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Stability and instability of asymptotic profiles of solutions for fast diffusion equations

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Abstract

In this note, we review the authors' recent results in [1] on the stability analysis of asymptotic profiles of solutions to the Cauchy-Dirichlet problem for the fast diffusion equation.

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We are concerned with the Cauchy-Dirichlet problem for the fast diffusion equation of the form

$$\partial_t \left(|u|^{m-2}u\right) = \Delta u \quad \text{in } \Omega \times (0,\infty), \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{3}$$

where $\partial_t = \partial/\partial t$ and Δ denotes the N-dimensional Laplace operator. Throughout this note, we assume that

$$2 < m < 2^* := \begin{cases} 2N/(N-2) & \text{if } N \geq 3, \\ \infty & \text{if } N = 1,2 \end{cases} \quad \text{and} \quad u_0 \in H_0^1(\Omega) \tag{4}$$

(then $H_0^1(\Omega)$ is compactly embedded in $L^m(\Omega)$). By putting $w = |u|^{m-2}u$, Equation (1) can be rewritten in a usual form of fast diffusion equation,

$$\partial_t w = \Delta \left(|w|^{r-2}w\right) \qquad \text{in } \Omega \times (0,\infty)$$

with the exponent $r = m/(m-1) < 2$. Fast diffusion equations arise in the studies of plasma physics (see [3]), kinetic theory of gases, solid state physics and so on.

It is well known that every solution of (1)–(3) vanishes in finite time (see [15], [2]). Moreover, Berryman and Holland [4] studied asymptotic profiles of solutions as well as the explicit rate of the extinction of solutions.

In this note, we address ourselves to the stability and instability of each asymptotic profile. Namely, our question is the following: For any initial data $u_0 \in H_0^1(\Omega)$ sufficiently close to an asymptotic profile ϕ, does the asymptotic profile of the unique solution $u = u(x,t)$ for (1)–(3) also coincide with ϕ or not ? In [4] and [12], the stability of the unique positive asymptotic profile is discussed for nonnegative initial data in some special cases (e.g., $N = 1$). However, to the best of our knowledge, the notions of stability and instability of asymptotic profiles for (1)–(3) have not been precisely defined so far, and moreover, the stability analysis has not been done for (possibly) sign-changing initial data. In this note, we give precise definitions of the stability and instability of profiles for (possibly) sign-changing initial data, and furthermore, we present criteria for the stability and for the instability. We also perform the stability analysis in several concrete cases of the domain Ω and the exponent m.

Our method of analysis is based on a dynamical system generated by a rescaled problem,

$$\partial_s \left(|v|^{m-2}v\right) = \Delta v + \lambda_m |v|^{m-2}v \quad \text{in} \quad \Omega \times (0,\infty) \tag{5}$$

with some constant $\lambda_m > 0$ and a transformed time-variable s, on a surface $\mathcal{X}$ in the energy space $H_0^1(\Omega)$. The gradient structure of the equation above and variational features of a Lyapunov energy function over the phase surface $\mathcal{X}$ play a crucial role. Moreover, as a by-product, we classify the whole of the energy space $H_0^1(\Omega)$ for initial data in terms of large-time behaviors of solutions to the Cauchy-Dirichlet problem for (5).

Notation. Let $H_0^1(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in the usual Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$. Let us denote by $\|\cdot\|_m$ the usual norm of $L^m(\Omega)$-space, and moreover, $\|\cdot\|_{1,2} := \|\nabla\cdot\|_2$ stands for the norm of $H_0^1(\Omega)$. For a function $u = u(x,t) : \Omega \times (0,\infty) \to \mathbb{R}$, we often write $u(t) := u(\cdot,t)$, which is a function from Ω into $\mathbb{R}$, for a fixed time $t > 0$.

2 Asymptotic profiles of vanishing solutions

Throughout this note, we are concerned with solutions of (1)–(3) defined by

Definition 2.1 (Solution of (1)–(3)). *A function $u : \Omega \times (0,\infty) \to \mathbb{R}$ is said to be a (weak) solution of* (1)–(3)*, if the following conditions are all satisfied:*

- $u \in C([0,\infty); H_0^1(\Omega))$ *and* $|u|^{m-2}u \in C^1([0,\infty); H^{-1}(\Omega))$.
- *For all* $t \in (0,\infty)$ *and* $\psi \in C_0^\infty(\Omega)$,

$$\left\langle \frac{d}{dt}\left(|u|^{m-2}u\right)(t), \psi \right\rangle_{H_0^1} + \int_\Omega \nabla u(x,t) \cdot \nabla \psi(x)\,dx = 0.$$

- $u(\cdot,t) \to u_0$ *strongly in* $H_0^1(\Omega)$ *as* $t \to +0$.

Then for any $u_0 \in H_0^1(\Omega)$, the problem (1)–(3) admits a unique solution (see, e.g., [5] and [17, 18]).

Moreover, every solution $u = u(x,t)$ of (1)–(3) for $u_0 \neq 0$ vanishes at a finite time $t_* > 0$ at the rate $(t_* - t)^{1/(m-2)}$ (see [4], [12], [16]).

Proposition 2.2 (Extinction rate of solutions). *Assume that* $2 < m \le 2^*$. *Then for any* $u_0 \in H_0^1(\Omega) \setminus \{0\}$, *the unique solution* $u = u(x,t)$ *of* (1)–(3) *vanishes at a finite time* $t_* = t_*(u_0) > 0$. *Moreover, it holds that*

$$(t_* - t)^{1/(m-2)} \le C_1 \|u(t)\|_m \le C_2 \|u(t)\|_{1,2} \le C_3 (t_* - t)^{1/(m-2)}$$

with some constants C_i $(i = 1,2,3)$. *Hence* $\|u(t)\|_{1,2}$ *and* $\|u(t)\|_m$ *vanish at the rate of* $(t_* - t)^{1/(m-2)}$.

The finite time $t_* = t_*(u_0)$ is called *extinction time* (of the unique solution u) for a data u_0. Here t_* can be regarded as a functional defined on $H_0^1(\Omega)$ with value in $[0,\infty)$:

$$\begin{array}{cccc} t_* : & H_0^1(\Omega) & \to & [0,\infty), \\ & u_0 & \mapsto & t_*(u_0). \end{array}$$

From the explicit rate of the extinction of all solutions, one can define *asymptotic profiles* $\phi = \phi(x)$ of solutions for (1)–(3) by

Definition 2.3 (Asymptotic profiles [1]). *Let* $u_0 \in H_0^1(\Omega) \setminus \{0\}$ *and let* $u = u(x,t)$ *be a solution for* (1)–(3) *vanishing at a finite time* $t_* > 0$. *A function* $\phi \in H_0^1(\Omega) \setminus \{0\}$ *is called an asymptotic profile of* u *if there exists an increasing sequence* $t_n \to t_*$ *such that*

$$\lim_{t_n \nearrow t_*} \|(t_* - t_n)^{-1/(m-2)} u(t_n) - \phi\|_{1,2} = 0.$$

In order to characterize ϕ, we apply the following transformation:

$$v(x,s) := (t_* - t)^{-1/(m-2)} u(x,t) \quad \text{and} \quad s := \log(t_*/(t_* - t)) \ge 0. \tag{6}$$

Then s tends to infinity as $t \nearrow t_*$. Moreover, the asymptotic profile $\phi = \phi(x)$ of $u = u(x,t)$ is reformulated as $\phi(x) := \lim_{s_n \nearrow \infty} v(x, s_n)$ in $H_0^1(\Omega)$. Furthermore, the Cauchy-Dirichlet problem (1)–(3) for $u = u(x,t)$ is rewritten as the following rescaled problem:

$$\partial_s \left(|v|^{m-2} v\right) = \Delta v + \lambda_m |v|^{m-2} v \quad \text{in } \Omega \times (0,\infty), \tag{7}$$

$$v = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{8}$$

$$v(\cdot,0) = v_0 \quad \text{in } \Omega, \tag{9}$$

where the initial data v_0 and the constant λ_m are given by

$$v_0 = t_*(u_0)^{-1/(m-2)} u_0 \quad \text{and} \quad \lambda_m = (m-1)/(m-2) > 0. \tag{10}$$

Then we have:

Theorem 2.4 (Existence of asymptotic profiles and their characterization [1]). *For any sequence $s_n \to \infty$, there exist a subsequence (n') of (n) and $\phi \in H_0^1(\Omega) \setminus \{0\}$ such that $v(s_{n'}) \to \phi$ strongly in $H_0^1(\Omega)$. Moreover, ϕ is a nontrivial stationary solution of* (7)–(9)*, that is, ϕ solves the Dirichlet problem,*

$$-\Delta\phi = \lambda_m |\phi|^{m-2}\phi \quad in \ \ \Omega, \qquad \phi = 0 \quad on \ \ \partial\Omega. \tag{11}$$

Remark 2.5. (i) Berryman and Holland [4] first proved the existence of asymptotic profiles for positive classical solutions of (1)–(3), and then, Kwong [12] extended their result to nonnegative weak solutions. Furthermore, Savaré and Vespri [16] proved the convergence of $v(s_n)$ strongly in $L^m(\Omega)$ as $s_n \to \infty$ for sign-changing solutions. Combining their methods of proof, one can prove the theorem stated above.

(ii) If ϕ is a nontrivial solution of (11), then the function $U(x,t) = (1-t)_+^{1/(m-2)}\phi(x)$ solves (1)–(3) with $U(0) = \phi(x)$. Hence $t_*(\phi) = 1$ and the profile of $U(x,t)$ is $\phi(x)$.

(iii) By Theorem 2.4 and (ii), the set of all asymptotic profiles of solutions for (1)–(3) coincides with the set of all nontrivial solutions of (11). We shall denote these sets by $\mathcal{S}$.

3 Stability and instability of asymptotic profiles

Our stability analysis is based on the transformation (6) and the rescaled problem (7)–(9). Taking account of the relation, $v_0 = t_*(u_0)^{-1/(m-2)}u_0$, and introducing the set

$$\mathcal{X} := \left\{ t_*(u_0)^{-1/(m-2)}u_0 \colon u_0 \in H_0^1(\Omega) \setminus \{0\} \right\},$$

we define the (asymptotic) stability and instability of each profile as follows:

Definition 3.1 (Stability and instability of profiles [1]). *Let $\phi \in H_0^1(\Omega)$ be an asymptotic profile of vanishing solutions for* (1)–(3).

(i) *ϕ is said to be stable, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that any solution v of* (7)–(9) *satisfies*

$$v(0) \in \mathcal{X} \cap B(\phi;\delta) \quad \Rightarrow \quad \sup_{s\in[0,\infty)} \|v(s) - \phi\|_{1,2} < \varepsilon,$$

where $B(\phi;\delta) := \{w \in H_0^1(\Omega) \colon \|\phi - w\|_{1,2} < \delta\}$.

(ii) *ϕ is said to be unstable, if ϕ is not stable.*

(iii) *ϕ is said to be asymptotically stable, if ϕ is stable, and moreover, there exists $\delta_0 > 0$ such that any solution v of* (7)–(9) *satisfies*

$$v(0) \in \mathcal{X} \cap B(\phi;\delta_0) \quad \Rightarrow \quad \lim_{s\nearrow\infty} \|v(s) - \phi\|_{1,2} = 0.$$

Here we enumerate several properties of the set $\mathcal{X}$ in the following:

(i) If $v_0 \in \mathcal{X}$, then $v(s) \in \mathcal{X}$ for all $s \geq 0$.

(ii) $\mathcal{X} = \{v_0 \in H_0^1(\Omega) \colon t_*(v_0) = 1\}$, which is homeomorphic to a unit sphere in $H_0^1(\Omega)$.

(iii) $\mathcal{S} := \{$ nontrivial solutions of (11)$\} \subset \mathcal{X}$ (because, $t_*(\phi) = 1$ for $\phi \in \mathcal{S}$ by (ii) of Remark 2.5).

(iv) If $v_0 \in \mathcal{X}$, then $v(s_n) \to \phi$ strongly in $H_0^1(\Omega)$ with some $\phi \in \mathcal{S}$ along some sequence $s_n \to \infty$ (by Theorem 2.4).

Hence (7)–(9) generates a dynamical system in the phase surface $\mathcal{X}$. Then solutions of (11) can be regarded as stationary points of the dynamical system. Therefore the notions of stability and instability of asymptotic profiles defined above are regarded as those in Lyapunov's sense for the stationary points. Moreover, (7)–(9) can be written as a (generalized) gradient system,

$$\frac{d}{ds}|v|^{m-2}v(s) = -\nabla J(v(s)), \quad s > 0, \qquad v(0) = v_0 \in \mathcal{X},$$

where ∇J stands for the Fréchet derivative of the functional

$$J(w) = \frac{1}{2}\|w\|_{1,2}^2 - \frac{\lambda_m}{m}\|w\|_m^m \quad \text{for } w \in H_0^1(\Omega).$$

Then one can prove that $s \mapsto J(v(s))$ is non-increasing by multiplying (7) by $\partial_s v(x,s)$ and integrating this over Ω. Here let us recall that ϕ is an asymptotic profile if and only if ϕ is a nontrivial solution of (11) (equivalently, $\nabla J(\phi) = 0$ and $\phi \neq 0$). Therefore one can reveal the stability/instability of profiles by investigating variational properties of the functional J over $\mathcal{X}$. However, some difficulties may arise due to the lack of explicit representation of the functional $t_*(\cdot)$ (cf. we can obtain upper and lower estimates for $t_*(\cdot)$ in terms of initial data).

Remark 3.2. Since $m > 2$, J forms a mountain pass structure over the whole of $H_0^1(\Omega)$. Hence 0 is the unique local minimizer of J and all nontrivial critical points are saddle points of J. However, our stability analysis will be carried out on the surface $\mathcal{X}$ in $H_0^1(\Omega)$. Hence our conclusion on the stability of profiles will differ from this observation, and moreover, it would be troublesome to show the instability of profiles due to the restriction of $\mathcal{X}$.

4 Stability criteria

Let d_1 be the *least energy* of J over nontrivial solutions, i.e.,

$$d_1 := \inf_{v \in \mathcal{S}} J(v) \quad \text{with} \quad \mathcal{S} = \{ \text{ nontrivial solutions of (11)}\}.$$

A *least energy solution* ϕ of (11) means $\phi \in S$ satisfying $J(\phi) = d_1$. One can prove that every least energy solution of (11) is sign-definite by using the strong maximum principle.

In [1], the authors obtained the following criteria for the stability and instability of asymptotic profiles:

Theorem 4.1 (Stability of profiles [1]). *Let ϕ be a least energy solution of* (11). *Then it follows that*

(i) *ϕ is a stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other least energy solutions.*

(ii) *ϕ is an asymptotically stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other sign-definite solutions.*

Theorem 4.2 (Instability of profiles [1]). *Let ϕ be a sign-changing solution of* (11). *Then it follows that*

(i) *ϕ is not an asymptotically stable profile.*

(ii) *ϕ is an unstable profile, if ϕ is isolated in $H_0^1(\Omega)$ from any $\psi \in S$ satisfying $J(\psi) < J(\phi)$.*

Let us exhibit several examples of Ω and m that satisfy assumptions stated above. We first note that sign-definite solutions are isolated in $H_0^1(\Omega)$ from all sign-changing solutions. Moreover, least energy solutions are also isolated from all sign-definite ones in the following cases:

Corollary 4.3 (Examples of asymptotically stable profiles [1]). *Least energy solutions of* (11) *are asymptotically stable profiles in the following cases*:

- *Ω is a ball and $2 < m < 2^*$ (see Gidas-Ni-Nirenberg* [10]).
- *$\Omega \subset \mathbb{R}^2$ is bounded and convex and $2 < m < 2^*$ (see Lin* [13] *and also Dancer* [6], *Pacella* [14]).
- *$\Omega \subset \mathbb{R}^N$ is bounded and $2 < m < 2 + \delta$ (see Dancer* [7] *and also Zou* [19]).
- *$\Omega \subset \mathbb{R}^N$ is symmetric with respect to the planes $[x_i = 0]$ and convex in the axes x_i for all $i = 1, 2, \ldots, N$ and $2^* - \delta < m < 2^*$ (see Grossi* [11]).

As for the instability of sign-changing solutions, we have:

Corollary 4.4 (Instability of sign-changing least energy profiles [1]). *Least energy solutions among sign-changing solutions (sign-changing least energy solutions, for short) of* (11) *are unstable profiles.*

Since $m < 2^*$ and Ω is bounded, one can always assure the existence of sign-changing least energy solutions of (11). Moreover, sign-changing least energy solutions are distinct from all nontrivial solutions of (11) with lower energies.

Furthermore, in the one-dimensional case, one can explicitly solve (11),

$$-\phi'' = \lambda_m |\phi|^{m-2}\phi \text{ in } (0,1), \quad \phi(0) = \phi(1) = 0, \tag{12}$$

and then, the set $\mathcal{S}$ of all nontrivial solutions for (12) consists of the sign-definite ones $\pm\phi_1$ and the sign-changing ones $\pm\phi_n$ with $(n-1)$ zeros in $(0,1)$ for $n = 2, 3, \ldots$. Moreover, it follows that

$$J(\pm\phi_1) < J(\pm\phi_2) < \cdots < J(\pm\phi_n) \to \infty \quad \text{as} \quad n \to \infty.$$

In particular, they are distinct from each other. Then all the asymptotic profiles turned out to be either asymptotically stable or unstable.

Corollary 4.5 (Stability and instability of profiles in $N = 1$). *Sign-definite profiles are asymptotically stable. All the other profiles are unstable.*

5 Sketch of proof of Theorem 4.1

In order to prove Theorem 4.1, we employ the following facts: From the continuous dependence of solutions on initial data, we have,

Proposition 5.1 (Continuity of $t_*(\cdot)$ [1]). *If $u_{0,n} \to u_0$ weakly in $H^1_0(\Omega)$ and $(u_{0,n})$ is bounded in $H^1_0(\Omega)$, then $t_*(u_{0,n}) \to t_*(u_0)$.*

Let us recall that $\mathcal{X} = \{w \in H^1_0(\Omega) \colon t_*(w) = 1\}$. By the proposition above, we obtain the following lemma.

Lemma 5.2 (Closedness of $\mathcal{X}$ [1]). *If $u_n \in \mathcal{X}$ and $u_n \to u$ weakly in $H^1_0(\Omega)$, then $u \in \mathcal{X}$.*

Furthermore, we obtain,

Lemma 5.3 (Variational feature of $\mathcal{X}$ [1]). *Let $d_1 = \inf_{\mathcal{S}} J$. Then it follows that*

$$\mathcal{X} \subset [d_1 \le J] := \left\{ v_0 \in H^1_0(\Omega) \colon d_1 \le J(v_0) \right\}.$$

Moreover, if $v_0 \in \mathcal{X}$ and $J(v_0) = d_1$, then $\nabla J(v_0) = 0$.

This lemma also holds in the Sobolev-critical case $m = 2^*$. Here we give a simpler proof only for the subcritical case $m < 2^*$.

Proof. Let $v_0 \in \mathcal{X}$ and assume $J(v_0) < d_1$. Let $v(s)$ be a solution of (7)–(9) with $v(0) = v_0$. Then by Theorem 2.4 one can take a sequence $s_n \to \infty$ such that

$$v(s_n) \to \phi \quad \text{strongly in } H^1_0(\Omega) \quad \text{and} \quad \phi \in \mathcal{S}.$$

Since $s \mapsto J(v(s))$ is non-increasing, we deduce that

$$J(v_0) \ge J(v(s)) \ge J(\phi) \ge d_1,$$

which implies a contradiction. Hence $d_1 \le J(v_0)$.

If $v_0 \in \mathcal{X}$ and $J(v_0) = d_1$, then $J(v_0) = \min_{\mathcal{X}} J$. Hence $v(s) \equiv v_0$. ☐

Denote by $\mathcal{LES}$ the set of all least energy solutions of (11). Now, let us assume that

$$B(\phi; r) \cap \mathcal{LES} = \{\phi\} \tag{13}$$

with some $r > 0$.

Claim 5.1. *For any $\varepsilon \in (0, r)$, it holds that*

$$c := \inf\{J(v) \colon v \in \mathcal{X},\ \|v - \phi\|_{1,2} = \varepsilon\} > d_1.$$

Remark 5.4. Here we remark that the infimum of J over any small sphere centered at ϕ in $H_0^1(\Omega)$ is never greater than d_1, because ϕ is a saddle point. This claim would be essential in our proof.

Assume on the contrary that $c = d_1$, i.e., there exits $v_n \in \mathcal{X}$ such that

$$\|v_n - \phi\|_{1,2} = \varepsilon \quad \text{and} \quad J(v_n) \to d_1.$$

Since $m < 2^*$, it entails that, up to a subsequence,

$$v_n \to v_\infty \quad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^m(\Omega).$$

By Lemmas 5.2 and 5.3, we obtain

$$v_\infty \in \mathcal{X}, \qquad \text{and hence,} \qquad d_1 \le J(v_\infty).$$

Therefore it follows that

$$\begin{aligned}\frac{1}{2}\|v_n\|_{1,2}^2 &= J(v_n) + \frac{\lambda_m}{m}\|v_n\|_m^m \\ &\to d_1 + \frac{\lambda_m}{m}\|v_\infty\|_m^m \le J(v_\infty) + \frac{\lambda_m}{m}\|v_\infty\|_m^m = \frac{1}{2}\|v_\infty\|_{1,2}^2.\end{aligned}$$

By using the weak lower semicontinuity,

$$\liminf_{s_n \to \infty} \|v_n\|_{1,2} \ge \|v_\infty\|_{1,2},$$

and the uniform convexity of $\|\cdot\|_{1,2}$, we deduce that $v_n \to v_\infty$ strongly in $H_0^1(\Omega)$. Hence $\|v_\infty - \phi\|_{1,2} = \varepsilon$ and $J(v_\infty) = d_1$.

We have proved that

$$v_\infty \in \mathcal{X}, \quad J(v_\infty) = d_1 \quad \text{and} \quad \|v_\infty - \phi\|_{1,2} = \varepsilon.$$

Hence $v_\infty \in \mathcal{LES}$ by Lemma 5.3. However, the fact that $\|v_\infty - \phi\|_{1,2} = \varepsilon < r$ contradicts (13). □

Let $\varepsilon \in (0, r)$ be arbitrarily given. Choose $\delta \in (0, \varepsilon)$ so small that

$$J(v) < c \quad \text{for all } v \in B(\phi; \delta).$$

Here it is possible, because $c > d_1 = J(\phi)$ by Claim 5.1, and J is continuous in $H_0^1(\Omega)$. For any $v_0 \in \mathcal{X} \cap B(\phi; \delta)$, let $v(s)$ be a solution of (7)–(9). Then $v(s) \in \mathcal{X}$.

Claim 5.2. *For any $s \geq 0$, $v(s) \in B(\phi;\varepsilon)$, and hence ϕ is stable.*

Assume on the contrary that $v(s_0) \in \partial B(\phi;\varepsilon)$ at some $s_0 > 0$. By the definition of c, it holds that

$$c \leq J(v(s_0)).$$

However, it contradicts the fact that $J(v(s_0)) \leq J(v_0) < c$.

Moreover, if ϕ is isolated from all sign-definite solutions of (11), then $v(s_n)$ converges strongly in $H_0^1(\Omega)$ to ϕ along some sequence $s_n \to \infty$.

6 Sketch of proof of Theorem 4.2

Let ϕ be a sign-changing solution of (11) (hence ϕ admits more than two nodal domains).

Claim 6.1. *The function ϕ is not an asymptotically stable profile.*

Let D be a nodal domain of ϕ and define

$$\phi_\mu(x) := \begin{cases} \mu\phi(x) & \text{if } x \in D, \\ \phi(x) & \text{if } x \in \Omega \setminus D \end{cases} \qquad \text{for } \mu \geq 0$$

(Note: ϕ_μ might not belong to $\mathcal{X}$). Then one can observe that

- $\phi_\mu \to \phi$ strongly in $H_0^1(\Omega)$ as $\mu \to 1$,
- if $\mu \neq 1$, then $J(c\phi_\mu) < J(\phi)$ for any $c \geq 0$.

Moreover, we set

$$u_{0,\mu} := \phi_\mu, \quad \tau_\mu := t_*(u_{0,\mu}), \quad v_{0,\mu} := \tau_\mu^{-1/(m-2)} u_{0,\mu} \in \mathcal{X}.$$

It then follows that

- $\tau_\mu \to t_*(\phi) = 1$ and $v_{0,\mu} \to \phi$ strongly in $H_0^1(\Omega)$ as $\mu \to 1$,
- if $\mu \neq 1$, then $J(v_{0,\mu}) < J(\phi)$.

Hence the solution $v_\mu(s)$ of (7)–(9) with $v_\mu(0) = v_{0,\mu}$ never converges to ϕ as $s \to \infty$. Therefore ϕ is not an asymptotically stable profile.

Remark 6.1 (Another deformation of ϕ)**.** One can also deform ϕ in a simpler way such as $\psi_\mu := \mu\phi$. Then we observe

$$J(\psi_\mu) < J(\phi) \quad \text{if} \quad \mu \neq 1,$$

and this fact entails that every nontrivial solution is not a (local) minimizer of J over $H_0^1(\Omega)$. However, we find that $\phi_\mu \notin \mathcal{X}$, and moreover,

$$\hat{v}_{0,\mu} := t_*(\psi_\mu)^{-1/(m-2)}\psi_\mu$$

belongs to $\mathcal{X}$ and coincides with ϕ for any $\mu \in \mathbb{R}$. Hence $J(\hat{v}_{0,\mu}) = J(\phi)$, and the argument above might not follow.

In addition, let us assume that

$$\overline{B(\phi;R)} \cap \{\psi \in \mathcal{S}\colon J(\psi) < J(\phi)\} = \emptyset \tag{14}$$

with some $R > 0$.

Claim 6.2. *If $\mu \neq 1$, then $v_\mu(s) \notin \overline{B(\phi;R)}$ for any $s \gg 1$.*

Assume on the contrary that $v_\mu(s_n) \in \overline{B(\phi;R)}$ with some sequence $s_n \to \infty$. Then by Theorem 2.4, we deduce that, up to a subsequence,

$$v_\mu(s_n) \to \psi \quad \text{strongly in } H_0^1(\Omega)$$

with some $\psi \in \overline{B(\phi;R)} \cap \mathcal{S}$. Moreover, we have

$$J(\psi) \leq J(v_{0,\mu}) < J(\phi),$$

which contradicts (14). Thus ϕ is an unstable profile.

7 Global dynamics for the rescaled problem

The final section is devoted to further discussion of the surface $\mathcal{X}$, which was a phase space in our stability analysis, and then, we shall reveal the global dynamics of solutions to the rescaled problem (7)–(9) for any data $v_0 \in H_0^1(\Omega)$.

The following proposition classifies the whole of the energy space $H_0^1(\Omega)$ in terms of large-time behaviors of solutions for (7)–(9) (cf. see [9] for the semilinear heat equation), and in particular, $\mathcal{X}$ is a separatrix between the stable and unstable sets.

Proposition 7.1 (Characterization of $\mathcal{X}$). *Let $v(s)$ be a solution of* (7)–(9) *with $v(0) = v_0$. Then it follows that*

- (i) *If $v_0 \in \mathcal{X} = \{v_0 \in H_0^1(\Omega)\colon t_*(v_0) = 1\}$, then $v(s_n)$ converges to some nontrivial solution ϕ of* (11) *strongly in $H_0^1(\Omega)$ along some sequence $s_n \to \infty$.*
- (ii) *If $v_0 \in \mathcal{X}^+ := \{v_0 \in H_0^1(\Omega)\colon t_*(v_0) > 1\}$, then $v(s)$ blows up in infinite time. Hence $\mathcal{X}^+$ is an unstable set.*
- (iii) *If $v_0 \in \mathcal{X}^- := \{v_0 \in H_0^1(\Omega)\colon t_*(v_0) < 1\}$, then $v(s)$ vanishes in finite time. Hence $\mathcal{X}^-$ is a stable set.*

Moreover, $\mathcal{X}$ does not coincide with the Nehari manifold of J,

$$\mathcal{N} := \left\{w \in H_0^1(\Omega)\colon \langle \nabla J(w), w\rangle = 0\right\}.$$

Furthermore, $\mathcal{X}$ is surrounded by $\mathcal{N}$ (i.e., $\mathcal{N} \subset \mathcal{X} \cup \mathcal{X}^+$) and $\mathcal{N} \cap \mathcal{X} = \mathcal{S}$.

We finally give conceptual diagrams of the mountain pass structure of the Lyapunov functional J and two surfaces $\mathcal{X}$ and $\mathcal{N}$ with critical points of J on $H_0^1(\Omega)$ for the one-dimensional case in Fig.1, where the thick gray curve and the dashed curve denote the sets $\mathcal{X}$ and $\mathcal{N}$, respectively. Moreover, the thick arrows mean the flow of the dynamical system generated by the rescaled problem (7)–(9) in the whole of $H_0^1(\Omega)$. The dots stand for critical points of J; in particular, $\pm\phi_1$ denote least energy solutions (i.e., positive and negative ones) and $\pm\phi_2$ denote the sign-changing least energy solutions (with only one node in $(0,1)$).

The least energy solutions $\pm\phi_1$ are saddle points of J over $H_0^1(\Omega)$. However, the flow of the dynamical system generated by (7)–(9) is invariant over $\mathcal{X}$, and therefore, $\pm\phi_1$ become (asymptotically) stable in our stability analysis.

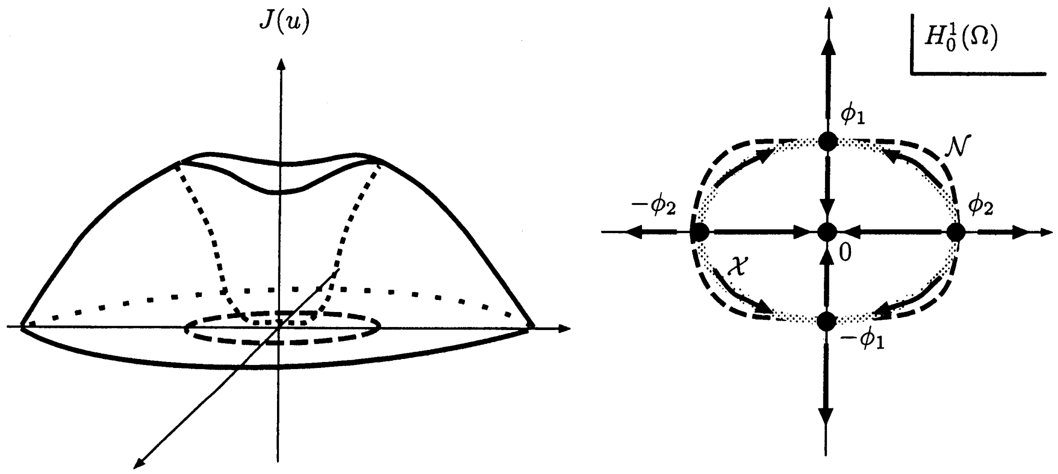

Fig.1. The mountain pass structure of the Lyapunov functional J and two surfaces $\mathcal{X}$ and $\mathcal{N}$ with critical points of J on $H_0^1(\Omega)$.

References

[1] Akagi G. and Kajikiya, R., Stability analysis of asymptotic profiles for sign-changing solutions to fast diffusion equations, submitted.

[2] Bénilan, P. and Crandall, M.G., The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$, Indiana Univ. Math. J. **30** (1981), 161–177.

[3] Berryman, J.G. and Holland, C.J., Nonlinear diffusion problem arising in plasma physics, Phys. Rev. Lett. **40** (1978), 1720–1722.

[4] Berryman, J.G. and Holland, C.J., Stability of the separable solution for fast diffusion, Arch. Rational Mech. Anal. **74** (1980), 379–388.

[5] Brézis, H., Monotonicity methods in Hilbert spaces and some applications to non-linear partial differential equations, Contributions to Nonlinear Functional Analysis, ed. Zarantonello, E., Academic Press, New York-London, 1971, pp.101–156.

[6] Dancer, E.N., The effect of the domain shape on the number of positive solutions of certain nonlinear equations, J. Differential Equations **74** (1988), 120–156.

[7] Dancer, E.N., Real analyticity and nondegeneracy, Math. Ann. **325** (2003), 369–392.

[8] DiBenedetto, E., *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York, 1993.

[9] Gazzola, F. and Weth, T., Finite time blow-up and global solutions for semilinear parabolic equations with initial data at high energy level, Differential Integral Equations **18** (2005), 961–990.

[10] Gidas, B., Ni, W.-M. and Nirenberg, L., Symmetry and related properties via the maximum principle, Comm. Math. Phys. **68** (1979), 209–243.

[11] Grossi, M., A uniqueness result for a semilinear elliptic equation in symmetric domains, Adv. Differential Equations **5** (2000), 193–212.

[12] Kwong, Y.C., Asymptotic behavior of a plasma type equation with finite extinction, Arch. Rational Mech. Anal. **104** (1988), 277–294.

[13] Lin, C.S., Uniqueness of least energy solutions to a semilinear elliptic equation in $\mathbb{R}^2$, Manuscripta Math. **84** (1994), 13–19.

[14] Pacella, F., Uniqueness of positive solutions of semilinear elliptic equations and related eigenvalue problems, Milan J. Math. **73** (2005), 221–236.

[15] Sabinina, E.S., On a class of non-linear degenerate parabolic equations, Dokl. Akad. Nauk SSSR **143** (1962), 794–797.

[16] Savaré, G. and Vespri, V., The asymptotic profile of solutions of a class of doubly nonlinear equations, Nonlinear Anal. **22** (1994), 1553–1565.

[17] Vázquez, J.L., *Smoothing and decay estimates for nonlinear diffusion equations. Equations of porous medium type*, Oxford Lecture Series in Mathematics and its Applications, 33. Oxford University Press, Oxford, 2006.

[18] Vázquez, J.L., *The porous medium equation. Mathematical theory*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2007.

[19] Zou, H., On the effect of the domain geometry on the uniqueness of positive solutions of $\Delta u + u^p = 0$, Ann. Sc. Norm. Sup., **21** (1995), 343–356.