<table>
<thead>
<tr>
<th>Title</th>
<th>A Consideration on Functions Preserving Set Inclusion Relation (Clone Theory and Discrete Mathematics・Algebra and Logic Related to Computer Science)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Takagi, Noboru</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2013), 1846: 94-112</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195047">http://hdl.handle.net/2433/195047</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
A Consideration on Functions Preserving Set Inclusion Relation

Noboru Takagi
Department of Electronics and Informatics, Toyama Prefectural University

Abstract—This paper discusses functions over the set of non-empty subsets of \( \{0, 1, \ldots, r-1\} \) that are monotonic in the set inclusion relation. Min, Max and Literal operations play an important role in multiple-valued logic design/circuits because they can realize any function over \( \{0, 1, \ldots, r-1\} \). Operations over the set of non-empty subsets of \( \{0, 1, \ldots, r-1\} \) that preserve the set inclusion relation are introduced from Min, Max and Literal operations over \( \{0, 1, \ldots, r-1\} \). Then, this paper proves some of mathematical properties of functions over the set of non-empty subsets of \( \{0, 1, \ldots, r-1\} \) that are composed of the operations introduced.

Keywords: Multiple-Valued Logic Design/Circuits, Set Inclusion Relation, Clone Theory

1 Introduction

S. C. Kleene [1] first introduced regularity into ternary operations over the set of truth values \( \{0, 1, u\} \) in the following way.

A truth table for a ternary operation is regular if it satisfies the condition that

"A given column (row) contains 1 in the u row (column), only if the column (row)
consists entirely of 1's; and likewise for 0".

Kleene's regularity is one of the ways how binary operations can be expanded into ternary operations. Table 1 is the truth tables of regular ternary operations, which are given from the traditional binary operations AND, OR and NOT.

It is worth to notice that M. Goto [2] independently introduced ternary operations that are identical with the Kleene's ternary operations in Table 1. He showed that the ternary operations can be a model for analyzing undetermined behavior existing in binary systems, such as hazards in binary logic circuits. After Goto's work, M. Mukaidono studied mathematical properties of functions over \( \{0, 1, u\} \) that can be expressed by a formula composed of the three ternary operations (He called the ternary functions regular ternary logic functions). One of Mukaidono's main results[3] is that a function \( f \) over \( \{0, 1, u\} \) is a regular ternary logic function if and only if the function \( f \) is monotonic in the partial ordered relation, defined by Figure 1. I. G. Rosenberg [8] indicated that the set of regular ternary logic functions is this clone generated by the Kleene's ternary logic, i.e., the clone is identical with the clone over the 3-element universe \( \{\{0\}, \{1\}, \{0, 1\}\} \) that preserves the set inclusion relation \( \subseteq \).

This paper discusses functions over the set of non-empty subsets of \( \{0, 1, \ldots, r-1\} \) when \( r \) is more than 2. In the following, \( E_r \) and \( P_r \) denote the \( r \)-valued set \( \{0, 1, \ldots, r-1\} \) and the set of non-empty subsets of \( E_r \), respectively.

Table 1: Truth Tables of Regular Ternary Operations NOT, AND and OR

<table>
<thead>
<tr>
<th>NOT</th>
<th>AND</th>
<th>OR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>u</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>u</td>
<td>u</td>
<td>u</td>
</tr>
</tbody>
</table>

\( \{0, 1, u\} \)
Preliminaries

If only capital non-empty Delta selected section constants multiple-valued operations singletons mathematical expressed subsets Rosenberg operations Literal they functions necessary subsets operations setting consisting operations partial obtained then formula equals important section denoted function elements properties operation partial expanded functions they discuss the focus is this paper. This paper then clarifies mathematical properties of functions over $P_r$, which are expressed by formulas composed of the operations given from Min, Max, and Literal operations over $E_r$.

This paper is organized below. Section 2 is for preliminaries. This section shows the definition for expanding operations over $E_r$ into operations over $P_r$, and then gives some of their mathematical properties. Section 3 focuses on Min, Max, and Delta Literal operations over $E_r$. They are expanded into operations over $P_r$, and then this section proves a necessary and sufficient condition for a function over $P_r$ to be expressed by a formula composed of these operations. Section 4 shows examples for the results obtained in Section 3. Section 5 discusses mathematical properties of functions over $P_r$ when we selected Min, Max, and Universal Literal operations over $E_r$. Then, Section 6 gives examples for the results appeared in Section 5 Section 7 concludes the paper.

2 Preliminaries

Let $E_r$ be the $r$-valued set $\{0, \ldots, r - 1\}$, and let $P_r$ be the set of all non-empty subsets of $E_r$, i.e., $P_r = 2^{E_r} - \{\emptyset\}$, where $2^{E_r}$ is the power set of $E_r$. If a subset of $E_r$ consists of only one element, then it is called a singleton. The set of all singletons of $E_r$ is denoted by $S_r$, i.e., $S_r = \{\{0\}, \ldots, \{r - 1\}\}$. It is evident that the set $P_r$ is a partial ordered set in the set inclusion $\subseteq$. In this paper, elements of the set $E_r$ are denoted by small letters such as $a, b, c, x, y$, etc., while elements of the set $P_r$ (i.e., non-empty subsets of $E_r$) are denoted by capital letters such as $A, B, C, X, Y$ etc.

**Definition 1** Let $o$ be an $n$-ary operation on $E_r$. Then, an $n$-ary operation $\hat{o}$ on $P_r$ with respect to $o$ is defined by setting

$$\hat{o}(A_1, \ldots, A_n) = \{o(a_1, \ldots, a_n) | a_i \in A_1, \ldots, a_n \in A_n\}$$

for any element $(A_1, \ldots, A_n) \in P_r^n$. (End of Definition)

The following three operations play an important role in multiple-valued logic design because $r$-valued functions consisting of these operations and the constants $0, \ldots, r - 1$ are
Table 2: Truth Table of $\wedge$

<table>
<thead>
<tr>
<th>$X \backslash Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>01</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>02</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>012</td>
</tr>
</tbody>
</table>

Table 3: Truth Table of $\sqcup$

<table>
<thead>
<tr>
<th>$X \backslash Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>01</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>02</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>012</td>
</tr>
</tbody>
</table>

Table 4: Truth Table of $X^S$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$X^0$</th>
<th>$X^1$</th>
<th>$X^2$</th>
<th>$X^{01}$</th>
<th>$X^{02}$</th>
<th>$X^{12}$</th>
<th>$X^{012}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>01</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>02</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>012</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

functionally complete on the set $E_r$ [5].

\[
a \cdot b = \min(a, b),
\]

\[
a + b = \max(a, b),
\]

\[
x^S = \begin{cases} r - 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}
\]

where $a, b \in E_r$ and $S \subseteq E_r$. The unary operations $x^S$ are often called the universal literals. However, when $S$ is a singleton, $x^S$ is sometimes called a delta literal.

For simplicity, in writing elements of $P_r$, we will remove brackets and put an underline if no confusion arises. That is, for example, $\underline{0}$, $\underline{02}$ and $\underline{012}$ stand for $\{0\}$, $\{0, 2\}$ and $\{0, 1, 2\}$, respectively. Tables 2, 3 and 4 are truth tables of operations on $P_r$ with respect to $\cdot$, $+$ and $x^S$, respectively. Because this paper focuses on the operations on $P_r$ with respect to $\cdot$, $+$ and $x^S$, they are denoted by $\wedge$, $\sqcup$ and $X^S$, respectively. 1

This paper does not allow any kinds of compositions of the operations $\wedge$, $\sqcup$ and $X^S$ on $P_r$. Compositions are restricted by the form of the formulas defined below.

**Definition 2** Formulas are defined inductively in the following way.

1. Constants $\{0\}, \ldots, \{r - 1\}$ and literals $X_i^S$ ($i = 1, \ldots, n$ and $S \in P_r$) are formulas.
2. If $G$ and $H$ are formulas, then $(G \wedge H)$ and $(G \sqcup H)$ are also formulas.
3. It is a formula if and only if we get it from (1) and (2) in a finite number of steps.

(End of Definition)

1 The operations $\wedge$ and $\sqcup$ do not satisfy the absorption laws and the distributive laws. Thus, the algebraic system $(P_r, \wedge, \sqcup)$ do not form a lattice.
In writing formulas, we sometimes omit the operation $\land$ for simplicity.

It is evident that every formula expresses a function on $P_r$ when each variable $X_i$ takes an element of $P_r$. Furthermore, it is easy to verify that the formulas can not express all of the functions on $P_r$, i.e., the functions on $P_r$ expressed by the formulas are not functionally complete on $P_r$. Thus, one of the main subjects of the paper is to clear what functions on $P_r$ can be expressed by the formulas.

In the following, for any elements $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ of $P_r^n$, $(A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)$ stands for $A_i \subseteq B_i$ for all $i$'s. Moreover, $(A_1, \ldots, A_n) \cap (B_1, \ldots, B_n) = \emptyset$ stands for $A_i \cap B_i = \emptyset$ for some $i$.

**Theorem 1** Suppose a function $f$ on $P_r$ can be expressed by a formula. Then, $f(A_1, \ldots, A_n) \in S_r$ holds for any element $(A_1, \ldots, A_n) \in S_r^n$.

**Theorem 2** Suppose a function $f$ on $P_r$ can be expressed by a formula. Then, $f(A_1, \ldots, A_n) \subseteq f(B_1, \ldots, B_n)$ holds for any elements $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ of $P_r^n$ such that $(A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)$.

### 3 Functions Expressed by Formulas Composed of $\land$, $\lor$, and Delta Literals

This section shows a necessary and sufficient condition for functions on $P_r$ that can be expressed by formulas with the operations $\land$, $\lor$, and delta literals.

**Theorem 3** Let $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ be elements of $P_r$. If a function $f$ on $P_r$ is expressed by a formula, then the least element of $f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n)$ (which is a subset of $E_r$) is equal to the least element of $f(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ for any elements $A$ and $B$ of $P_r - S_r$, i.e.,

$$\min f(a_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) = \min f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$$

holds for any elements $A$ and $B$ of $P_r - S_r$.

From Theorems 1, 2 and 3, any function $f$ on $P_r$ expressed by a formula satisfies the following Condition A.

**Condition A:** Let $f$ be a function on $P_r$.

1. If $(A_1, \ldots, A_n) \in S_r^n$, then $f(A_1, \ldots, A_n) \in S_r$.
2. For any elements $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ of $P_r^n$, $(A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)$ implies $f(A_1, \ldots, A_n) \subseteq f(B_1, \ldots, B_n)$.
3. Let $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ be elements of $P_r$. Then, the least element of $f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n)$ is equal to the least element of $f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$ for any elements $A$ and $B$ of $P_r - S_r$, i.e.,

$$\min f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) = \min f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$$

holds for any elements $A$ and $B$ of $P_r - S_r$.

---

2All of the proofs in this paper are omitted because of the limitation of the space.
In the remainder of this section, it is proven that Condition A is a necessary and sufficient condition for a function on \( P_r \) to be expressed by a formula with the operations \( \wedge, \sqcup \), and delta literals.

**Definition 3** Let \( f \) be a function on \( P_r \) and let \( A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n) \) be an element of \( S_r^{n-1} \). Then, we define one-variable functions \( \check{f}_A^i(X) \) and \( \hat{f}_A^i(X) \) \((i = 1, \ldots, n)\) expressed by the following formulas.

\[
\check{f}_A^i(X) = \bigcup_{s \in E_r} \left\{ \{s\} \land \bigcup_{B \in P_A^+(s)} X^B \right\}, \quad (1)
\]

where \( P_A^+(s) \) is the set of all maximal elements of the set

\[
P_A^i(s) = \{ B \in P_r \mid \min f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) = s \}, \quad (2)
\]

and

\[
\hat{f}_A^i(X) = \bigcup_{S \in P_r - S_f} \left( \bigcup_{t \in S} \left\{ \{t\} \land \bigcup_{B \in Q_A^+(S)} \left( \bigwedge_{e \in B} X^e \right) \right\} \right), \quad (3)
\]

where \( Q_A^+(S) \) is the set of all minimal elements of

\[
Q_A^i(S) = \{ B \in P_r - S_f \mid f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) = S \}. \quad (4)
\]

In the formulas (1) and (3), if \( P_A^i(s) \) and \( Q_A^i(S) \) are the empty set, then

\[
\bigcup_{B \in P_A^+(s)} X^B \text{ and } \bigcup_{B \in Q_A^+(S)} \left( \bigwedge_{e \in B} X^e \right)
\]

are defined as \( \{0\} \), respectively. Moreover, in the formula (1), when \( B = E_r \), then \( X^B \) is the constant \( \{r-1\} \).

In the formula (1), if \( f \) is a function satisfying Condition A, then any subset \( P_A^i(s) \) is a subset of \( S_r \), or it is equal to \( \{E_r\} \). Now, let us show this property. Suppose an element \( B \) of \( P_r - S_r \) is a member of \( P_A^i(s) \). Then, by Condition A(3), it follows that \( E_r \) is also a member of \( P_A^i(s) \). Therefore, when an element of \( P_r - S_r \) is a member of \( P_A^i(s) \), then \( E_r \) is also a member of \( P_A^i(s) \). This fact implies that \( P_A^i(s) \) is a subset of \( S_r \), or it is equal to \( \{E_r\} \). So, the formula (1) is well-defined when \( f \) is a function satisfying Condition A.

**Lemma 1** Let \( A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n) \) be an element of \( S_r^{n-1} \). Then, for a function \( f \) satisfying Condition A,

\[
f_A^i(B) = \begin{cases} f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) & \text{if } f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) \in S_r \\ K & \text{otherwise} \end{cases}
\]

holds for any element \( B \) of \( P_r \), where \( K \) is an element of \( P_r \) such that

\[
\{f_0\} \subseteq K \subseteq f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)
\]

and \( f_0 \) is the least element of \( f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) \). (End of Lemma)
Lemma 2 Let \(A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)\) be an element of \(S_r^{n-1}\). Then, for a function \(f\) satisfying Condition \(A\),

\[
\hat{f}_A^i(B) = \begin{cases} 
\{0\} & \text{if } f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) \in S_r \\
\{0\} \cup f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) & \text{otherwise}
\end{cases}
\]

holds for any element \(B \in P_r\).

Lemma 3 Let \(A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)\) be an element of \(S_r^{n-1}\). Then, for a function \(f\) satisfying Condition \(A\),

\[
\hat{f}_A^i(B) \cup \check{f}_A^i(B) = f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)
\]

holds for any element \(B \in P_r\).

In the following, this section proves that any function satisfying Condition \(A\) can be expressed by a formula, and also shows a method how a formula can be formulated by a function satisfying Condition \(A\).

Definition 4 Let \(f\) be a function on \(P_r\). Then, \(f_1\) is defined as a function on \(P_r\) expressed by the following formula.

\[
f_1(X_1, \ldots, X_n) = \bigcup_{i=1}^{n} f^i(X_1, \ldots, X_n),
\]

where

\[
f^i(X_1, \ldots, X_n) = \bigcup_{A=(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n) \in S_r^{n-1}} \left( \bigwedge_{j=1(j\neq i)}^{n} X_j^{A_j} \land \left( \check{f}_A^i(X_i) \cup \hat{f}_A^i(X_i) \right) \right).
\]

(End of Definition)

Here, let us introduce a subset of \(P_r^n\), which will be denoted by \(I(r, n)\), below.

\[
I(r, n) = \bigcup_{i=1}^{n} \{(A_1, \ldots, A_n) \in P_r^n \mid A_i \in P_r - S_r \text{ and } A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \in S_r\}
\]

That is, each element \((A_1, \ldots, A_n)\) of \(I(r, n)\) consists of elements of \(S_r\), but except for one.

Lemma 4 Let \(F\) be a function satisfying Condition \(A\). Then,

\[
f_1(A_1, \ldots, A_n) = \begin{cases} f(A_1, \ldots, A_n) & \text{if } f(A_1, \ldots, A_n) \in S_r \cup I(r, n) \\
K & \text{otherwise}
\end{cases}
\]

holds for any element \((A_1, \ldots, A_n) \in P_r^n\), where \(K\) is an element of \(P_r\) such that \(\{0\} \subseteq K \subseteq \{0\} \cup f(A_1, \ldots, A_n)\).

(End of Lemma)
Definition 5 Let $f$ be a function on $P_r$, let $S$ be an element of $P_r - S_r$, and let $\hat{T}(f, S)$ is the set of all minimal elements of the following subset of $P^n_r$.

$$T(f, S) = \{(A_1, \ldots, A_n) \in P^n_r \mid f(A_1, \ldots, A_n) = S \text{ and } (A_1, \ldots, A_n) \notin S^n_r \cup I(r, n)\}$$ (6)

Then, $f_2$ is defined as a function on $P_r$ expressed by the following formula.

$$f_2(X_1, \ldots, X_n) = \{s_0\} \cup \left[ \bigcup_{(A_1, \ldots, A_n) \in \hat{T}(f, S)} \left\{ \bigwedge_{b \in A_1} X_1^{(b)} \land \cdots \land \bigwedge_{b \in A_n} X_n^{(b)} \right\} \right],$$ (7)

where

$$f_S(X_1, \ldots, X_n) = \begin{cases} \{s_0\} & \text{if } (A_1, \ldots, A_n) \in S^n_r \cup I(r, n) \\
\{0\} & \text{otherwise} \end{cases}$$ (8)

and $s_0$ is the least element of $\bigcup_{(A_1, \ldots, A_n) \in P^n_r} f(A_1, \ldots, A_n)$. (End of Definition)

Lemma 5 Let $f$ be a function on $P_r$ satisfying Condition A. Then,

$$f_2(A_1, \ldots, A_n) = \begin{cases} \{s_0\} & \text{if } (A_1, \ldots, A_n) \in S^n_r \cup I(r, n) \\
\{0\} & \text{otherwise} \end{cases}$$

holds for any element $(A_1, \ldots, A_n) \in P^n_r$, where $s_0$ is the least element of the union $\bigcup_{(A_1, \ldots, A_n) \in P^n_r} f(A_1, \ldots, A_n)$. (End of Lemma)

Theorem 4 Let $f$ be a function on $P_r$ satisfying Condition A. Then,

$$f(A_1, \ldots, A_n) = f_1(A_1, \ldots, A_n) \cup f_2(A_1, \ldots, A_n)$$

holds for any element $(A_1, \ldots, A_n) \in P^n_r$, where $f_1$ and $f_2$ are the formulas (5) and (7), respectively. (End of Theorem)

4 Examples of Functions Satisfying Condition A

Consider the function $f$ on $P_3$ whose truth table is given in Table 5. It is not difficult to verify that $f$ satisfies Condition A. Then, this section illustrates how we can form the formula that expresses the function $f$.

Example 1 Let us first consider the formulas (1) and (3). It follows by Eq. (2) that we have the following three subsets of $P_3$.

$$P_3[0](0) = \{B \in P_3 \mid \min f(B, 0) = 0\} = \{0, 2, 01, 02, 012\}$$
$$P_3[1](1) = \{B \in P_3 \mid \min f(B, 0) = 1\} = \{1\}$$
$$P_3[2](2) = \{B \in P_3 \mid \min f(B, 0) = 2\} = \emptyset$$
Table 5: Example of Function $f$ Satisfying Condition A

<table>
<thead>
<tr>
<th>$X \setminus Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>00</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>012</td>
</tr>
<tr>
<td>01</td>
<td>01</td>
<td>01</td>
<td>02</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>12</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
</tbody>
</table>

Thus, since $\hat{P}_0^1(0) = \{012\}$, $\hat{P}_0^1(1) = \{1\}$, and $\hat{P}_0^1(2) = \emptyset$, we have the formula $\hat{f}_0^1(X)$ by Eq. (1).

\[
\hat{f}_0^1(X) = (\underline{0} \wedge \underline{012}) \sqcup (\underline{1} \wedge X^1) \sqcup (\underline{2} \wedge \underline{0})
= \underline{1} \wedge X^1
\] (9)

In a similar way, we have the formulas $\hat{f}_1^1(X), \hat{f}_2^1(X), \hat{f}_0^2(Y), \hat{f}_1^2(Y)$, and $\hat{f}_2^2(Y)$, below.

\[
\begin{align*}
\hat{f}_1^1(X) &= \underline{1}X^1 \sqcup X^2 \\
\hat{f}_2^1(X) &= X^1 \\
\hat{f}_0^2(Y) &= \emptyset \\
\hat{f}_1^2(Y) &= \underline{1} \sqcup Y^2 \\
\hat{f}_2^2(Y) &= Y^1
\end{align*}
\] (10)

Moreover, it follows by Eq. (4) that we have

\[
\begin{align*}
Q_0^0(01) &= \{ B \in P_3 - S_3 \mid f(B, \underline{0}) = \underline{01} \} = \{01\}, \\
Q_0^0(02) &= \{ B \in P_3 - S_3 \mid f(B, \underline{0}) = \underline{02} \} = \emptyset, \\
Q_0^0(12) &= \{ B \in P_3 - S_3 \mid f(B, \underline{0}) = \underline{12} \} = \emptyset, \text{ and} \\
Q_0^0(012) &= \{ B \in P_1 - S_3 \mid f(B, \underline{0}) = \underline{012} \} = \{12, 012\}.
\end{align*}
\]

Thus, since $\hat{Q}_0^0(01) = \{01\}, \hat{Q}_0^0(02) = \hat{Q}_0^0(12) = \emptyset$ and $\hat{Q}_0^0(012) = \{12\}$, we have the formula $\hat{f}_0^1(X)$ by Eq. (3).

\[
\hat{f}_0^1(X) = (0X^0X^1 \sqcup X^0X^1) \sqcup \emptyset \sqcup \emptyset \sqcup (0X^2X^2 \sqcup X^2X^2 \sqcup 2X^2X^2)
= X^0X^1 \sqcup X^1X^1 \sqcup X^1X^2.
\] (11)

In a similar way, we have the formulas $\hat{f}_1^1(X), \hat{f}_2^1(X), \hat{f}_0^2(Y), \hat{f}_1^2(Y)$, and $\hat{f}_2^2(Y)$, below.

\[
\begin{align*}
\hat{f}_1^1(X) &= \underline{1}X^0X^1 \sqcup X^3Y^2 \sqcup X^1X^3 \sqcup X^1X^2, \\
\hat{f}_2^1(X) &= X^0X^1 \sqcup X^1X^2, \\
\hat{f}_0^2(Y) &= Y^1Y^2, \\
\hat{f}_1^2(Y) &= X^3Y^2 \sqcup X^3Y^2 \sqcup Y^3Y^2 \sqcup Y^1Y^2, \\
\hat{f}_2^2(Y) &= Y^3Y^2 \sqcup Y^1Y^2.
\end{align*}
\] (12)
Table 6: Truth Tables of $\check{f}_{A}^{1}(X)$ and $\hat{f}_{A}^{1}(X)$

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{0}^{1}(X)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>01</td>
<td>0</td>
<td>1</td>
<td>01</td>
</tr>
<tr>
<td>$f_{1}^{1}(X)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>$f_{2}^{1}(X)$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{0}^{1}(X)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>0</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>$f_{1}^{1}(X)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>$f_{2}^{1}(X)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
</tr>
</tbody>
</table>

Table 7: Truth Tables of $\check{f}_{A}^{2}(Y)$ and $\hat{f}_{A}^{2}(Y)$

<table>
<thead>
<tr>
<th>$Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\check{f}_{A}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\hat{f}_{A}^{2}(Y)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$\check{f}_{A}^{2}(Y)$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$\hat{f}_{A}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\check{f}_{A}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>0</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>$\hat{f}_{A}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
</tr>
</tbody>
</table>

Table 8: Truth Table of $f_{A}^{i} = \check{f}_{A}^{i} \sqcup \hat{f}_{A}^{i}$

<table>
<thead>
<tr>
<th>X or Y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{0}^{1}(X)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>01</td>
<td>0</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>$f_{1}^{1}(X)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>$f_{2}^{1}(X)$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{0}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_{1}^{2}(Y)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$f_{2}^{2}(Y)$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
</tr>
</tbody>
</table>

Tables 6 and 7 show the truth tables of $\check{f}_{A}$ and $\hat{f}_{A}$ for which $i = 1, 2$ and $A \in \{0, 1, 2\}$.

(End of Example)

It follows by Lemma 3 that

$$f(X, B) = \check{f}_{A}^{1}(X) \sqcup \hat{f}_{A}^{1}(X) \quad \text{and} \quad f(B, Y) = \check{f}_{A}^{2}(Y) \sqcup \hat{f}_{A}^{2}(Y)$$

hold for every $A \in \{0, 1, 2\}$ and every $B \in P_{3}$, where $\check{f}_{A}^{1}(X)$, $f_{A}^{1}(X)$, $\check{f}_{A}^{2}(Y)$ and $\hat{f}_{A}^{2}(Y)$ have been obtained in Eqs. (9), (10), (11) and (12). Table 8 shows the truth tables of $\check{f}_{A} \sqcup \hat{f}_{A}$, where $i = 1, 2$ and $A \in \{0, 1, 2\}$.

**Example 2** Let us next consider the formula (5) in Definition 4. It follows by Eqs. (9) and (11) that we have the formula $\check{f}_{A}^{1}(X) \sqcup \hat{f}_{A}^{1}(X)$ below.

$$\check{f}_{A}^{1}(X) \sqcup \hat{f}_{A}^{1}(X) = X^{1} \sqcup X^{2} \sqcup X^{1}X^{2} \sqcup X^{1}X^{2} = X^{1} \sqcup X^{2}$$

In a similar way, by Eqs. (10), (11) and (12), we have the formulas

$$\check{f}_{A}^{1}(X) \sqcup \hat{f}_{A}^{1}(X) = X^{1} \sqcup X^{2},$$

$$f_{A}^{2}(X) \sqcup \check{f}_{A}^{2}(X) = X^{1},$$

$$\check{f}_{A}^{2}(Y) \sqcup \hat{f}_{A}^{2}(Y) = Y^{1}Y^{2},$$

$$f_{A}^{2}(Y) \sqcup \check{f}_{A}^{2}(Y) = Y^{1},$$

Therefore, the formula $f_{1}(X, Y)$ of (5) in Definition 4 is given as

$$f_{1}(X, Y) = f^{1}(X, Y) \sqcup f^{2}(X, Y),$$

(13)
Table 9: Truth Table of $f_1(X, Y)$

<table>
<thead>
<tr>
<th>X \ Y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>01</td>
<td>01</td>
<td>01</td>
<td>02</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>02</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>12</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
</tbody>
</table>

Table 10: Truth Table of $f_2(X, Y)$

<table>
<thead>
<tr>
<th>X \ Y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>01</td>
<td>02</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>01</td>
<td>01</td>
<td>01</td>
<td>02</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>12</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
</tbody>
</table>

where

\[
f^1(X, Y) = Y^0 \left( f_{02}^0(X) \cup f_{02}^1(X) \right) \cup Y^1 \left( f_{02}^2(X) \right) \cup Y^2 \left( f_{02}^3(X) \right) \quad \text{and}
\]

\[
f^2(X, Y) = X^0 \left( f_{02}^0(Y) \cup f_{02}^1(Y) \right) \cup X^1 \left( f_{02}^2(Y) \right) \cup X^2 \left( f_{02}^3(Y) \right).
\]

Table 9 is the truth table of $f_1(X, Y)$.

**Example 3**

In this example, let us consider the formula (7) in Definition 5. It follows by Eq. (6) that we have the following subsets of $P_3^2$.

\[
T(f, 01) = \emptyset
\]

\[
T(f, 02) = \{(02, 01), (02, 12), (02, 012)\}
\]

\[
T(f, 12) = \emptyset
\]

\[
T(f, 012) = \{(A, B) | A \in \{01, 12, 012\} \text{ and } B \in P_3 - S_3\}
\]

Therefore, since we have

\[
\hat{T}(f, 01) = \emptyset,
\]

\[
\hat{T}(f, 02) = \{(02, 01), (02, 12)\},
\]

\[
\hat{T}(f, 12) = \emptyset, \text{ and}
\]

\[
\hat{T}(f, 012) = \{(01, 01), (01, 02), (01, 12), (12, 01), (12, 02), (12, 12)\},
\]

it follows by Eq. (8) that we have the following formulas.

\[
f_{01}(X, Y) = \underline{0}
\]

\[
f_{02}(X, Y) = X^0X^2Y^2Y^2 \cup X^0X^2Y^2Y^2 \cup X^0X^2Y^2Y^2 \cup X^1X^2Y^2Y^2 \cup X^1X^2Y^2Y^2 \cup X^0X^2Y^2Y^2
\]

Thus, the formula $f_2(X, Y)$ of (7) in Definition 5 is obtained as the formula below.

\[
f_2(X, Y) = \underline{0} \cup \{0f_{02}(X, Y) \cup 2f_{02}(X, Y)\} \cup \{0f_{02}(X, Y) \cup 1f_{012}(X, Y) \cup 2f_{012}(X, Y)\}
\]

\[
= f_{02}(X, Y) \cup 1f_{012}(X, Y) \cup f_{012}(X, Y)
\]

(14)

Table 10 is the truth table of $f_2(X, Y)$.

(End of Example)

It follows by Theorem 4 that the function f of Table 5 can be expressed by the formula

\[f_1(X, Y) \cup f_2(X, Y),\]

where $f_1(X, Y)$ and $f_2(X, Y)$ are the formulas given in (13) and (14), respectively.
5 Functions Expressed by Formulas Composed of $\wedge$, $\sqcup$ and Universal Literals

This section discusses functions on $P_r$ expressed by formulas, which are composed of the operations $\wedge$, $\sqcup$ and universal literals. Then, a necessary and sufficient condition for a function on $P_r$ to be expressed by a formula when $r$ is equal to 3.

**Theorem 5** Let $f$ be a function on $P_r$. If $f$ can be expressed by a formula, then

$$\bigcap_{A \in P_r - S_r} f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) \neq \emptyset$$

holds for any elements $A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n$ of $P_r$. 

(End of Theorem)

By Theorems 1, 2 and 5, any function $f$ on $P_r$ expressed by a formula satisfies the following Condition B.

**Condition B:** Let $f$ be a function on $P_r$.

1. If $(A_1, \ldots, A_n) \in S^n_r$, then $f(A_1, \ldots, A_n) \in S_r$.
2. For any elements $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ of $P^n_r$, $(A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)$ implies $f(A_1, \ldots, A_n) \subseteq f(B_1, \ldots, B_n)$.
3. $\bigcap_{A \in P_r - S_r} f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) \neq \emptyset$ holds for any elements $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ of $P_r$.

In the following, this section proves that Condition B is a necessary and sufficient condition for a function on $P_3$ to be expressed by a formula with the operations $\wedge$, $\sqcup$, and universal literals.

**Definition 6** Let $(A_1, \ldots, A_n)$ be any element of $P^n_r$. Then, $\alpha = X_1^{A_1} \wedge \cdots \wedge X_n^{A_n}$ is said to be the type-1 term corresponding to $(A_1, \ldots, A_n)$. Next, let $(B_1, \ldots, B_n)$ be any element of $P^n_r - S^n_r$. Then, $\beta = \bigwedge_{e \in B_1} X_1^{\{e\}} \wedge \cdots \wedge \bigwedge_{e \in B_n} X_n^{\{e\}}$ is said to be the type-2 term corresponding to $(B_1, \ldots, B_n)$. 

(End of Definition)

Let $S$ be an element of $P_r$, and let $T$ be an element of $P_r - S_r$. Then, it is easy to verify that the following two equations are valid.

$$X^S = \begin{cases} \{r-1\} & \text{if } X \subseteq S \\ \{0\} & \text{if } X \cap S = \emptyset \\ \{0, r-1\} & \text{otherwise} \end{cases} \quad (15)$$

$$\bigwedge_{e \in T} X^{\{e\}} = \begin{cases} \{0, r-1\} & \text{if } T \subseteq X \\ \{0\} & \text{otherwise} \end{cases} \quad (16)$$

Therefore, for any type-1 term $\alpha$ and any type-2 term $\beta$, $\alpha(A_1, \ldots, A_n) = \{r-1\}$, $\{0, r-1\}$, or $\{0\}$, and $\beta(A_1, \ldots, A_n) = \{0, r-1\}$ or $\{0\}$ hold for any element $(A_1, \ldots, A_n) \in P^n_r$.

**Lemma 6** For any type-1 term $\alpha$ corresponding to $(A_1, \ldots, A_n) \in P^n_r$,
(1) \((B_1, \ldots, B_n) \subseteq (A_1, \ldots, A_n)\) iff \(\alpha(B_1, \ldots, B_n) = \{r-1\}\),

(2) \((A_1, \ldots, A_n) \cap (B_1, \ldots, B_n) = \emptyset\) iff \(\alpha(B_1, \ldots, B_n) = \{0\}\),

(3) \((B_1, \ldots, B_n) \not\in (A_1, \ldots, A_n)\) and \((A_1, \ldots, A_n) \cap (B_1, \ldots, B_n) = \emptyset\) iff \(\alpha(B_1, \ldots, B_n) = \{0, r-1\}\) hold for any \((B_1, \ldots, B_n) \in \mathcal{P}_r^n\).

(End of Lemma)

**Lemma 7** For any type-2 term \(\alpha\) corresponding to \((A_1, \ldots, A_n) \in \mathcal{P}_r^n - \mathcal{S}_r^n\),

(1) \((A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)\) iff \(\alpha(B_1, \ldots, B_n) = \{0, r-1\}\),

(2) \((A_1, \ldots, A_n) \not\subset (B_1, \ldots, B_n)\) iff \(\alpha(B_1, \ldots, B_n) = \{0\}\) hold for any \((B_1, \ldots, B_n) \in \mathcal{P}_r^n\).

(End of Lemma)

Let \(f\) be a function satisfying Condition B, and let \(S\) be an element of \(\mathcal{P}_r\). Then, define two subsets of \(\mathcal{P}_r^n\), denoted by \(\mathcal{L}(f, S)\) and \(\mathcal{U}(f, S)\), below:

\[
\mathcal{L}(f, S) = \{(A_1, \ldots, A_n) \in \mathcal{P}_r^n | f(A_1, \ldots, A_n) \subseteq S\}
\]

\[
\mathcal{U}(f, S) = \{(A_1, \ldots, A_n) \in \mathcal{P}_r^n | f(A_1, \ldots, A_n) \cap S \neq \emptyset\}.
\]

Let \(\mathcal{L}(f, S)\) and \(\mathcal{U}'(f, S)\) be the sets of all maximal elements of \(\mathcal{L}(f, S)\) and of all minimal elements of \(\mathcal{U}(f, S)\), respectively. Further, let \(\mathcal{L}(f, S) = \mathcal{U}'(f, S) - \mathcal{S}_r^n\).

**Lemma 8** Let \(f\) be a function satisfying Condition B, and let \(S\) be an element of \(\mathcal{P}_r\). Then, \((f)^S\) can be expressed by the following formula.

\[
(f)^S = \begin{cases} 
\bigcup_{A \in \mathcal{L}(f, S)} \alpha_A \cup \bigcup_{A \in \mathcal{U}'(f, S)} \beta_A & \text{if } \mathcal{L}(f, S) \neq \emptyset \text{ or } \mathcal{U}'(f, S) \neq \emptyset \\
\{0\} & \text{otherwise}
\end{cases}
\]

(17)

where \(\alpha_A\) and \(\beta_A\) are the type-1 and type-2 terms corresponding to \(A\), respectively.

(End of Lemma)

Now, let us consider formulas of one-variable functions satisfying Condition B. Any one-variable function \(f\) satisfying Condition B is in at least one of the following three cases:

(B-1) \(f(A) \neq 01\) holds for any element \(A \in \mathcal{P}_3 - \mathcal{S}_3\).

(B-2) \(f(A) \neq 02\) holds for any element \(A \in \mathcal{P}_3 - \mathcal{S}_3\).

(B-3) \(f(A) \neq 12\) holds for any element \(A \in \mathcal{P}_3 - \mathcal{S}_3\).

---

Footnote: If \(f\) is in neither one of the cases (B-1), (B-2), (B-3), then it implies that we have three distinct elements \(A, B, C\) in \(\mathcal{P}_3 - \mathcal{S}_3\) such that \(f(A) = 01\), \(f(B) = 02\), and \(f(C) = 12\). However, this contradicts to the fact that \(f\) satisfies Condition B(3).
Property 1 Any one-variable function $f$ satisfying Condition B can be expressed by the following formula.

$$f(X) = \begin{cases} 
    f^{12}(X) \land (1 \cup f^{02}(X)) & \text{if } f \text{ is in the case (B-1)} \\
    (1 \land f^{1}(X)) \cup f^{2}(X) \cup (1 \land f^{12}(X)) & \text{if } f \text{ is in the case (B-2)} \\
    (1 \land f^{1}(X)) \cup f^{2}(X) & \text{if } f \text{ is in the case (B-3)} 
\end{cases} \quad (18)$$

(End of Property)

By Property 1, every one-variable function satisfying Condition B can be expressed by a formula.

Next, let us consider the case where functions satisfying Condition B depend more than one variable. Then, any function $f$ satisfying Condition B is in at least one of the three cases below.

(B-4) $f(A_1, \ldots, A_n) \neq 01$ holds for any element $(A_1, \ldots, A_n) \in (P_3 - S_3)^n$.

(B-5) $f(A_1, \ldots, A_n) \neq 12$ holds for any element $(A_1, \ldots, A_n) \in (P_3 - S_3)^n$.

(B-6) $\bigcap_{(A_1, \ldots, A_n) \in (P_3 - S_3)^n} f(A_1, \ldots, A_n) = 1$.

Tables 11, 12 and 13 are examples of two-variable functions being in the cases (B-4), (B-5), and (B-6), respectively.

Then, we can prove Properties 1 ~ 6, which show a way for constructing formulas of $n$-variable functions satisfying Condition B.

Let $A$ be an element $(A_1, \ldots, A_{i-1}, X, A_{i+1}, \ldots, A_n)$ of $P_r^{n-1}$. Then, denote the one-variable function $f(A_1, \ldots, A_{i-1}, X, A_{i+1}, \ldots, A_n)$ by $f_A^i(X)$. 
Property 2 Suppose a function $f$ satisfying Condition B is in the case (B-5). Let $f'$ be a function expressed by the formula

$$f' = p^1 \sqcup \cdots \sqcup p^n,$$

(19)

where

$$p^i(X_1, \ldots, X_n) = \bigcup_{A=(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n) \in \mathcal{S}^{n-1}_3} (f_A^i(X_i) \land X_1^{A_1} \land \cdots \land X_{i-1}^{A_{i-1}} \land X_{i+1}^{A_{i+1}} \land \cdots \land X_n^{A_n}).$$

(20)

Then, for any element $(A_1, \ldots, A_n) \in \mathcal{P}^n_3$,

$$f'(A_1, \ldots, A_n) = \begin{cases} f(A_1, \ldots, A_n) & \text{if } (A_1, \ldots, A_n) \not\in (P_3 - S_3)^n \\ K & \text{otherwise} \end{cases}$$

where $K$ is an element of $P_r$ such that $\{0\} \subseteq K \subseteq \{0\} \cup F(A_1, \ldots, A_n)$. (End of Property)

Property 3 Suppose a function $f$ satisfying Condition B is in the case (B-5). Let $f''$ be a function expressed by the formula

$$f''(X_1, \ldots, X_n) = \bigcup_{S \in P_3 - S_3} \left\{ \bigcup_{t \in \hat{T}(S)} (\{t\} \land f_{\hat{T}(S)}(X_1, \ldots, X_n)) \right\},$$

(21)

where $\hat{T}(S)$ is the set of all minimal elements of the set

$$T(S) = \{(A_1, \ldots, A_n) \in (P_3 - S_3)^n \mid f(A_1, \ldots, A_n) = S\}$$

and

$$f_{\hat{T}(S)}(X_1, \ldots, X_n) = \bigcup_{(A_1, \ldots, A_n) \in \hat{T}(S)} \left\{ \bigwedge_{e \in A_1} X^{[e]} \land \cdots \land \bigwedge_{e \in A_n} X^{[e]} \right\}.$$  

(22)

Then, for any element $(A_1, \ldots, A_n) \in \mathcal{P}^n_3$,

$$f''(A_1, \ldots, A_n) = \begin{cases} \{0\} & \text{if } (A_1, \ldots, A_n) \not\in (P_3 - S_3)^n \\ f(A_1, \ldots, A_n) & \text{otherwise} \end{cases}$$

(End of Property)

Property 4 Any function $f$ satisfying Condition B can be expressed by $f = f' \sqcup f''$, if $f$ is in the case (B-5). (End of Property)

Property 5 Suppose a function $f$ satisfying Condition B is in the case (B-4). Let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of $S_3^{n-1}$. Then, define $g_A^i$ and $h$ as functions on $P_r$ expressed by the following formulas.

$$g_A^i(X_1, \ldots, X_n) = f_A^i(X_i) \sqcup (X_1^{A_1} \land \cdots \land X_{i-1}^{A_{i-1}} \land X_{i+1}^{A_{i+1}} \land \cdots \land X_n^{A_n}),$$

(23)

where $A'_j = E_3 - A_j$ ($j = 1, \ldots, i-1, i+1, \ldots, n$).

$$h(X_1, \ldots, X_n) = \left\{ \left( \bigcup_{A \in \mathcal{S}_3} X_i^A \right) \cup f^{12}(X_1, \ldots, X_n) \right\} \land \left\{ \left( \bigcup_{A \in \mathcal{S}_3} X_i^A \right) \cup f^{02}(X_1, \ldots, X_n) \cup \bot \right\}.$$  

(24)
Then, $f$ can be expressed by the following formula.

$$f(X_1, \ldots, X_n) = G(X_1, \ldots, X_n) \land h(X_1, \ldots, X_n),$$

where $G$ is $\land$-ing of all the $g^i_A$s of Eq. (23), i.e.,

$$G(X_1, \ldots, X_n) = \bigwedge_{i=1}^{n} \left( \bigwedge_{A \in S_{3}^{n-1}} g^i_A(X_1, \ldots, X_n) \right).$$

(25)

(End of Property)

Property 6 Suppose a function $f$ satisfying Condition B is in the case (B-6). Then, $f$ is in either one of the following two cases.

1. $f(A) \neq \underline{02}$ holds for any element $A \in P_3^n$, or
2. $f(A) = \underline{02}$ holds for some element $A \in P_3^n$.

If $f$ is in the case (1), then $f$ can be expressed by

$$f(X_1, \ldots, X_n) = (1 \land f^{1}(X_1, \ldots, X_n)) \cup f^{2}(X_1, \ldots, X_n) \cup (1 \land f^{12}(X_1, \ldots, X_n)).$$

(27)

Let $w$ be a function expressed by the following formula.

$$w(X_1, \ldots, X_n) = \bigcup_{(A_1, \ldots, A_n) \in Q_{\underline{02}}} \xi_1(A_1) \cup \cdots \cup \xi_n(A_n),$$

(28)

where $Q_{\underline{02}} = \{(A_1, \ldots, A_n) \in P_3^n \mid f(A_1, \ldots, A_n) = \underline{02}\}$.

Then, $f$ can be expressed by the following formula, if $f$ is in the case (2).

$$f(X_1, \ldots, X_n) = G(X_1, \ldots, X_n) \land w(X_1, \ldots, X_n),$$

(29)

where $G(X_1, \ldots, X_n)$ is given by Eq. (26).

(End of Property)

6 Examples of Function Satisfying Condition B

This section shows examples of 2-variable functions satisfying Condition B, and illustrates how they can be expressed by formulas.

Example 4 Consider the function $f$ defined by Table 12, which is in the case (B-5). The formula expressing $f$ is given by Properties 2, 3, and 4. First, consider the formulas $f^1_A(X)$ and $f^2_A(Y)$, which appear in Eq. (20). Table 14 shows the truth tables of the six one-variable functions $f^1_A(X)$, $f^1_B(X)$, $f^2_A(X)$, $f^2_B(Y)$, $f^3_Y(Y)$ and $f^2_Y(Y)$. Since $f^2_A(X)$ and $f^2_A(Y)$ are in (B-2) (or (B-3)), $f^1_A(X)$ is in (B-3), and $f^2_Y(Y)$ is in (B-1), it follows by Eq. (17) and (18) that these one-variable functions are expressed by the following formulas.

---

4 $Q_{\underline{02}} \cap (P_3 - S_3)^n = \emptyset$ holds, since $f$ is in the case (B-6).
Table 14: One-Variable Functions $f^1_A$ and $f^2_A$ of Example 4

<table>
<thead>
<tr>
<th>$X$ or $Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f^1_A(X)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f^2_A(X)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>$f^1_A(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f^2_A(Y)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
</tbody>
</table>

Table 15: Truth Table of $f'$ of Example 4

<table>
<thead>
<tr>
<th>$X$ \ $Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>01</td>
<td>0</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>02</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>01</td>
<td>0</td>
<td>01</td>
<td>0</td>
<td>01</td>
<td>0</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>02</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>12</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>012</td>
<td>0</td>
<td>012</td>
<td>01</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>012</td>
<td>0</td>
<td>012</td>
<td>01</td>
<td>012</td>
<td>012</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$X$ \ $Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>01</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>02</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Therefore, it follows by Eq. (19) that $f'$ is expressed by the following formula.

$$f' = X^0f^2_A(Y) \sqcup X^1f^2_A(Y) \sqcup X^2f^2_A(Y) \sqcup f^1_A(X)Y^1 \sqcup f^1_A(X)Y^2$$

(30)

Table 15 is the truth table of $f'$. Next, consider $f''$ in Eq. (21). Since

$$\hat{T}(01) = \{(01,01), (01,12)\},$$
$$\hat{T}(02) = \hat{T}(12) = 0,$$ and
$$\hat{T}(012) = \{(02,01), (02,02), (02,12), (12,01), (12,02), (12,12)\},$$

it follows by Eq. (22) that we have the following formulas.

$$f_{\hat{T}(01)}(X,Y) = X^0X^1Y^0Y^1 \sqcup X^0X^1Y^1Y^2$$
$$f_{\hat{T}(012)}(X,Y) = X^0X^2Y^0Y^1 \sqcup X^0X^2Y^0Y^2 \sqcup X^0X^2Y^1Y^2 \sqcup X^1X^2Y^0Y^1 \sqcup X^1X^2Y^1Y^2 \sqcup X^1X^2Y^2Y^2$$

We then have $f''(X,Y)$ below by Eq. (21).

$$f''(X,Y) = 1f_{\hat{T}(01)}(X,Y) \sqcup 1f_{\hat{T}(012)}(X,Y) \sqcup f_{\hat{T}(012)}(X,Y)$$

(31)

Table 16 is the truth table of $f''$. It follows by Property 4 that $f(X,Y) = f'(X,Y) \sqcup f''(X,Y)$.

(End of Example)
Example 5 Consider the function $f$ defined by Table 11, which is in (B-4). The formula expressing $f$ is given by Property 5. It follows by Eq. (17) and (18) that the one-variable functions $f_A^1(X)$ and $f_A^2(Y)$ are obtained below.

$$
\begin{align*}
  f_A^1(X) &= X^1, & f_A^1(Y) &= 1 \cup X^{12}, & f_A^2(X) &= 1 \cup X^{01}, & f_A^2(Y) &= Y^{12}(1 \cup Y^{01}). \\
  f_A^1(X, Y) &= X^1 \cup Y^{12}, & f_A^1(X, Y) &= 1 \cup X^{12} \cup Y^{02}, & f_A^1(X, Y) &= 1 \cup X^{01} \cup Y^{01}, & f_A^2(X, Y) &= X^{01} \cup Y^{12}(1 \cup Y^{01}). \\
  f_A^2(X, Y) &= X^{12} \cup 1X^{12} \cup Y^{12}, & f_A^2(X, Y) &= 2, & f_A^2(X, Y) &= Y^{12}(1 \cup Y^{01}).
\end{align*}
$$

Then, by Eq. (23), we have

$$
\begin{align*}
  g_0^1(X, Y) &= X^1 \cup Y^{12}, & g_1^1(X, Y) &= 1 \cup X^{12} \cup Y^{02}, & g_2^1(X, Y) &= 1 \cup X^{01} \cup Y^{01}, & g_2^2(X, Y) &= X^{01} \cup Y^{12} \cup Y^{01}.
\end{align*}
$$

By Eq. (26), we have the function $G(X, Y)$ expressed by $\wedge$-ing of all the above $g_A^i(X, Y)$'s. Table 17 is the truth table of $G(X, Y)$. Next, consider $h$ in Eq. (24). It follows by Eq. (17) that the functions $f_A^{12}(X, Y)$ and $f_A^{02}(X, Y)$ are expressed by the following formulas.

$$
\begin{align*}
  f_A^{12}(X, Y) &= X^1 \cup X^{12} \cup Y^{12} \cup Y^{01} \cup X^{01} \cup X^{12} \cup Y^{01} \\
  f_A^{02}(X, Y) &= X^{01} \cup X^{12} \cup Y^{01} \cup Y^{12} \cup Y^{01} \cup Y^{12}.
\end{align*}
$$

Thus, by Eq. (24), we have

$$
\begin{align*}
  h(X, Y) &= (v(X, Y) \cup f_A^{12}(X, Y)) \wedge (v(X, Y) \cup f_A^{02}(X, Y) \cup 1),
\end{align*}
$$

where $v(X, Y) = X^0 \cup X^1 \cup X^2 \cup Y^0 \cup Y^1 \cup Y^2$. Table 18 is the truth table of $h(X, Y)$. Lastly, by Eq. (25), $f(X, Y)$ are expressed by the following formula.

$$
\begin{align*}
  f(X, Y) &= G(X, Y) \wedge h(X, Y) \\
  &= g_0^0(X, Y) \wedge g_1^1(X, Y) \wedge g_2^2(X, Y) \wedge f_A^{12}(X, Y) \wedge g_A^1(X, Y) \wedge g_A^2(X, Y) \wedge h(X, Y).
\end{align*}
$$

Example 6 Consider the function $f$ define by Table 13, which is in (B-6). The formula expressing $f$ is given by Property 6. Since $f$ is in the case (2) of Property 6, $f$ is expressed by the formula given in Eq. (29).

First, consider the formula $G(X, Y)$. It follows by Eq. (17) and (18) that one-variable functions $f_A^1(X)$ and $f_A^2(Y)$ are obtained below.
Thus, we obtain \( g_{A}^{1}(X, Y) \) and \( g_{A}^{2}(X, Y) \) of Eq. (23) below.

\[
\begin{align*}
g_{A}^{0}(X, Y) &= Y^{12} \\
g_{A}^{1}(X, Y) &= \{1 \land X^{12}\} \cup X^{2} \\
g_{A}^{2}(X, Y) &= X^{12} \land \{1 \lor X^{02}\}
\end{align*}
\]

By Eq. (26), we have the function \( G(X, Y) \) expressed by \( \land \)-ing of all the above \( g_{A}^{i}(X, Y) \)'s. Table 19 is the truth table of \( G \).

Next, consider \( w(X, Y) \) of Eq. (28). Because

\[
Q_{02} = \{(2, 01), (2, 02), (2, 012), (02, 02)\},
\]

it follow by Eq. (28) that the formula \( w \) is obtained below.

\[
w(X, Y) = 1 \lor X^{2} \lor Y^{2}
\]

Table 20 is the truth table of \( w \). Lastly, it follows by Eq. (29) that the following formula expresses the function \( f \).

\[
f(X, Y) = G(X, Y) \land w(X, Y) = g_{A}^{0}(X, Y) \land g_{A}^{1}(X, Y) \land g_{A}^{2}(X, Y) \land g_{A}^{3}(X, Y) \land g_{A}^{4}(X, Y) \land w(X, Y).
\]

(End of Example)

7 Conclusions

This paper discussed functions over \( P_{r} \) that preserves the set inclusion relation \( \subseteq \). We referred the three kinds of operations Min, Max, and Literals over \( E_{r} \), because they are functionally complete on the \( r \)-valued set \( E_{r} \). This paper then proved some of the mathematical properties of functions over \( P_{r} \) that can be expressed by formulas. It is one of the open problems that which set of operations \( o_{1}, o_{2}, \ldots, o_{m} \) over \( P_{r} \) can realize any function over \( P_{r} \) preserving the set inclusion relation \( \subseteq \).
References


