ has *singularities*. And then, *step-2* will be rather nontrivial—for *singular Casimir elements*, we have to generalize the notion of *functional derivatives*. The present effort is devoted to such an extension of the notion of Casimir elements in an infinite-dimensional non-canonical Hamiltonian system. We will invoke the Euler equation of ideal fluid mechanics as an explicit example of infinite-dimensional non-canonical Hamiltonian system [2, 3, 4, 5]; in Sec. 2, we will formulate the determining equation in a rigorously solvable Hamiltonian system [6]. In Sec. 3.1, we will analyze the kernel of the corresponding symplectic operator $\mathcal{J}(u)$ and its singularity. A singular Casimir element and its appropriate generalized functional derivative (gradient) will be given in Sec. 3.2.

Remark 1 *In (5), the operator $\mathcal{J}(u)$ must be evaluated at the common u of $\partial_u H(u)$, thus $\mathcal{J}(u)\partial_u H(u)$ is a nonlinear operator with respect to u. However, the application of $\mathcal{J}(u)$ (or $\mathcal{J}(u)\partial_u$) may be regarded as a linear operator in the sense that $\mathcal{J}(u)(av + bw) = a\mathcal{J}(u)v + b\mathcal{J}(u)w$ (or $\mathcal{J}(u)\partial_u[aF(u) + bG(u)] = a\mathcal{J}(u)\partial_u F(u) + b\mathcal{J}(u)\partial_u G(u)$). Note that $\mathcal{J}(u)v$ (for $v = \partial_u F(u)$) is* not *$\mathcal{J}(v)v$.*

2 Hamiltonian Form of the Euler Equation

2.1 Vorticity Equation

We consider the Euler equation of motion of incompressible inviscid fluid:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p \ \ (\text{in } \Omega), \tag{8}$$

$$\nabla \cdot \boldsymbol{u} = 0 \ \ (\text{in } \Omega), \tag{9}$$

$$\boldsymbol{n} \cdot \boldsymbol{u} = 0 \ \ (\text{on } \Gamma), \tag{10}$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n = 2$ or 3) with a sufficiently smooth ($C^{2+\epsilon}$-class) boundary Γ ($\boldsymbol{n}$ is the unit normal vector onto Γ), $\boldsymbol{u}$ is an n-dimensional vector field representing the flow velocity, and p is a scalar field representing the fluid pressure (or specific enthalpy); all fields are real-valued functions of time t and coordinate $\boldsymbol{x} \in \Omega$.

We may rewrite (8) as

$$\partial_t \boldsymbol{u} = \boldsymbol{u} \times \boldsymbol{\omega} - \nabla \tilde{p} \ \ (\text{in } \Omega), \tag{11}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ (vorticity) and $\tilde{p} = p + u^2/2$ (total specific energy). The curl derivative of (11) reads as the *vorticity equation*

$$\partial_t \boldsymbol{\omega} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) \ \ (\text{in } \Omega). \tag{12}$$

We prepare basic function spaces pertinent to the mathematical formulation of the Euler equation. Let $L^2(\Omega)$ be the Hilbert space of Lebesgue-measurable and square-integrable real vector functions on Ω, which is endowed with the standard inner product $(\boldsymbol{a}, \boldsymbol{b}) = \int_\Omega \boldsymbol{a} \cdot \boldsymbol{b}\, dx$ and the norm $\|\boldsymbol{a}\| = (\boldsymbol{a}, \boldsymbol{a})^{1/2}$. We will also use the standard notation of Sobolev spaces. We define

$$L^2_\sigma(\Omega) = \{\boldsymbol{u} \in L^2(\Omega);\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{n} \cdot \boldsymbol{u} = 0\}, \tag{13}$$

where $\boldsymbol{n} \cdot \boldsymbol{u}$ denotes the trace of the normal component of $\boldsymbol{u}$ onto the boundary Γ, which is a continuous map from $\{\boldsymbol{u} \in L^2(\Omega);\ \nabla \cdot \boldsymbol{u} \in L^2(\Omega)\}$ to $H^{-1/2}(\Gamma)$. We have an orthogonal decomposition

$$L^2(\Omega) = L^2_\sigma(\Omega) \oplus \{\nabla\theta;\ \theta \in H^1(\Omega)\}. \tag{14}$$

Every $\boldsymbol{u} \in L^2_\sigma(\Omega)$ satisfies the conditions (9) and (10), thus we will consider that (11) is an evolution equation in the function space $L^2_\sigma(\Omega)$ (cf. Appendix A).

Hereafter, we assume that the space dimension $n = 2$ and $\Omega \subset \mathbb{R}^2$ is a smoothly bounded and simply connected (genus=0) region [7]. For the convenience of formulating equations, we immerse $\Omega \subset \mathbb{R}^2$ in $\mathbb{R}^3$ with adding a "perpendicular" coordinate z, and write $\boldsymbol{e}_\perp = \nabla z$.

Lemma 1 *Every two-dimensional vector field $\boldsymbol{u}$ satisfying the incompressibility condition (9) and the vanishing normal boundary condition (10) can be written as*

$$\boldsymbol{u} = \nabla\varphi \times \boldsymbol{e}_\perp \tag{15}$$

with a single-value function φ such that $\varphi|_\Gamma = 0$ [8], i.e,

$$L^2_\sigma(\Omega) = \{\nabla\varphi \times \boldsymbol{e}_\perp;\ \varphi \in H^1_0(\Omega)\}. \tag{16}$$

Proof For the convenience of the reader, we sketch the proof of this well-known lemma. Evidently, $\nabla\cdot(\nabla\varphi\times\boldsymbol{e}_\perp) = \nabla\cdot[\nabla\times(\varphi\boldsymbol{e}_\perp)] = 0$, and $\boldsymbol{n}\cdot(\nabla\varphi\times\boldsymbol{e}_\perp) = (\boldsymbol{e}_\perp\times\boldsymbol{n})\cdot\nabla\varphi = 0$ if $\varphi\in H_0^1(\Omega)$. Thus, the linear space $X = \{\nabla\varphi\times\boldsymbol{e}_\perp;\ \varphi\in H_0^1(\Omega)\}$ is contained in $L^2_\sigma(\Omega)$. And, the orthogonal complement of X in $L^2_\sigma(\Omega)$ is only zero vector: Suppose that $\boldsymbol{u}\in L^2_\sigma(\Omega)$ satisfies

$$(\boldsymbol{u}, \nabla\varphi\times\boldsymbol{e}_\perp) = 0 \quad \forall\varphi\in H_0^1(\Omega). \tag{17}$$

By the generalized Stokes formula, we find $(\boldsymbol{u}, \nabla\varphi\times\boldsymbol{e}_\perp) = (\boldsymbol{e}_\perp\cdot\nabla\times\boldsymbol{u}, \varphi)$. Since $\nabla\times\boldsymbol{u}$ has only the $\boldsymbol{e}_\perp$ component, (17) implies $\nabla\times\boldsymbol{u} = 0$. Since $\boldsymbol{u}\in L^2_\sigma(\Omega)$, we also have $\nabla\cdot\boldsymbol{u} = 0$ and $\boldsymbol{n}\cdot\boldsymbol{u} = 0$. In a simply connected Ω, such $\boldsymbol{u}$ is only zero vector. Hence, we have (16). ∎

Using the representation (15), we may formally calculate as

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{u} = (-\Delta\varphi)\boldsymbol{e}_\perp \equiv \omega\boldsymbol{e}_\perp.$$

The vorticity equation (12) simplifies as a single ($\boldsymbol{e}_\perp$-component) equation [9]:

$$\partial_t\omega = \{\omega, \mathcal{K}\omega\} \ \ (\text{in } \Omega), \tag{18}$$

where

$$\{a, b\} = -\nabla a\times\nabla b\cdot\boldsymbol{e}_\perp = \partial_y a\cdot\partial_x b - \partial_x a\cdot\partial_y b,$$

and $\mathcal{K}$ is the inverse map of $-\Delta$ with the Dirichlet boundary condition, i.e., $\mathcal{K}:\ \omega\mapsto\varphi$ gives the solution of the Laplace equation

$$-\Delta\varphi = \omega \ (\text{in } \Omega), \qquad \varphi = 0 \ (\text{on } \Gamma). \tag{19}$$

As well known, $\mathcal{K}:\ L^2(\Omega)\to H_0^1(\Omega)\cap H^2(\Omega)$ is a self-adjoint compact operator. For $\varphi\in H_0^1(\Omega)$, we define $\omega = -\Delta\varphi$ as a member of $H^{-1}(\Omega)$, the dual space of $H_0^1(\Omega)$ with respect to the inner-product of $L^2(\Omega)$. The inverse map (weak solution), then, defines $\mathcal{K}:\ H^{-1}(\Omega)\to H_0^1(\Omega)$.

Theorem 1 *We regard the vorticity equation (18) as an evolution equation in $H^{-1}(\Omega)$, i.e., we consider the weak form:*

$$(\partial_t\omega - \{\omega, \mathcal{K}\omega\}, \phi) = 0 \quad \forall\phi\in H_0^1(\Omega). \tag{20}$$

By the relations $\varphi = \mathcal{K}\omega$, $\boldsymbol{u} = \nabla\varphi\times\boldsymbol{e}_\perp$ and $\boldsymbol{\omega} = \omega\boldsymbol{e}_\perp$, (20) is equivalent to the Euler equation (11) as an evolution equation in $L^2_\sigma(\Omega)$.

Proof In the topology of $L^2_\sigma(\Omega)$, the Euler equation (11) reads as

$$(\partial_t\boldsymbol{u} - \boldsymbol{u}\times\boldsymbol{\omega} + \nabla\tilde{p}, \boldsymbol{v}) = 0 \quad \forall\boldsymbol{v}\in L^2_\sigma(\Omega). \tag{21}$$

By (14), the left-hand side of (21) reduces into $(\partial_t\boldsymbol{u} - \boldsymbol{u}\times\boldsymbol{\omega}, \boldsymbol{v})$. By Lemma 1, we may put $\boldsymbol{v} = \nabla\phi\times\boldsymbol{e}_\perp = \nabla\times(\phi\boldsymbol{e}_\perp)$ with $\phi\in H_0^1(\Omega)$. Plugging this representation into (21), we obtain

$$\begin{aligned}(\partial_t\boldsymbol{u} - \boldsymbol{u}\times\boldsymbol{\omega}, \nabla\times(\phi\boldsymbol{e}_\perp)) &= (\boldsymbol{e}_\perp\cdot\nabla\times(\partial_t\boldsymbol{u} - \boldsymbol{u}\times\boldsymbol{\omega}), \phi) \\ &= (\partial_t\omega - \{\omega, \varphi\}, \phi).\end{aligned}$$

Hence, (21) is equivalent to (20). ∎

2.2 Hamiltonian

Now we cast the vorticity equation (18) —to be precise, its "weak form" (20)— into a Hamiltonian form (for the Hamiltonian form of the Euler equation (11); see Appendix A). First, we formulate the Hamiltonian. A natural choice should be $H = \|\boldsymbol{u}\|^2/2$, the "energy" of the flow $\boldsymbol{u}$. By the relation $\boldsymbol{u} = \nabla\varphi \times \boldsymbol{e}_\perp$, we may rewrite $H = \|\nabla\varphi\|^2/2 = (\varphi, -\Delta\varphi)/2$. Selecting the vorticity ω as the state variable, we define (by relating $\varphi = \mathcal{K}\omega$)

$$H(\omega) = \frac{1}{2}\int_\Omega (\mathcal{K}\omega)\cdot\omega\, dx, \tag{22}$$

which is a continuous functional on $H^{-1}(\Omega)$ (in fact, equivalent to the square of the norm of $H^{-1}(\Omega)$, i.e., the *negative norm* induced by $H_0^1(\Omega)$).

2.3 Gradient in Hilbert Space

Next, we formulate the gradient of a functional in function space. Let $\Phi(x)$ be a functional on a Hilbert space X. A small perturbation $\epsilon\tilde{x} \in X$ ($|\epsilon| \ll 1$, $\|\tilde{x}\| = 1$) will cause a variation $\delta\Phi(x;\tilde{x}) = \Phi(x + \epsilon\tilde{x}) - \Phi(x)$. If there exists $g \in X^* = X$ such that $\delta\Phi(x;\tilde{x}) = \epsilon(g,\tilde{x}) + O(\epsilon^2)$ for every $\tilde{x}$, then we define $\partial_x\Phi(x) = g$, and call it the gradient of $\Phi(x)$. Evidently, the variation $|\delta\Phi(x;\tilde{x})|$ is maximized, at each x, by $\tilde{x} = \partial_x\Phi(x)/\|\partial_x\Phi(x)\|$. The notion of gradient will be extended for a class of "rugged" functionals (to define singular Casimir elements) in Sec. 3.2. As for the Hamiltonian, however, we may assume that the functional is smooth. The pertinent Hilbert space is $L^2(\Omega)$, on which the Hamiltonian $H(\omega)$ is differentiable; using the self-adjointness of $\mathcal{K}$, we obtain

$$\partial_\omega H(\omega) = \mathcal{K}\omega. \tag{23}$$

We note that the gradient $\partial_\omega H(\omega)$ may be evaluated for every $\omega \in H^{-1}(\Omega)$ with the value in $H_0^1(\Omega)$.

2.4 Non-Canonical Symplectic Operator

Finally, we define the non-canonical symplectic operator $\mathcal{J}(\omega)$. Formally, we put

$$\mathcal{J}(\omega)\psi = \left[(\partial_y\omega)\partial_x - (\partial_x\omega)\partial_y\right]\psi = \{\omega,\psi\}. \tag{24}$$

In this representation, ω must be a "differentiable" function. However, we will need to reduce the regularity requirement on ω. We recourse to a *weak formulation* that is amenable to the interpretation of the evolution in $H^{-1}(\Omega)$ (see Theorem 1). Formally, we may calculate as

$$(\mathcal{J}(\omega)\psi,\phi) = (\{\omega,\psi\},\phi) = (\omega,\{\psi,\phi\}). \tag{25}$$

The right-hand side is finite (well-defined) as far as $\omega \in C(\Omega)$ and $\psi,\phi \in H_0^1(\Omega)$. In fact,

$$\begin{aligned} |(\omega,\{\psi,\phi\})| &\le \|\omega\|_{sup}\int_\Omega |\{\psi,\phi\}|\,dx \\ &\le \|\omega\|_{sup}\int_\Omega |\nabla\psi|\,|\nabla\phi|\,dx \\ &\le \|\omega\|_{sup}\|\nabla\psi\|\,\|\nabla\phi\|, \end{aligned}$$

where $||\omega||_{sup} = \sup_{x\in\Omega} |\omega(x)|$. Hence, we may consider that the right-hand side of (25) is a bounded linear functional of $\phi \in H_0^1(\Omega)$, including two parameters ω and ψ; let us denote $(\omega, \{\psi, \phi\}) \equiv F(\omega, \psi; \phi)$. By this functional, we "define" $\mathcal{J}(\omega)\psi$ on the left-hand side of (25) as a member of $H_0^1(\Omega)^* = H^{-1}(\Omega)$, i.e., we put

$$(\mathcal{J}(\omega)\psi, \phi) \equiv F(\omega, \psi; \phi) \quad \forall \phi \in H_0^1(\Omega).$$

For a given $\omega \in C(\Omega)$, we may regard that $\mathcal{J}(\omega)$ is a bounded linear map operating on ψ, i.e., $\mathcal{J}(\omega) : H_0^1(\Omega) \to H^{-1}(\Omega)$. Evidently, $(\mathcal{J}(\omega)\psi, \phi) = -(\psi, \mathcal{J}(\omega)\phi)$, i.e., $\mathcal{J}(\omega)$ is antisymmetric. The bracket defined by this symplectic operator satisfies the Jacobi's identity [1, 4, 5].

2.5 Hamiltonian Form of Vorticity Equation

Combining the forgoing definitions of the Hamiltonian $H(\omega)$, the gradient ∂_ω, and the non-canonical symplectic operator $\mathcal{J}(\omega)$, we can write the vorticity equation (18) in the form of

$$\partial_t \omega = \mathcal{J}(\omega)\partial_\omega H(\omega). \tag{26}$$

As remarked in Theorem 1, (26) is an evolution equation in $H^{-1}(\Omega)$; see Appendix A for an alternative formulation of the Euler equation as a Hamiltonian system governing $\boldsymbol{u}(t)$ in $L^2_\sigma(\Omega)$.

For every fixed $\omega \in C(\Omega)$, $\mathcal{J}(\omega)$ may be regarded as a bounded linear map of $H_0^1(\Omega) \to H^{-1}(\Omega)$ (the bound changes as a function of ω). And $\partial_\omega H(\omega)$ is a bounded linear map of $H^{-1}(\Omega) \to H_0^1(\Omega)$. Hence, each element composing the right-hand side (generator) of the evolution equation (26) is separately regular. However, their nonlinear connection can create problem: As noted in Remark 1, we have to evaluate the operator $\mathcal{J}(\omega)$ at the common ω of $\partial_\omega H(\omega)$. While $\partial_\omega H(\omega)$ can be evaluated for every $\omega \in H^{-1}(\Omega)$ with its range $= H_0^1(\Omega)$ (= domain of $\mathcal{J}(\omega)$, *if* $\mathcal{J}(\omega)$ is defined), we can define the operator $\mathcal{J}(\omega)$ only for $\omega \in C(\Omega)$. The difficulty of this nonlinear system is now delineated by the singular behavior of the symplectic operator $\mathcal{J}(\omega)$ as a function of ω —if the orbit $\omega(t)$ (in the function space $H^{-1}(\Omega)$) runs away to increase $||\omega||_{sup}$, the evolution equation (26) will breakdown.

To match the connection of $\mathcal{J}(\omega)$ and $\partial_\omega H(\omega)$, the domain of the total generator $\mathcal{J}(\omega)\partial_\omega H(\omega)$ must be restricted in $C(\Omega)$. Fortunately, this domain is not too small; the regular (classical) solutions for an appropriate initial condition lives in this domain, i.e., if a sufficiently smooth initial condition is given, the orbit stays in the region where $||\omega||_{sup}$ is bounded [10].

3 Casimir Elements

3.1 The Kernel of $\mathcal{J}(\omega)$

We begin with a general representation of the kernel of the non-canonical symplectic operator $\mathcal{J}(\omega)$.

Proposition 1 *For a given $\omega \in C(\Omega)$, ψ ($\in H^1_0(\Omega)$ = domain of $\mathcal{J}(\omega)$) is an element of* $\mathrm{Ker}(\mathcal{J}(\omega))$, *iff*

$$\omega \nabla \psi = \nabla \theta \quad \exists \theta \in H^1(\Omega), \tag{27}$$

which implies that

$$\mathrm{Ker}(\mathcal{J}(\omega)) = \{\psi \in H^1_0(\Omega);\ \omega \nabla \psi \in L^2_\sigma(\Omega)^\perp\}. \tag{28}$$

Proof By the definition (24), $\psi \in \mathrm{Ker}(\mathcal{J}(\omega))$ implies $\{\omega, \psi\} = 0$ in the topology of $H^{-1}(\Omega)$, i.e.,

$$(\{\omega, \psi\}, \phi) \equiv -(\omega \nabla \psi, \nabla \phi \times \boldsymbol{e}_\perp) = 0 \quad \forall \phi \in H^1_0(\Omega). \tag{29}$$

By Lemma 1, (29) implies that

$$(\omega \nabla \psi, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in L^2_\sigma(\Omega). \tag{30}$$

Remembering (14), we obtain (27) and (28). ∎

To construct a Casimir element from $\psi \in \mathrm{Ker}(\mathcal{J}(\omega))$, we will need a more "explicit" relation between ω and ψ. It is available for a sufficiently regular ω.

Let us start by assuming $\omega \in C^1(\Omega)$. Then, we may evaluate $\mathcal{J}(\omega)\psi$ as $\{\omega, \psi\} \equiv -\nabla \omega \times \nabla \psi \cdot \boldsymbol{e}_\perp \in L^2(\Omega)$. Therefore, $\psi \in H^1_0(\Omega)$ belongs to $\mathrm{Ker}(\mathcal{J}(\omega))$, iff

$$\{\omega, \psi\} = 0 \quad [\in L^2(\Omega)]. \tag{31}$$

Equation (31) implies that two vectors $\nabla \omega \in C(\Omega)$ and $\nabla \psi \in L^2(\Omega)$ must align almost everywhere in Ω, excepting the region in which one of them is zero. Such relation of ω and ψ can be represented, by invoking a certain scalar $\zeta(x, y)$, as

$$\omega = f(\zeta), \quad \psi = g(\zeta). \tag{32}$$

The simplest solution is given by $\psi = g(\omega)$ (i.e., f = identity). In later discussion, we shall invoke a nontrivial f to represent a wider class of solutions.

We note that the condition $\psi \in H^1_0(\Omega)$ (= domain of $\mathcal{J}(\omega)$; see Sec. 2.4) implies a boundary condition $\psi|_\Gamma = 0$. For (31) to be compatible with this boundary condition, $0 \equiv \boldsymbol{n} \times (\nabla \omega \times \nabla \psi) = (\boldsymbol{n} \times \nabla \omega) \times \nabla \psi$ on Γ ($\boldsymbol{n}$ is the normal vector onto Γ). Hence, to find ψ such that $\nabla \psi \neq 0$ on Γ, we need $\omega|_\Gamma$ = constant. Otherwise (i.e., if $\omega \neq$ constant on some $\Gamma' \subseteq \Gamma$), ψ must vanish in a neighborhood of Γ'. To formulate the solvability (compatibility) condition, we denote by $\Omega_\circ(\omega)$ the largest region in Ω (not necessarily a connected set) which is bounded by a level set (contour) of ω. If $\omega|_\Gamma \neq$ constant, $\Omega_\circ(\omega)$ is smaller than Ω, and then, every level set of ω in $\Omega \setminus \Omega_\circ(\omega)$ intersects Γ. Hence, $\mathrm{supp}(\psi) :=$ closure of $\{x \in \Omega;\ \psi(x) \neq 0\} \subseteq \Omega_\circ(\omega)$. We shall assume that $\Omega_\circ(\omega) \neq \emptyset$ for the existence of nontrivial $\psi \in \mathrm{Ker}(\mathcal{J}(\omega))$.

Now we make a moderate generalization about the regularity: Suppose that ω is Lipschitz continuous, i.e., $\omega \in C^{0,1}(\Omega)$. Then, ω has a classical gradient $\nabla \omega$ almost everywhere in Ω, and $|\nabla \omega|$ is bounded (ω may fail to have a classical $\nabla \omega$ on a measure-zero subset $\Omega_s \subset \Omega$, but at $\boldsymbol{x} \in \Omega_s$, we may define a set-valued *generalized gradient*; see [11]). With a Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$, we can solve (27) by

$$\psi = g(\omega), \quad \theta = \vartheta(\omega), \tag{33}$$

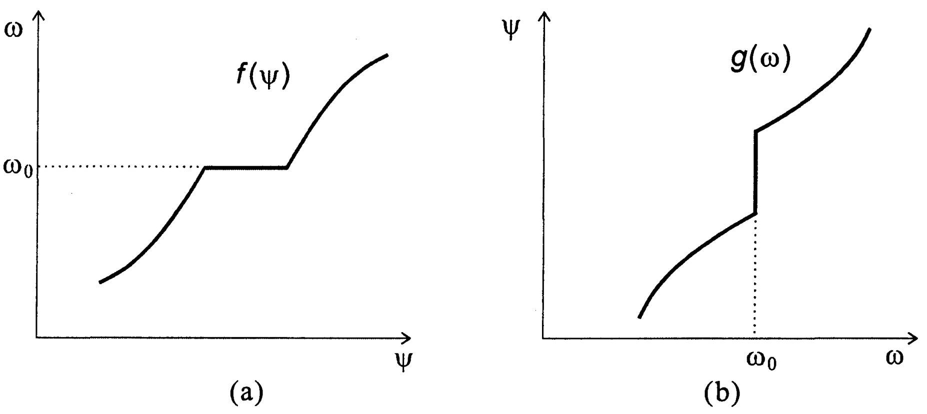

Figure 1: (a) The dual representation $\psi = f(\omega)$ of the plateau singularity. (b) Singular (discontinuous) function $\omega = g(\psi)$ representing the kernel element of that stems from a "plateau" of ω.

where $\vartheta(\xi) = g(\xi)\xi - \int g(\xi)d\xi$. To meet the boundary condition $\psi|_\Gamma = 0$, g must satisfy

$$g(\omega(\boldsymbol{x})) = 0 \quad \forall \boldsymbol{x} \in \Gamma. \tag{34}$$

However, the solution (33) omits a different type of solution that emerges with a *singularity* of $\mathcal{J}(\omega)$: If ω has a "plateau," i.e., $\omega = \omega_0$ (constant) in a finite region $\Omega_p \subseteq \Omega$, the operator $\mathcal{J}(\omega)$ trivializes as $\mathcal{J}(\omega) = \{\omega_0, \cdot\} = 0$ in Ω_p (i.e., the "Rank" drops to zero; remember the example of Sec. 1), and, in Ω_p, we can solve (27) by an arbitrary ψ and $\theta = \omega_0\psi$. Notice that the foregoing solution (33) restricts $\psi = g(\omega_0)$ = constant in Ω_p. To remove this degeneracy, we have to abandon the continuity of g. Let us assume that ω has a single plateau for simplicity. First we invoke the reversed form [cf. (32)]:

$$\omega = f(\psi). \tag{35}$$

Here we assume that f is a Lipschitz continuous monotonic function. Denoting $\vartheta(\eta) = \int f(\eta)d\eta$, we may write $\omega\nabla\psi = \nabla\vartheta(\psi)$ (the gradients on both sides evaluate classically almost everywhere in Ω, if ψ is Lipschitz continuous). If the function $f(\psi)$ is flat on some interval (i.e., $f(\psi) = \omega_0$ = constant for $\psi_- < \psi < \psi_+$), a plateau appears in the distribution of ω; see Fig. 1 (a). Since the present mission is to find ψ for a given ω, we transform (35) back to (33) with defining $g = f^{-1}$. A plateau in the graph of f will, then, appear as a "jump" in the graph of g; see Fig. 1 (b).

We now allow the function $g(\omega)$ to have a jump at $\omega_0 = \omega|_{\Omega_p}$. Formally, we consider g such that $g(\omega) = g_L(\omega) + \alpha Y(\omega - \omega_0)$ with a Lipschitz continuous $g_L(\omega)$ and a *step function* $Y(\omega - \omega_0)$ (α is a constant determining the width of the jump). We have to connect the graph of the step function by filling the gap between $\lim_{\omega\uparrow\omega_0} g(\omega) = \psi_-$ and $\lim_{\omega\downarrow\omega_0} g(\omega) = \psi_+ = \psi_- + \alpha$; see Fig. 1 (b). Since $g(\omega)$ is multi-valued at $\omega = \omega_0$, $\psi(\boldsymbol{x}) = g(\omega(\boldsymbol{x}))$ may vary arbitrarily, within the range of $[\psi_-, \psi_+]$, in the plateau Ω_p. Choosing a sufficiently smooth ψ in Ω_p, we may assume $\psi \in H^1(\Omega)$.

Summarizing the foregoing discussions (and making an obvious generalization), we have

Corollary 1 *Suppose that $\omega \in C^{0,1}(\Omega)$ and $\Omega_\circ(\omega) \neq \emptyset$. Then Ker($\mathcal{J}(\omega)$) contains non-trivial elements, and a part of them can be represented as*

$$\psi = g(\omega), \tag{36}$$

where $g(\xi)$ is an arbitrary function that satisfies the boundary condition (34) and such that

$$g(\xi) = g_L(\xi) + \sum_{\ell=1}^{\nu} \alpha_\ell Y(\xi - \omega_\ell), \tag{37}$$

where ω_ℓ denotes the hight of a plateau of ω, $Y(\xi)$ is the "filled" step function, and α_ℓ is a constant.

Remark 2 *Obviously the form (37) of $g(\omega)$ is rather restrictive:*

(i) In the plateau region, (27) has a wider class of solutions. In fact, ψ may be an arbitrary (H^1-class) function whose range may exceed the interval $[\psi_-, \psi_+]$. Then, the graph of $g(\omega)$ has a "thorn" at ω_0. However, we may not integrate such a function to define a Casimir element $G(\omega)$; see Sec. 3.2.

(ii) In (37), we restrict the continuous part $g_L(\xi)$ to be Lipschitz continuous, by which $\psi = g(\omega)$ ($\omega \in C^{0,1}(\Omega)$) is assured of Lipschitz continuity (thus, $\psi \in H^1(\Omega)$). However, this condition may be weakened, depending on a specific ω, as far as $g'(\omega)\nabla\omega \in H^1(\Omega)$.

3.2 Construction of Casimir Elements

Our next mission is to "integrate" the kernel element $\psi \in \text{Ker}(\mathcal{J}(\omega))$ as a function of ω, and define a Casimir element $C(\omega)$, i.e., to find a functional $C(\omega)$ such that $\partial_\omega C(\omega) \in \text{Ker}(\mathcal{J}(\omega))$. To this end, the parameterized solution (36) will be invoked, where the function $g(\omega)$ may have singularities as described in Corollary 1. The central issue of this section, then, will be to consider an appropriate "generalized functional derivative" by which we can define "singular Casimir elements" pertinent to the singularities of the non-canonical symplectic operator $\mathcal{J}(\omega)$.

Let us start by a *regular* Casimir element generated by $g(\xi) \in C(\mathbb{R})$. Denoting $G(\xi) = \int g(\xi)\, d\xi$, we define

$$C_G(\omega) = \int G(\omega)\, dx. \tag{38}$$

The gradient of this functional can be calculated by the definition given in Sec. 2.3: Perturbing ω by $\epsilon\tilde{\omega}$ results in

$$\delta C_G(\omega, \tilde{\omega}) = \epsilon \int_\Omega g(\omega)\tilde{\omega}\, dx + O(\epsilon^2).$$

Hence, we obtain $\partial_\omega C_G(\omega) = g(\omega)$, proving that $C_G(\omega)$ of (38) is the Casimir element corresponding to $g(\omega) \in \text{Ker}(\mathcal{J}(\omega))$.

Now we construct a *singular* Casimir element corresponding to a general $g(\omega)$ that may have "jumps" at the singularity of $\mathcal{J}(\omega)$ (i.e., the plateaus of ω). The formal primitive function $G(\xi)$ of such $g(\xi)$ has "kinks" where the classical differential does not apply

—this problem leads to the requirement of an appropriately generalized gradient of the functional $C_G(\omega) = \int G(\omega)\,dx$ generated by a kinked $G(\xi)$.

Here we invoke the *Clarke gradient* [11]: The generalized gradient of a Lipschitz-continuous function (or functional) F at x, denoted by $\tilde{\partial}_x F(x)$, is the convex hull of the set of limits of the form

$$\lim_{j\to\infty} \partial_x F(x+\delta_j) \quad (\lim_{j\to\infty} \delta_j = 0). \tag{39}$$

It is evident that $\tilde{\partial}_x F(x)$ gives the classical gradient $\partial_x F(x)$, if $F(x)$ is continuously differentiable in the neighborhood of x. It is also evident that a "kink" a one-dimensional $F : \mathbb{R} \to \mathbb{R}$ yields on the graph of $\tilde{\partial}_x F(x)$ a "jump" with the gap filled; see Fig. 1 (b). When $F(x)$ is a convex functional on a Hilbert space X, $\tilde{\partial}_x F(x)$ is equal to the *sub-differential*:

$$\tilde{\partial}_x F(x) : \ x \mapsto \{g;\ F(x+\delta) - F(x) \geq (g, x),\ \forall \delta \in X\}, \tag{40}$$

which gives the *maximally monotone* (i.e., the gap-filled) function [12].

Following conclusion is readily deducible:

Proposition 2 *Suppose that $\omega \in C^{0,1}(\Omega)$ and $\Omega_\circ(\omega) \neq \emptyset$. By $g(\xi)$ satisfying (34) and (37), we define $G(\xi)$ such that $g(\xi) = \tilde{\partial}_\xi G(\xi)$. Then, $C_G(\omega) = \int G(\omega)\,dx$ is a generalized Casimir element, i.e., $\tilde{\partial}_\omega C_G(\omega) \in Ker(\mathcal{J}(\omega))$.*

4 Concluding Remarks

Formulating the Euler equation of two-dimensional incompressible inviscid flow in a Hamiltonian form, we have studied the center of the Lie-Poisson algebra, i.e., the kernel of the non-canonical symplectic operator $\mathcal{J}(\omega)$; a Casimir element $C(\omega)$ is given by "integrating" the "differential equation"

$$\mathcal{J}(\omega)\partial_\omega C(\omega) = 0.$$

If the state vector ω belongs to a finite-dimension space, $\mathcal{P} := \mathcal{J}(\omega)\partial_\omega$ is a linear partial differential operator. Non-triviality arises in the *singularity* of $\mathcal{P}$, from which an inherent structure emerges. As given in Introduction, a simple example is $\mathcal{P} = ix\partial_x$, which generates a *hyper-function* Casimir $C(x) = Y(x)$. In a finite dimension space, the theory finds its way in the algebraic analysis —in the language of D-module theory, Casimir elements constitute $\mathrm{Ker}(\mathcal{P}) = \mathrm{Hom}_{\mathcal{D}}(\mathrm{Coker}(\mathcal{P}), F)$, where $\mathcal{D}$ is the ring of partial differential operators and F is the function space on which $\mathcal{P}$ operates; $\mathrm{Coker}(\mathcal{P}) = \mathcal{D}/\mathcal{D}\mathcal{P}$ is the D-module corresponding to the equation $\mathcal{P}C(\omega) = 0$. In the present study, ω is a member of an infinite-dimension Hilbert space, thus $\mathcal{P}$ may be regarded as an infinite-dimensional generalization of partial differential operators. From the singularity of an infinite-dimensional (or *functional*) differential operator $\mathcal{P} = \mathcal{J}(\omega)\partial_\omega$, we have unearthed *singular* Casimir elements —to justify the operation of $\mathcal{P}$ on singular elements, we invoked a generalized functional derivative (Clarke differential or sub-differential), which we denoted by $\tilde{\partial}_\omega$.

In an infinite-dimension system, we cannot "count" the dimensions of $\mathrm{Ker}(\mathcal{P})$ and $\mathrm{Ker}(\mathcal{J})$. It is, however, evident that dim-$\mathrm{Ker}(\mathcal{P})$ < dim-$\mathrm{Ker}(\mathcal{J})$, if $\mathcal{J}$ has singularities, i.e., singularities create "non-integrable" elements of $\text{Ker}(\mathcal{J})$. As shown in Corollary 1, a *plateau* in ω causes a singularity of $\mathcal{J}(\omega)$ and generates new elements of $\text{Ker}(\mathcal{J}(\omega))$, which can be integrated to produce singular Casimir elements (Proposition 2). However, as noted in Remark 2 (i), more general elements of $\text{Ker}(\mathcal{J}(\omega))$ stem in the plateau singularity, which are *not integrable*. Moreover, we had to assume Lipschitz continuity for ω to obtain an explicit relation between $\psi \in \text{Ker}(\mathcal{J}(\omega))$ and ω —otherwise we cannot *integrate* ψ with respect to ω to construct a Casimir element. In the general definition of $\mathcal{J}(\omega)$, however, ω may be non-differentiable (we assume only continuity), and then, a general $\psi \in \text{Ker}(\mathcal{J}(\omega))$ may not have an integrable relation to ω (see Proposition 1).

We end this paper with remaking the relation between the (generalized) Casimir elements and the fixed points (the stationary ideal flows); cf. [6]. Generalizing (4) to an infinite-dimension space, we may find an extended set of fixed points by solving

$$\tilde{\partial}_u H_\mu(u) = \tilde{\partial}_u [H(u) - \mu C(u)] \ni 0. \tag{41}$$

We have to note, however, that we are uncertain whether every fixed point can be derived by Casimir elements or not. For example, let us consider a simple Hamiltonian $H(u) = \|u\|^2/2$ (in Appendix A, the Hamiltonian of the Euler equation is given in this form). Then, $\partial_u H(u) = u$, thus (5) reads as

$$\partial_t u = \mathcal{J}(u)u. \tag{42}$$

The totality of fixed points is $\text{Ker}(\mathcal{J}(u))$. For $v \in \text{Ker}(\mathcal{J}(u))$ to be characterized by (41), which now simplifies as $u = \mu\partial_u C(u)$, we encounter the "integration problem", i.e., we have to construct $C(u)$ such that $v(u) = \partial_u C(u)$ for every $v(u) \in \text{Ker}(\mathcal{J}(u))$ —this may not be always possible. On the other hand, even for a given $C(u)$, the "nonlinear equation" $u = \mu\partial_u C(u)$ does not necessarily have a solution [6].

A singular (kinked) Casimir yields multivalued (set-valued) gradient $\tilde{\partial}_\omega C(\omega)$, encompassing an infinite degree of freedom stemming in the plateau singularity (in the plateau of ω, $\psi \in \text{Ker}(\mathcal{J}(\omega))$ is freed from ω and may distribute arbitrarily). The component $g_L(\omega)$ of (37), built in the Casimir $C(\omega)$, represents explicitly the regular degree of freedom in $\text{Ker}(\mathcal{J}(\omega))$. In contrast, the undetermined degree of freedom pertinent to the singularity $\omega = \omega_0$ is "implicitly" included in the step-function component of (37), or in the kink of $C(\omega)$. Providing a Hamiltonian $H(\omega)$ (embodying the dynamics), however, a specific relation between $\varphi \in \tilde{\partial}_\omega C(\omega)$ and ω emerges.

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Appendix A. Formulation in terms of flow $u \in L^2_\sigma(\Omega)$

Here we formulate the Euler equation (for both $n = 2$ and 3) as an evolution equation in $L^2_\sigma(\Omega)$ (see Sec. 2.1), and write it in a Hamiltonian form. The difference from the formulation in Sec. 2 is in that the state variable will be the flow velocity $\boldsymbol{u}$ (instead of the vorticity ω).

As noted in Sec. 2.1, $L^2_\sigma(\omega)$ is a closed subspace of $L^2(\Omega)$, and we have the orthogonal decomposition (14). We denote by P_σ the orthogonal projection onto $L^2_\sigma(\Omega)$. Applying

P_σ to the both sides of (11), we obtain

$$\partial_t \boldsymbol{u} = -P_\sigma(\nabla \times \boldsymbol{u}) \times \boldsymbol{u}, \tag{43}$$

which may be interpreted as an evolution equation in $L^2_\sigma(\Omega)$ (the incompressibility (9) and the boundary condition (10) are included in the condition $\boldsymbol{u} \in L^2_\sigma(\Omega)$).

For $\boldsymbol{u} \in L^2_\sigma(\Omega)$, we define the Hamiltonian

$$H(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u}\|^2. \tag{44}$$

Fixing a sufficiently smooth $\boldsymbol{u}$ (we assume $\nabla \times \boldsymbol{u} \in C(\Omega)$) as a parameter, we define, for $\boldsymbol{v} \in L^2_\sigma(\Omega)$, the non-canonical symplectic operator

$$\mathcal{J}(\boldsymbol{u})\boldsymbol{v} = -P_\sigma(\nabla \times \boldsymbol{u}) \times \boldsymbol{v}. \tag{45}$$

As a linear operator (Remark 1), $\mathcal{J}(\boldsymbol{u})$ consists of the vector multiplication by $(\nabla \times \boldsymbol{u})$ and the projection by P_σ. Evidently, $\mathcal{J}(\boldsymbol{u})$ is antisymmetric:

$$(\mathcal{J}(\boldsymbol{u})\boldsymbol{v}, \boldsymbol{v}') = -(\boldsymbol{v}, \mathcal{J}(\boldsymbol{u})\boldsymbol{v}') \quad \forall \boldsymbol{v}, \boldsymbol{v}' \in L^2_\sigma(\Omega).$$

In fact, $i\mathcal{J}(\boldsymbol{u})$ (for every fixed smooth $\boldsymbol{u}$) is a self-adjoint bounded operator in $L^2_\sigma(\Omega)$.

With these $H(\boldsymbol{u})$, $\mathcal{J}(\boldsymbol{u})$, and the gradient $\partial_{\boldsymbol{u}}$ (see Sec. 2.3), we may write (43) as

$$\partial_t \boldsymbol{u} = \mathcal{J}(\boldsymbol{u})\partial_{\boldsymbol{u}} H(\boldsymbol{u}). \tag{46}$$

Hereafter, $n = 2$: By Lemma 1, we may put $\boldsymbol{u} = \nabla\varphi \times \boldsymbol{e}_\perp$ and $\omega = -\Delta\varphi$ with $\varphi \in H^1_0(\Omega)$. Fixing $\omega \in C(\Omega)$ as a parameter, and putting $\boldsymbol{v} = \nabla\psi \times \boldsymbol{e}_\perp$ with $\psi \in H^1_0(\Omega)$, we may write

$$\mathcal{J}(\boldsymbol{u})\boldsymbol{v} = -P_\sigma\left[\omega\boldsymbol{e}_\perp \times (\nabla\psi \times \boldsymbol{e}_\perp)\right] = -P_\sigma\left[\omega\nabla\psi\right].$$

By (14), $\boldsymbol{v} \in \mathrm{Ker}(\mathcal{J}(\boldsymbol{u}))$ iff

$$\omega\nabla\psi = \nabla\theta \quad \exists\theta \in H^1(\Omega), \tag{47}$$

which is equivalent to (27). Just like the argument of Sec. 3.1, we find solutions of (47) such as

$$\psi = g(\omega). \tag{48}$$

The Casimir element constructed from (48) is

$$C_G(\boldsymbol{u}) = \int_\Omega G(\boldsymbol{e}_\perp \cdot \nabla \times \boldsymbol{u})\, dx = \int_\Omega G(\omega)\, dx. \tag{49}$$

Perturbing $\boldsymbol{u}$ by $\epsilon\tilde{\boldsymbol{u}}$ (we restrict $\boldsymbol{n} \cdot \tilde{\boldsymbol{u}} = 0$ on Γ) yields $\delta\omega = \epsilon\boldsymbol{e}_\perp \cdot \nabla \times \tilde{\boldsymbol{u}}$, and

$$\begin{aligned}\delta C_G(\boldsymbol{u}; \tilde{\boldsymbol{u}}) &= \epsilon \int G'(\omega)\boldsymbol{e}_\perp \cdot \nabla \times \tilde{\boldsymbol{u}}\, dx + O(\epsilon^2) \\ &= \epsilon \int \nabla G'(\omega) \times \boldsymbol{e}_\perp \cdot \tilde{\boldsymbol{u}}\, dx + O(\epsilon^2).\end{aligned}$$

Hence, we obtain, denoting $g(\omega) = G'(\omega)$,

$$\partial_{\boldsymbol{u}} C_G(\boldsymbol{u}) = \nabla g(\omega) \times \boldsymbol{e}_\perp. \tag{50}$$

By (48), it is evident that $\mathcal{J}(\boldsymbol{u})\partial_{\boldsymbol{u}} C_G(\boldsymbol{u}) = -P_\sigma[\omega\nabla g(\omega)] = 0$, confirming $\partial_{\boldsymbol{u}} C_G(\boldsymbol{u}) \in \mathrm{Ker}(\mathcal{J}(\boldsymbol{u}))$.

In the present formulation, $H(\boldsymbol{u}) = \|\boldsymbol{u}\|^2/2$, thus, as noted in Sec. 1, we may find a fixed point by solving $\boldsymbol{u} = \mu\partial_{\boldsymbol{u}} C_G(\boldsymbol{u})$.

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[7] Generalization to a multiply connected region is not difficult: We decompose $L^2_H(\Omega) = \{\boldsymbol{u} \in L^2_\sigma(\Omega);\ \nabla\times\boldsymbol{u} = 0\}$ from $L^2_\sigma(\Omega)$ and put $L^2_\sigma(\Omega) = L^2_H(\Omega) \oplus L^2_\Sigma(\Omega)$. The dimension of the subspace $L^2_H(\Omega)$ is equal to the genus of Ω. The projection of $\boldsymbol{u}$, which obeys (8)-(10), is shown to be constant through the evolution, so we may regard (11) as an evolution equation in $L^2_\Sigma(\Omega)$.

[8] The function φ is called a Clebsch potential. To represent an incompressible flow of dimension n, we need $n-1$ *Clebsch potentials* $\varphi_1, \cdots, \varphi_{n-1}$, while each φ_j does not have a uniquely determined boundary condition; see Z. Yoshida, J. Math. Phys. **50** (2009), 113101. Hence, the recourse to the vorticity equation is not effective in higher dimensions. In Appendix A, we invoke another method to eliminate the pressure term and formulate the problem in an alternative form, which applies in general space dimension.

[9] For sufficiently smooth $\boldsymbol{u}$, the two-dimensional vorticity equation (18) reads as a Liouville equation. The corresponding Hamilton's equation (characteristic ODE) is the streamline equation $d\boldsymbol{x}/dt = (\partial_y\varphi, -\partial_x\varphi) = \boldsymbol{u}$. By the boundary condition $\boldsymbol{n}\cdot\boldsymbol{u} = (\boldsymbol{e}_\perp \times \boldsymbol{n})\cdot\nabla(\mathcal{K}\omega) = 0$, the characteristic curves are confined in Ω. Hence, we do not need (or, cannot impose) a boundary condition on ω; the single equation (18) determines the evolution of ω (and $\boldsymbol{u} = \nabla(\mathcal{K}\omega)\times\boldsymbol{e}_\perp$).

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