A Consideration on Functions Preserving Set Inclusion Relation

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Abstract—This paper discusses functions over the set of non-empty subsets of \{0,1,\ldots,r-1\} that are monotonic in the set inclusion relation. Min, Max and Literal operations play an important role in multiple-valued logic design/circuits because they can realize any function over \{0,1,\ldots,r-1\}. Operations over the set of non-empty subsets of \{0,1,\ldots,r-1\} that preserve the set inclusion relation are introduced from Min, Max and Literal operations over \{0,1,\ldots,r-1\}. Then, this paper proves some of mathematical properties of functions over the set of non-empty subsets of \{0,1,\ldots,r-1\} that are composed of the operations introduced.

Keywords: Multiple-Valued Logic Design/Circuits, Set Inclusion Relation, Clone Theory

1 Introduction

S. C. Kleene [1] first introduced regularity into ternary operations over the set of truth values \(\{0,1,u\}\) in the following way.

A truth table for a ternary operation is regular if it satisfies the condition that

"A given column (row) contains 1 in the u row (column), only if the column (row) consists entirely of 1's; and likewise for 0'."

Kleene's regularity is one of the ways how binary operations can be expanded into ternary operations. Table 1 is the truth tables of regular ternary operations, which are given from the traditional binary operations AND, OR and NOT.

It is worth to notice that M. Goto [2] independently introduced ternary operations that are identical with the Kleene's ternary operations in Table 1. He showed that the ternary operations can be a model for analyzing undetermined behavior existing in binary systems, such as hazards in binary logic circuits. After Goto's work, M. Mukaidono studied mathematical properties of functions over \(\{0,1,u\}\) that can be expressed by a formula composed of the three ternary operations (He called the ternary functions regular ternary logic functions). One of Mukaidono's main results[3] is that a function \(f\) over \(\{0,1,u\}\) is a regular ternary logic function if and only if the function \(f\) is monotonic in the partial ordered relation, defined by Figure 1. I. G. Rosenberg [8] indicated that the set of regular ternary logic functions is this clone generated by the Kleene's ternary logic, i.e., the clone is identical with the clone over the 3-element universe \(\{\{0\},\{1\},\{0,1\}\}\) that preserves the set inclusion relation \(\subseteq\).

This paper discusses functions over the set of non-empty subsets of \(\{0,1,\ldots,r-1\}\) when \(r\) is more than 2. In the following, \(E_r\) and \(P_r\) denote the \(r\)-valued set \(\{0,1,\ldots,r-1\}\) and the set of non-empty subsets of \(E_r\), respectively.

| Table 1: Truth Tables of Regular Ternary Operations NOT, AND and OR |
|------------------|------------------|------------------|
| NOT | AND | OR |
| 0  | 1   | 0 | 1 | u |
| 1  | 0   | 1 | 1 | u |
| u  | u   | u | u | u |
First, this paper shows a definition for expanding operations over \( E_r \) into operations over \( P_r \). This definition is identical with the Kleene's regularity when \( r \) is equal to 2, and it has already been shown by M. Mukaidono [4] and I. G. Rosenberg [8]. Min, Max, and Literal operations play an important role in multiple-valued logic design/circuits, because they can realize any multiple-valued logic function over \( E_r \). Therefore, Min, Max, and Literal operations are focused on in this paper. This paper then clarifies mathematical properties of functions over \( P_r \), which are expressed by formulas composed of the operations given from Min, Max, and Literal operations over \( E_r \).

This paper is organized below. Section 2 is for preliminaries. This section shows the definition for expanding operations over \( E_r \) into operations over \( P_r \), and then gives some of their mathematical properties. Section 3 focuses on Min, Max, and Delta Literal operations over \( E_r \). They are expanded into operations over \( P_r \), and then this section proves a necessary and sufficient condition for a function over \( P_r \) to be expressed by a formula composed of these operations. Section 4 shows examples for the results obtained in Section 3. Section 5 discusses mathematical properties of functions over \( P_r \) when we selected Min, Max, and Universal Literal operations over \( E_r \). Then, Section 6 gives examples for the results appeared in Section 5 Section 7 concludes the paper.

## 2 Preliminaries

Let \( E_r \) be the \( r \)-valued set \( \{0, \ldots, r - 1\} \), and let \( P_r \) be the set of all non-empty subsets of \( E_r \), i.e., \( P_r = 2^{E_r} - \{\emptyset\} \), where \( 2^{E_r} \) is the power set of \( E_r \). If a subset of \( E_r \) consists of only one element, then it is called a singleton. The set of all singletons of \( E_r \) is denoted by \( S_r \), i.e., \( S_r = \{0\}, \ldots, \{r - 1\} \). It is evident that the set \( P_r \) is a partial ordered set in the set inclusion \( \subseteq \). In this paper, elements of the set \( E_r \) are denoted by small letters such as \( a, b, c, x, y \), etc., while elements of the set \( P_r \) (i.e., non-empty subsets of \( E_r \)) are denoted by capital letters such as \( A, B, C, X, Y \) etc.

**Definition 1** Let \( o \) be an \( n \)-ary operation on \( E_r \). Then, an \( n \)-ary operation \( \hat{o} \) on \( P_r \) with respect to \( o \) is defined by setting

\[
\hat{o}(A_1, \ldots, A_n) = \{o(a_1, \ldots, a_n) \mid a_i \in A_1, \ldots, a_n \in A_n\}
\]

for any element \( (A_1, \ldots, A_n) \in P_r^n \).  

(End of Definition)

The following three operations play an important role in multiple-valued logic design because \( r \)-valued functions consisting of these operations and the constants \( 0, \ldots, r - 1 \) are
Table 2: Truth Table of $\wedge$

<table>
<thead>
<tr>
<th>$X \setminus Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>01</td>
<td>01</td>
<td>1</td>
<td>01</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>01</td>
<td>02</td>
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<tr>
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<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>1</td>
<td>12</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
</tbody>
</table>

Table 3: Truth Table of $\cup$

<table>
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<th>$X \setminus Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
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<td>0</td>
</tr>
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<td>0</td>
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<td>1</td>
<td>01</td>
<td>01</td>
<td>1</td>
<td>01</td>
</tr>
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<td>1</td>
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<td>01</td>
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<td>01</td>
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<tr>
<td>012</td>
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<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
</tbody>
</table>

Table 4: Truth Table of $X^S$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$\overline{0}$</th>
<th>1</th>
<th>2</th>
<th>$\overline{01}$</th>
<th>$\overline{02}$</th>
<th>$\overline{12}$</th>
<th>$\overline{012}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^2$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
</tr>
<tr>
<td>$X^4$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
</tr>
<tr>
<td>$X^8$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
</tr>
<tr>
<td>$X^{16}$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
</tr>
<tr>
<td>$X^{32}$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
</tr>
<tr>
<td>$X^{64}$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
</tr>
<tr>
<td>$X^{128}$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
<td>$\overline{02}$</td>
</tr>
</tbody>
</table>

The operations $\wedge$ and $\cup$ do not satisfy the absorption laws and the distributive laws. Thus, the algebraic system $(P_r, \wedge, \cup)$ do not form a lattice.

**Definition 2** Formulas are defined inductively in the following way.

1. Constants $\{0\}, \ldots, \{r-1\}$ and literals $X_i^S$ ($i = 1, \ldots, n$ and $S \subseteq E_r$) are formulas.
2. If $G$ and $H$ are formulas, then $(G \wedge H)$ and $(G \cup H)$ are also formulas.
3. It is a formula if and only if we get it from (1) and (2) in a finite number of steps.

(End of Definition)
In writing formulas, we sometimes omit the operation \( \wedge \) for simplicity.

It is evident that every formula expresses a function on \( \mathcal{P} \), when each variable \( X_i \) takes an element of \( \mathcal{P} \). Furthermore, it is easy to verify that the formulas can not express all of the functions on \( \mathcal{P} \), i.e., the functions on \( \mathcal{P} \) expressed by the formulas are not functionally complete on \( \mathcal{P} \). Thus, one of the main subjects of the paper is to clear what functions on \( \mathcal{P} \) can be expressed by the formulas.

In the following, for any elements \((A_1, \ldots, A_n)\) and \((B_1, \ldots, B_n)\) of \( \mathcal{P}^n \), \((A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)\) stands for \( A_i \subseteq B_i \) for all \( i \)'s. Moreover, \((A_1, \ldots, A_n) \cap (B_1, \ldots, B_n) = \emptyset\) stands for \( A_i \cap B_i = \emptyset \) for some \( i \).

**Theorem 1** Suppose a function \( f \) on \( \mathcal{P} \) can be expressed by a formula. Then, \( f(A_1, \ldots, A_n) \in S_r \) holds for any element \((A_1, \ldots, A_n) \in \mathcal{P}^n \).

**Theorem 2** Suppose a function \( f \) on \( \mathcal{P} \) can be expressed by a formula. Then, \( f(A_1, \ldots, A_n) \subseteq f(B_1, \ldots, B_n) \) holds for any elements \((A_1, \ldots, A_n)\) and \((B_1, \ldots, B_n)\) of \( \mathcal{P}^n \) such that \((A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)\).

### 3 Functions Expressed by Formulas Composed of \( \wedge, \cup, \) and Delta Literals

This section shows a necessary and sufficient condition for functions on \( \mathcal{P} \), that can be expressed by formulas with the operations \( \wedge, \cup \) and delta literals.

**Theorem 3** Let \( A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \) be elements of \( \mathcal{P} \). If a function \( f \) on \( \mathcal{P} \) is expressed by a formula, then the least element of \( f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) \) (which is a subset of \( E_r \)) is equal to the least element of \( f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) \) for any elements \( A \) and \( B \) of \( \mathcal{P} - \mathcal{S}_r \), i.e.,

\[
\min f(a_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) = \min f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)
\]

holds for any elements \( A \) and \( B \) of \( \mathcal{P} - \mathcal{S}_r \).

From Theorems 1, 2 and 3, any function \( f \) on \( \mathcal{P} \) expressed by a formula satisfies the following Condition A.

**Condition A:** Let \( f \) be a function on \( \mathcal{P} \).

1. If \((A_1, \ldots, A_n) \in S_r^n \), then \( f(A_1, \ldots, A_n) \in S_r \).
2. For any elements \((A_1, \ldots, A_n)\) and \((B_1, \ldots, B_n)\) of \( \mathcal{P}^n \), \((A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)\) implies \( f(A_1, \ldots, A_n) \subseteq f(B_1, \ldots, B_n) \).
3. Let \( A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \) be elements of \( \mathcal{P} \). Then, the least element of \( f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) \) is equal to the least element of \( f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) \) for any elements \( A \) and \( B \) of \( \mathcal{P} - \mathcal{S}_r \), i.e.,

\[
\min f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) = \min f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)
\]

holds for any elements \( A \) and \( B \) of \( \mathcal{P} - \mathcal{S}_r \).

\(^2\)All of the proofs in this paper are omitted because of the limitation of the space.
In the remainder of this section, it is proven that Condition A is a necessary and sufficient condition for a function on $P_{r}$ to be expressed by a formula with the operations $\wedge$, $\sqcup$, and delta literals.

**Definition 3** Let $f$ be a function on $P_{r}$, and let $A = (A_{1}, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{n})$ be an element of $S_{r}^{n-1}$. Then, we define one-variable functions $\hat{f}_{A}^{i}(X)$ and $\check{f}_{A}^{i}(X)$ (i = 1, \ldots, n) expressed by the following formulas.

\[
\check{f}_{A}^{i}(X) = \bigcup_{s \in E_{r}} \left( \{s\} \wedge \bigcup_{B \in \check{P}_{A}^{i}(s)} X^B \right),
\]

where $\check{P}_{A}^{i}(s)$ is the set of all maximal elements of the set

\[
P_{A}^{i}(s) = \{B \in P_{r} | \min f(A_{1}, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_{n}) = s\},
\]

and

\[
\hat{f}_{A}^{i}(X) = \bigcup_{s \in P_{r}-S_{r}} \left( \bigcup_{t \in S} \left( \{t\} \wedge \bigcup_{B \in Q_{A}^{i}(S)} \left( \bigwedge_{e \in B} X^e \right) \right) \right),
\]

where $Q_{A}^{i}(S)$ is the set of all minimal elements of

\[
Q_{A}^{i}(S) = \{B \in P_{r} - S_{r} | f(A_{1}, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_{n}) = S\}.
\]

In the formulas (1) and (3), if $\check{P}_{A}^{i}(s)$ and $Q_{A}^{i}(S)$ are the empty set, then $\check{P}_{A}^{i}(s)$ is a subset of $S_{r}$, or it is equal to $\{E_{r}\}$. So, the formula (1) is well-defined when $f$ is a function satisfying Condition A.

**Lemma 1** Let $A = (A_{1}, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{n})$ be an element of $S_{r}^{n-1}$. Then, for a function $f$ satisfying Condition A,

\[
\check{f}_{A}^{i}(B) = \begin{cases} f(A_{1}, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_{n}) & \text{if } f(A_{1}, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_{n}) \in S_{r} \\ K & \text{otherwise} \end{cases}
\]

holds for any element $B$ of $P_{r}$, where $K$ is an element of $P_{r}$ such that

\[
\{f_{0}\} \subseteq K \subseteq f(A_{1}, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_{n})
\]

and $f_{0}$ is the least element of $f(A_{1}, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_{n})$.
Lemma 2 Let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of $S_r^{n-1}$. Then, for a function $f$ satisfying Condition $A$,

$$
\hat{f}_A^i(B) = \begin{cases} 
\{0\} & \text{if } f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) \in S_r 
\{0\} \cup f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) & \text{otherwise}
\end{cases}
$$

holds for any element $B \in P_r$. (End of Lemma)

Lemma 3 Let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of $S_r^{n-1}$. Then, for a function $f$ satisfying Condition $A$,

$$
\hat{f}_A^i(B) \sqcup \check{f}_A^i(B) = f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)
$$

holds for any element $B \in P_r$.

In the following, this section proves that any function satisfying Condition A can be expressed by a formula, and also shows a method how a formula can be formulated by a function satisfying Condition A.

Definition 4 Let $f$ be a function on $P_r$. Then, $f_1$ is defined as a function on $P_r$ expressed by the following formula.

$$
f_1(X_1, \ldots, X_n) = \bigcup_{i=1}^n f_i(X_1, \ldots, X_n), \quad (5)
$$

where

$$
f_i(X_1, \ldots, X_n) = \bigcup_{A=(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n) \in S_r^{n-1}} \left( \bigwedge_{j=1(j \neq i)}^n X_j^{A_j} \land (\hat{f}_A^i(X_i) \sqcup \check{f}_A^i(X_i)) \right).
$$

(End of Definition)

Here, let us introduce a subset of $P_r^n$, which will be denoted by $I(r, n)$, below.

$$
I(r, n) = \bigcup_{i=1}^n \{(A_1, \ldots, A_n) \in P_r^n | A_i \in P_r - S_r \text{ and } A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \in S_r \}
$$

That is, each element $(A_1, \ldots, A_n)$ of $I(r, n)$ consists of elements of $S_r$, but except for one.

Lemma 4 Let $F$ be a function satisfying Condition A. Then,

$$
f_1(A_1, \ldots, A_n) = \begin{cases} 
f(A_1, \ldots, A_n) & \text{if } f(A_1, \ldots, A_n) \in S_r \cup I(r, n) 
k & \text{otherwise}
\end{cases}
$$

holds for any element $(A_1, \ldots, A_n) \in P_r^n$, where $K$ is an element of $P_r$ such that $\{0\} \subseteq K \subseteq \{0\} \cup f(A_1, \ldots, A_n)$. (End of Lemma)
Definition 5 Let $f$ be a function on $P_r$, let $S$ be an element of $P_r - S_r$, and let $\hat{T}(f, S)$ is the set of all minimal elements of the following subset of $P^n_r$.

$$T(f, S) = \{(A_1, \ldots, A_n) \in P^n_r | f(A_1, \ldots, A_n) = S \text{ and } (A_1, \ldots, A_n) \notin S^n_r \cup I(r, n)\}$$ (6)

Then, $f_2$ is defined as a function on $P_r$ expressed by the following formula.

$$f_2(X_1, \ldots, X_n) = \{s_0\} \cup \left[\bigcup_{S \in P_r - S_r} \left\{\bigcup_{t \in S} \{t\} \land f_S(X_1, \ldots, X_n)\right\}\right],$$ (7)

where

$$f_S(X_1, \ldots, X_n) = \begin{cases} \bigcup_{(A_1, \ldots, A_n) \in \hat{T}(f, S)} \{X_1^{(b)} \land \cdots \land X_n^{(b)}\} & \text{if } \hat{T}(f, S) \neq \emptyset \\ \{0\} & \text{otherwise} \end{cases}$$ (8)

and $s_0$ is the least element of $\bigcup_{(A_1, \ldots, A_n) \in P^n_r} f(A_1, \ldots, A_n)$. (End of Definition)

Lemma 5 Let $f$ be a function on $P_r$ satisfying Condition A. Then,

$$f_2(A_1, \ldots, A_n) = \begin{cases} \{s_0\} & \text{if } (A_1, \ldots, A_n) \in S^n_r \cup I(r, n) \\ f(A_1, \ldots, A_n) & \text{otherwise,} \end{cases}$$

holds for any element $(A_1, \ldots, A_n) \in P^n_r$, where $s_0$ is the least element of the union $\bigcup_{(A_1, \ldots, A_n) \in P^n_r} f(A_1, \ldots, A_n)$. (End of Lemma)

Theorem 4 Let $f$ be a function on $P_r$ satisfying Condition A. Then,

$$f(A_1, \ldots, A_n) = f_1(A_1, \ldots, A_n) \cup f_2(A_1, \ldots, A_n)$$

holds for any element $(A_1, \ldots, A_n) \in P^n_r$, where $f_1$ and $f_2$ are the formulas (5) and (7), respectively. (End of Theorem)

4 Examples of Functions Satisfying Condition A

Consider the function $f$ on $P_3$ whose truth table is given in Table 5. It is not difficult to verify that $f$ satisfies Condition A. Then, this section illustrates how we can form the formula that expresses the function $f$.

Example 1 Let us first consider the formulas (1) and (3). It follows by Eq. (2) that we have the following three subsets of $P_3$.

| $P_0^1(0)$ | $\{B \in P_3 | \min f(B, 0) = 0\} = \{0, 2, 01, 02, 12, 012\}$ |
| $P_0^1(1)$ | $\{B \in P_3 | \min f(B, 0) = 1\} = \{1\}$ |
| $P_0^1(2)$ | $\{B \in P_3 | \min f(B, 0) = 2\} = \emptyset$ |
Table 5: Example of Function $f$ Satisfying Condition A

<table>
<thead>
<tr>
<th>$X \setminus Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>02</td>
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<td>1</td>
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<td>2</td>
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<td>02</td>
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<td>0</td>
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<td>02</td>
</tr>
<tr>
<td>12</td>
<td>012</td>
<td>012</td>
<td>02</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>012</td>
<td>012</td>
<td>012</td>
<td>02</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
</tbody>
</table>

Thus, since $\tilde{P}_{\underline{0}}^{0}(0) = \{012\}$, $\tilde{P}_{\underline{0}}^{0}(1) = \{1\}$, and $\tilde{P}_{\underline{0}}^{0}(2) = \emptyset$, we have the formula $\tilde{f}_{\underline{0}}^{0}(X)$ by Eq. (1).

$$\tilde{f}_{\underline{0}}^{0}(X) = (\underline{0} \wedge \underline{012}) \sqcup (1 \wedge X^1) \sqcup (2 \wedge 0) = 1 \wedge X^1$$

(9)

In a similar way, we have the formulas $\tilde{f}_{\underline{0}}^{1}(X), \tilde{f}_{\underline{0}}^{2}(Y), \tilde{f}_{\underline{1}}^{1}(X), \tilde{f}_{\underline{1}}^{2}(Y)$, below.

$$\tilde{f}_{\underline{1}}^{1}(X) = \underline{1}X^0X^1 \sqcup X^0Y^2 \sqcup \underline{1}X^1X^2 \sqcup X^1X^2 \sqcup X^1X^2$$

$$\tilde{f}_{\underline{2}}^{1}(X) = X^0X^1 \sqcup X^1X^2$$

$$\tilde{f}_{\underline{0}}^{2}(Y) = Y^1$$

$$\tilde{f}_{\underline{1}}^{2}(Y) = \underline{1}Y^0Y^2 \sqcup X^0Y^2 \sqcup \underline{1}Y^1Y^2 \sqcup Y^1Y^2$$

$$\tilde{f}_{\underline{2}}^{2}(Y) = Y^0Y^1 \sqcup Y^1Y^2$$

(10)

Moreover, it follows by Eq. (4) that we have

$$Q_{\underline{0}}^{0}(01) = \{B \in P_3 - S_3 \mid f(B, 0) = 01\} = \{01\},$$

$$Q_{\underline{0}}^{0}(02) = \{B \in P_3 - S_3 \mid f(B, 0) = 02\} = \emptyset,$$

$$Q_{\underline{0}}^{0}(12) = \{B \in P_3 - S_3 \mid f(B, 0) = 12\} = \emptyset,$$

and

$$Q_{\underline{0}}^{0}(012) = \{B \in P_3 - S_3 \mid f(B, 0) = 012\} = \{012\}.$$}

Thus, since $\hat{Q}_{\underline{0}}^{0}(01) = \{01\}, \hat{Q}_{\underline{0}}^{0}(02) = \hat{Q}_{\underline{0}}^{0}(12) = \emptyset$ and $\hat{Q}_{\underline{0}}^{0}(012) = \{012\}$, we have the formula $\hat{f}_{\underline{0}}^{0}(X)$ by Eq. (3).

$$\hat{f}_{\underline{0}}^{0}(X) = (0 \wedge 012) \sqcup (1 \wedge X^1) \sqcup (2 \wedge 0) = 1 \wedge X^1$$

(11)

In a similar way, we have the formulas $\hat{f}_{\underline{0}}^{1}(X), \hat{f}_{\underline{0}}^{2}(Y), \hat{f}_{\underline{1}}^{1}(X), \hat{f}_{\underline{1}}^{2}(Y)$, below.

$$\hat{f}_{\underline{1}}^{1}(X) = \underline{1}X^0X^1 \sqcup \underline{1}X^0X^2 \sqcup \underline{1}X^1X^2 \sqcup X^1X^2 \sqcup X^1X^2$$

$$\hat{f}_{\underline{2}}^{1}(X) = X^0X^1 \sqcup X^1X^2$$

$$\hat{f}_{\underline{0}}^{2}(Y) = Y^1Y^2$$

$$\hat{f}_{\underline{1}}^{2}(Y) = \underline{1}Y^0Y^2 \sqcup X^0Y^2 \sqcup \underline{1}Y^1Y^2 \sqcup Y^1Y^2$$

$$\hat{f}_{\underline{2}}^{2}(Y) = Y^0Y^1 \sqcup Y^1Y^2$$

(12)
Tables 6 and 7 show the truth tables of $\check{f}_{A}^{i}$ and $\hat{f}_{A}^{i}$ for which $i = 1, 2$ and $A \in \{0, 1, 2\}$.

(End of Example)

It follows by Lemma 3 that

$$f(X, B) = \check{f}_{A}^{1}(X) \sqcup \hat{f}_{A}^{1}(X) \quad \text{and} \quad f(B, Y) = \check{f}_{A}^{2}(Y) \sqcup \hat{f}_{A}^{2}(Y)$$

hold for every $A \in \{0, 1, 2\}$ and every $B \in P_{3}$, where $\check{f}_{A}^{1}(X)$, $\hat{f}_{A}^{1}(X)$, $\check{f}_{A}^{2}(Y)$ and $\hat{f}_{A}^{2}(Y)$ have been obtained in Eqs. (9), (10), (11) and (12). Table 8 shows the truth tables of $\check{f}_{A}^{i} \sqcup \hat{f}_{A}^{i}$, where $i = 1, 2$ and $A \in \{0, 1, 2\}$.

**Example 2** Let us next consider the formula (5) in Definition 4. It follows by Eqs. (9) and (11) that we have the formula $\check{f}_{A}^{1}(X) \sqcup \hat{f}_{A}^{1}(X)$ below.

$$\check{f}_{0}^{1}(X) \sqcup \hat{f}_{0}^{1}(X) = \underline{1}X^{1} \sqcup \underline{1}X^{0}X^{1} \sqcup \underline{1}X^{1}X^{2} \sqcup X^{1}X^{2} = \underline{1}X^{1} \sqcup X^{1}X^{2}$$

In a similar way, by Eqs. (10), (11) and (12), we have the formulas

$$\check{f}_{1}^{1}(X) \sqcup \hat{f}_{1}^{1}(X) = X^{1} \sqcup X^{2},$$
$$\check{f}_{2}^{1}(X) \sqcup \hat{f}_{2}^{1}(X) = X^{1},$$
$$\check{f}_{0}^{2}(Y) \sqcup \hat{f}_{0}^{2}(Y) = Y^{1}Y^{2},$$
$$\check{f}_{1}^{2}(Y) \sqcup \hat{f}_{1}^{2}(Y) = 1 \sqcup Y^{2},$$
$$\check{f}_{2}^{2}(Y) \sqcup \hat{f}_{2}^{2}(Y) = Y^{1}.$$

Therefore, the formula $f_{1}(X, Y)$ of (5) in Definition 4 is given as

$$f_{1}(X, Y) = f^{1}(X, Y) \sqcup f^{2}(X, Y), \quad (13)$$
Table 9: Truth Table of $f_1(X,Y)$

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
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<th>012</th>
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<td>012</td>
<td>012</td>
<td>012</td>
</tr>
</tbody>
</table>

$012$ | 012| 012| 012| 012| 012| 012| 012| 012 |

where

\[ f_1(X,Y) = Y^0 \left( f_\underline{01}^1(X) \cup f_\underline{02}^1(X) \right) \cup Y^1 \left( f_\underline{12}^1(X) \cup f_\underline{12}^1(X) \right) \cup Y^2 \left( f_\underline{012}^1(X) \cup f_\underline{012}^1(X) \right) \]

and

\[ f_2(X,Y) = X^0 \left( f_\underline{02}^2(Y) \cup f_\underline{02}^2(Y) \right) \cup X^1 \left( f_\underline{12}^2(Y) \cup f_\underline{12}^2(Y) \right) \cup X^2 \left( f_\underline{012}^2(Y) \cup f_\underline{012}^2(Y) \right). \]

Table 9 is the truth table of $f_1(X,Y)$. (End of Example)

**Example 3** In this example, let us consider the formula (7) in Definition 5. It follows by Eq. (6) that we have the following subsets of $P_3^2$.

\[
\begin{align*}
T(f, 01) &= \emptyset \\
T(f, 02) &= \{(02, 01), (02, 12), (02, 012)\} \\
T(f, 12) &= \emptyset \\
T(f, 012) &= \{(A, B) | A \in \{01, 12, 012\} \text{ and } B \in P_3 - S_3\}
\end{align*}
\]

Therefore, since we have

\[
\begin{align*}
\hat{T}(f, 01) &= \emptyset, \\
\hat{T}(f, 02) &= \{(02, 01), (02, 12)\}, \\
\hat{T}(f, 12) &= \emptyset, \text{ and} \\
\hat{T}(f, 012) &= \{(01, 01), (01, 02), (01, 12), (12, 01), (12, 02), (12, 12)\},
\end{align*}
\]

it follows by Eq. (8) that we have the following formulas.

\[
\begin{align*}
f_{01}(X,Y) &= 0 \\
f_{02}(X,Y) &= X^0X^2Y^2Y^1 \cup X^0X^2Y^1Y^2 \\
f_{12}(X,Y) &= 0 \\
f_{012}(X,Y) &= X^0X^1Y^2Y^1 \cup X^0X^1Y^2Y^2 \cup X^0X^1Y^1Y^2 \cup X^0X^1Y^2Y^2 \cup X^0X^1Y^2Y^2 \\
&\quad \cup X^0X^1Y^2Y^1 \cup X^0X^1Y^2Y^2 \cup X^0X^1X^2Y^2Y^2 \cup X^0X^1X^2Y^2Y^2 \cup X^0X^1X^2Y^2Y^2
\end{align*}
\]

Thus, the formula $f_2(X,Y)$ of (7) in Definition 5 is obtained as the formula below.

\[
\begin{align*}
f_2(X,Y) &= 0 \cup \{0f_{02}(X,Y) \cup 2f_{02}(X,Y)\} \cup \{0f_{012}(X,Y) \cup 1f_{012}(X,Y) \cup 2f_{012}(X,Y)\} \\
&= f_{02}(X,Y) \cup 1f_{012}(X,Y) \cup f_{012}(X,Y)
\end{align*}
\]

Table 10 is the truth table of $f_2(X,Y)$. (End of Example)

It follows by Theorem 4 that the function $f$ of Table 5 can be expressed by the formula $f_1(X,Y) \cup f_2(X,Y)$, where $f_1(X,Y)$ and $f_2(X,Y)$ are the formulas given in (13) and (14), respectively.
5 Functions Expressed by Formulas Composed of $\land$, $\lor$ and Universal Literals

This section discusses functions on $P_r$ expressed by formulas, which are composed of the operations $\land$, $\lor$ and universal literals. Then, a necessary and sufficient condition for a function on $P_r$ to be expressed by a formula when $r$ is equal to 3.

**Theorem 5** Let $f$ be a function on $P_r$. If $f$ can be expressed by a formula, then

$$\bigcap_{A \in P_r - S_r} f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) \neq \emptyset$$

holds for any elements $A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n$ of $P_r$.

(End of Theorem)

By Theorems 1, 2 and 5, any function $f$ on $P_r$ expressed by a formula satisfies the following Condition B.

**Condition B:** Let $f$ be a function on $P_r$.

(1) If $(A_1, \ldots, A_n) \in S_r^n$, then $f(A_1, \ldots, A_n) \in S_r$.

(2) For any elements $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ of $P_r^n$, $(A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)$ implies $f(A_1, \ldots, A_n) \subseteq f(B_1, \ldots, B_n)$.

(3) $\bigcap_{A \in P_r - S_r} f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) \neq \emptyset$ holds for any elements $A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n$ of $P_r$.

In the following, this section proves that Condition B is a necessary and sufficient condition for a function on $P_3$ to be expressed by a formula with the operations $\land$, $\lor$, and universal literals.

**Definition 6** Let $(A_1, \ldots, A_n)$ be any element of $P_r^n$. Then, $\alpha = X_1^{A_1} \land \cdots \land X_n^{A_n}$ is said to be the type-1 term corresponding to $(A_1, \ldots, A_n)$. Next, let $(B_1, \ldots, B_n)$ be any element of $P_r^n - S_r^n$. Then, $\beta = \bigwedge_{e \in B_1} X_1^{\{e\}} \land \cdots \land \bigwedge_{e \in B_n} X_n^{\{e\}}$ is said to be the type-2 term corresponding to $(B_1, \ldots, B_n)$.

(End of Definition)

Let $S$ be an element of $P_r$, and let $T$ be an element of $P_r - S_r$. Then, it is easy to verify that the following two equations are valid.

$$X^S = \begin{cases} \{r-1\} & \text{if } X \subseteq S \\ \{0\} & \text{if } X \cap S = \emptyset \\ \{0, r-1\} & \text{otherwise} \end{cases} \quad (15)$$

$$\bigwedge_{e \in T} X^{\{e\}} = \begin{cases} \{0, r-1\} & \text{if } T \subseteq X \\ \{0\} & \text{otherwise} \end{cases} \quad (16)$$

Therefore, for any type-1 term $\alpha$ and any type-2 term $\beta$, $\alpha(A_1, \ldots, A_n) = \{r-1\}$, $\{0, r-1\}$, or $\{0\}$, and $\beta(A_1, \ldots, A_n) = \{0, r-1\}$ or $\{0\}$ hold for any element $(A_1, \ldots, A_n) \in P_r^n$.

**Lemma 6** For any type-1 term $\alpha$ corresponding to $(A_1, \ldots, A_n) \in P_r^n$,
$(B_1, \ldots, B_n) \subseteq (A_1, \ldots, A_n)$ iff \(\alpha(B_1, \ldots, B_n) = \{r - 1\}\),

$(A_1, \ldots, A_n) \cap (B_1, \ldots, B_n) = \emptyset$ iff \(\alpha(B_1, \ldots, B_n) = \{0\}\),

$(B_1, \ldots, B_n) \not\in (A_1, \ldots, A_n)$ and $(A_1, \ldots, A_n) \cap (B_1, \ldots, B_n) = \emptyset$ iff \(\alpha(B_1, \ldots, B_n) = \{0, r - 1\}\) hold for any \((B_1, \ldots, B_n) \in P_r^n\).

(End of Lemma)

**Lemma 7** For any type-2 term \(\alpha\) corresponding to \((A_1, \ldots, A_n) \in P_r^n - S_r^n\),

(1) \((A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)\) iff \(\alpha(B_1, \ldots, B_n) = \{0, r - 1\}\),

(2) \((A_1, \ldots, A_n) \not\subset (B_1, \ldots, B_n)\) iff \(\alpha(B_1, \ldots, B_n) = \{0\}\)

hold for any \((B_1, \ldots, B_n) \in P_r^n\).

(End of Lemma)

Let \(f\) be a function satisfying Condition B, and let \(S\) be an element of \(P_r\). Then, define two subsets of \(P_r^n\), denoted by \(\check{L}(f, S)\) and \(U'(f, S)\), below.

\[
\check{L}(f, S) = \{ (A_1, \ldots, A_n) \in P_r^n \mid f(A_1, \ldots, A_n) \subseteq S \} \text{ and }
U'(f, S) = \{ (A_1, \ldots, A_n) \in P_r^n \mid f(A_1, \ldots, A_n) \cap S \neq \emptyset \}.
\]

Let \(\check{L}(f, S)\) and \(U'(f, S)\) be the sets of all maximal elements of \(L(f, S)\) and of all minimal elements of \(U(f, S)\), respectively. Further, let \(\hat{U}(f, S) = U'(f, S) - S_r^n\).

**Lemma 8** Let \(f\) be a function satisfying Condition B, and let \(S\) be an element of \(P_r\). Then, \((f)^S\) can be expressed by the following formula.

\[
(f)^S = \begin{cases} 
\bigcup_{A \in \check{L}(f, S)} \alpha_A \sqcup \bigcup_{A \in U'(f, S)} \beta_A & \text{if } \check{L}(f, S) \neq \emptyset \text{ or } U'(f, S) \neq \emptyset \\
\emptyset & \text{otherwise} 
\end{cases}
\]

(17)

where \(\alpha_A\) and \(\beta_A\) are the type-1 and type-2 terms corresponding to \(A\), respectively.

(End of Lemma)

Now, let us consider formulas of one-variable functions satisfying Condition B. Any one-variable function \(f\) satisfying Condition B is in at least one of the following three cases.

(B-1) \(f(A) \neq 01\) holds for any element \(A \in P_3 - S_3\).

(B-2) \(f(A) \neq 02\) holds for any element \(A \in P_3 - S_3\).

(B-3) \(f(A) \neq 12\) holds for any element \(A \in P_3 - S_3\).

---

3If \(f\) is in neither one of the cases (B-1), (B-2), (B-3), then it implies that we have three distinct elements \(A, B, C\) in \(P_3 - S_3\) such that \(f(A) = 01\), \(f(B) = 02\), and \(f(C) = 12\). However, this contradicts to the fact that \(f\) satisfies Condition B(3).
Table 11: Example of (B-4)
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x \setminus y & 0 & 1 & 2 & 01 & 02 & 012 & 012 \\
\hline
0 & 0 & 1 & 2 & 01 & 02 & 012 & 012 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 0 & 2 & 1 & 02 & 012 & 12 & 012 \\
01 & 02 & 12 & 2 & 012 & 012 & 012 & 012 \\
02 & 0 & 12 & 12 & 012 & 012 & 012 & 012 \\
012 & 0 & 02 & 12 & 012 & 012 & 012 & 012 \\
\hline
\end{array}
\]

Table 12: Example of (B-5)
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x \setminus y & 0 & 1 & 2 & 01 & 02 & 12 & 012 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 01 & 01 & 0 & 01 \\
2 & 0 & 2 & 1 & 02 & 012 & 12 & 012 \\
01 & 02 & 12 & 2 & 012 & 012 & 012 & 012 \\
02 & 0 & 12 & 12 & 012 & 012 & 012 & 012 \\
012 & 0 & 02 & 12 & 012 & 012 & 012 & 012 \\
\hline
\end{array}
\]

Table 13: Example of (B-6)
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x \setminus y & 0 & 1 & 2 & 01 & 02 & 12 & 012 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 01 & 01 & 0 & 01 \\
2 & 0 & 2 & 2 & 02 & 02 & 2 & 02 \\
0 & 01 & 012 & 12 & 012 & 012 & 012 & 012 \\
02 & 0 & 012 & 02 & 012 & 012 & 012 & 012 \\
12 & 0 & 12 & 12 & 012 & 012 & 012 & 012 \\
012 & 0 & 012 & 012 & 012 & 012 & 012 & 012 \\
\hline
\end{array}
\]

Property 1: Any one-variable function \( f \) satisfying Condition B can be expressed by the following formula.
\[
\begin{align*}
    f(X) &= \begin{cases} 
    f^{12}(X) \land (1 \cup f^{12}(X)) & \text{if } f \text{ is in the case (B-1)} \\
    (1 \land f^{1}(X)) \lor f^{2}(X) \lor (1 \land f^{12}(X)) & \text{if } f \text{ is in the case (B-2)} \\
    (1 \land f^{1}(X)) \lor f^{2}(X) & \text{if } f \text{ is in the case (B-3)} 
    \end{cases}
\end{align*}
\]

(End of Property)

By Property 1, every one-variable function satisfying Condition B can be expressed by a formula.

Next, let us consider the case where functions satisfying Condition B depend more than one variable. Then, any function \( f \) satisfying Condition B is in at least one of the three cases below.

(B-4) \( f(A_1, \ldots, A_n) \neq 01 \) holds for any element \((A_1, \ldots, A_n) \in (P_3 - S_3)^n\).

(B-5) \( f(A_1, \ldots, A_n) \neq 12 \) holds for any element \((A_1, \ldots, A_n) \in (P_3 - S_3)^n\).

(B-6) \( \bigcap_{(A_1, \ldots, A_n) \in (P_3 - S_3)^n} f(A_1, \ldots, A_n) = 1 \).

Tables 11, 12 and 13 are examples of two-variable functions being in the cases (B-4), (B-5), and (B-6), respectively.

Then, we can prove Properties 1 ~ 6, which show a way for constructing formulas of \( n \)-variable functions satisfying Condition B.

Let \( A \) be an element \((A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n) \in P_{f}^{n-1} \). Then, denote the one-variable function \( f(A_1, \ldots, A_{i-1}, X, A_{i+1}, \ldots, A_n) \) by \( f^i_A(X) \).
Property 2 Suppose a function $f$ satisfying Condition B is in the case (B-5). Let $f'$ be a function expressed by the formula
\[ f' = p^1 \sqcup \cdots \sqcup p^n, \]
where
\[ p^i(X_1, \ldots, X_n) = \bigcup_{A=(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n) \in S_{3}^{n-1}} (f_A^i(X_i) \wedge X_i^{A_i^1} \wedge \cdots \wedge X_{i-1}^{A_{i-1}} \wedge X_{i+1}^{A_{i+1}} \wedge \cdots \wedge X_n^{A_n}). \]
Then, for any element $(A_1, \ldots, A_n) \in P_3^n$,
\[ f'(A_1, \ldots, A_n) = \begin{cases} f(A_1, \ldots, A_n) & \text{if } (A_1, \ldots, A_n) \not\in (P_3 - S_3)^n \\ K & \text{otherwise} \end{cases} \]
where $K$ is an element of $P_r$ such that \( \{0\} \subseteq K \subseteq \{0\} \cup F(A_1, \ldots, A_n). \) (End of Property)

Property 3 Suppose a function $f$ satisfying Condition B is in the case (B-5). Let $f''$ be a function expressed by the formula
\[ f''(X_1, \ldots, X_n) = \bigcup_{S \in P_3 - S_3} \left( \bigcup_{t \in \hat{T}(S)} \left( \bigwedge_{e(A_1, \ldots, A_n) \in \hat{T}(S)} X^e \wedge \cdots \wedge \bigwedge_{e \in A_n} X^e \right) \right), \]
where $\hat{T}(S)$ is the set of all minimal elements of the set
\[ T(S) = \{(A_1, \ldots, A_n) \in (P_3 - S_3)^n \mid f(A_1, \ldots, A_n) = S\} \]
and
\[ f_{\hat{T}(S)}(X_1, \ldots, X_n) = \bigcup_{(A_1, \ldots, A_n) \in \hat{T}(S)} \left( \bigwedge_{e \in A_1} X^e \wedge \cdots \bigwedge_{e \in A_n} X^e \right). \]
Then, for any element $(A_1, \ldots, A_n) \in P_3^n$,
\[ f''(A_1, \ldots, A_n) = \begin{cases} \{0\} & \text{if } (A_1, \ldots, A_n) \not\in (P_3 - S_3)^n \\ f(A_1, \ldots, A_n) & \text{otherwise} \end{cases} \]
(End of Property)

Property 4 Any function $f$ satisfying Condition B can be expressed by $f = f' \sqcup f''$, if $f$ is in the case (B-5). (End of Property)

Property 5 Suppose a function $f$ satisfying Condition B is in the case (B-4). Let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of $S_3^{n-1}$. Then, define $g_A^i$ and $h$ as functions on $P_r$ expressed by the following formulas.
\[ g_A^i(X_1, \ldots, X_n) = f_A^i(X_i) \sqcup (X_1^{A_1^i} \wedge \cdots \wedge X_{i-1}^{A_{i-1}^i} \wedge X_{i+1}^{A_{i+1}^i} \wedge \cdots \wedge X_n^{A_n^i}), \]
where $A_j^i = E_3 - A_j$ ($j = 1, \ldots, i - 1, i + 1, \ldots, n$).
\[ h(X_1, \ldots, X_n) = \left\{ \left( \bigcup_{i=1}^n X_i^A \right) \sqcup f^{12}(X_1, \ldots, X_n) \right\} \wedge \left\{ \left( \bigcup_{i=1}^n X_i^A \right) \sqcup f^{02}(X_1, \ldots, X_n) \sqcup 1 \right\} \]
(24)
Then, $f$ can be expressed by the following formula.

$$f(X_1, \ldots, X_n) = G(X_1, \ldots, X_n) \land h(X_1, \ldots, X_n),$$  \hspace{1cm} (25)

where $G$ is the \land-ing of all the $g_A^i$'s of Eq. (23), i.e.,

$$G(X_1, \ldots, X_n) = \bigwedge_{i=1}^{n} \left( \bigwedge_{A \in S_3^{n-1}} g_A^i(X_1, \ldots, X_n) \right),$$  \hspace{1cm} (26)

(End of Property)

**Property 6** Suppose a function $f$ satisfying Condition B is in the case (B-6). Then, $f$ is in either one of the following two cases.

1. $f(A) \neq \underline{02}$ holds for any element $A \in P_3^n$, or
2. $f(A) = \underline{02}$ holds for some element $A \in P_3^n$.

If $f$ is in the case (1), then $f$ can be expressed by

$$f(X_1, \ldots, X_n) = (1 \land f^1(X_1, \ldots, X_n)) \cup f^2(X_1, \ldots, X_n) \cup (1 \land f^{12}(X_1, \ldots, X_n)).$$  \hspace{1cm} (27)

Let $w$ be a function expressed by the following formula.

$$w(X_1, \ldots, X_n) = \bigcup_{(A_1, \ldots, A_n) \in Q_{\underline{02}}} \xi_1(A_1) \cup \cdots \cup \xi_n(A_n),$$  \hspace{1cm} (28)

where $Q_{\underline{02}} = \{(A_1, \ldots, A_n) \in P_3^n \mid f(A_1, \ldots, A_n) = \underline{02}\}$, and

$$\xi_i(A) = \begin{cases} X^A_i & \text{if } A \in S_3 \\ 0 & \text{otherwise.} \end{cases}$$

Then, $f$ can be expressed by the following formula, if $f$ is in the case (2).

$$f(X_1, \ldots, X_n) = G(X_1, \ldots, X_n) \land w(X_1, \ldots, X_n),$$  \hspace{1cm} (29)

where $G(X_1, \ldots, X_n)$ is given by Eq. (26).  \hspace{1cm} (End of Property)

### 6 Examples of Function Satisfying Condition B

This section shows examples of 2-variable functions satisfying Condition B, and illustrates how they can be expressed by formulas.

**Example 4** Consider the function $f$ defined by Table 12, which is in the case (B-5). The formula expressing $f$ is given by Properties 2, 3, and 4. First, consider the formulas $f_A^1(X)$ and $f_A^2(Y)$, which appear in Eq. (20). Table 14 shows the truth tables of the six one-variable functions $f_1^1(X), f_1^2(X), f_2^1(X), f_2^2(Y), f_3^1(Y)$ and $f_3^2(Y)$. Since $f_1^1(X)$ and $f_3^1(Y)$ are in (B-2) (or (B-3)), $f_1^2(X)$ is in (B-3), and $f_3^2(Y)$ is in (B-1), it follows by Eq. (17) and (18) that these one-variable functions are expressed by the following formulas.

---

4 $Q_{\underline{02}} \cap (P_3 - S_3)^n = \emptyset$ holds, since $f$ is in the case (B-6).
Table 14: One-Variable Functions $f_{A}^{1}$ and $f_{A}^{2}$ of Example 4

<table>
<thead>
<tr>
<th>$X$ or $Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{A}^{1}(X)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_{A}^{2}(X)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>01</td>
<td>02</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>$f_{A}^{1}(Y)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>01</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_{A}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Therefore, it follows by Eq. (19) that $f'$ is expressed by the following formula.

$$f' = X^0 f_{A}^{2}(Y) \sqcup X^1 f_{A}^{1}(X) Y^0 \sqcup X^1 f_{A}^{2}(Y) \sqcup X^2 f_{A}^{1}(X) Y^2 \sqcup X^2 f_{A}^{2}(Y) \sqcup X^1 f_{A}^{1}(X) Y^1 \sqcup X^1 f_{A}^{2}(Y) \sqcup X^2 f_{A}^{1}(X) Y^2$$  \hspace{1cm} (30)

Table 15 is the truth table of $f'$. Next, consider $f''$ in Eq. (21). Since

$$\hat{T}(01) = \{(01, 01), (01, 12)\},$$

$$\hat{T}(02) = \hat{T}(12) = \emptyset,$$

$$\hat{T}(012) = \{(02, 01), (02, 02), (02, 12), (12, 01), (12, 02), (12, 12)\},$$

it follows by Eq. (22) that we have the following formulas.

$$f_{\hat{T}(01)}(X, Y) = X^0 X^1 Y^0 Y^1 \sqcup X^0 X^1 Y^1 Y^2$$

$$f_{\hat{T}(02)}(X, Y) = X^0 X^2 Y^0 Y^1 \sqcup X^0 X^2 Y^2 Y^2 \sqcup X^0 X^2 Y^0 Y^2 \sqcup X^1 X^2 Y^0 Y^1 \sqcup X^1 X^2 Y^2 Y^2$$

We then have $f''(X, Y)$ below by Eq. (21).

$$f''(X, Y) = 1 f_{\hat{T}(01)}(X, Y) \sqcup 1 f_{\hat{T}(02)}(X, Y) \sqcup f_{\hat{T}(012)}(X, Y)$$  \hspace{1cm} (31)

Table 16 is the truth table of $f''$. It follows by Property 4 that $f(X, Y) = f'(X, Y) \sqcup f''(X, Y)$.  

(End of Example)
Example 5 Consider the function $f$ defined by Table 11, which is in (B-4). The formula expressing $f$ is given by Property 5. It follows by Eq. (17) and (18) that the one-variable functions $f_{A}^{1}(X)$ and $f_{A}^{2}(Y)$ are obtained below.

Then, by Eq. (23), we have

By Eq. (26), we have the function $G(X, Y)$ expressed by $\wedge$-ing of all the above $g_{A}^{i}(X, Y)$'s. Table 17 is the truth table of $G(X, Y)$. Next, consider $h$ in Eq. (24). It follows by Eq. (17) that the functions $f^{12}(X, Y)$ and $f^{02}(X, Y)$ are expressed by the following formulas.

Thus, by Eq. (24), we have

where $v(X, Y) = X^{0} \sqcup X^{1} \sqcup X^{2} \sqcup Y^{0} \sqcup Y^{1} \sqcup Y^{2}$. Table 18 is the truth table of $h(X, Y)$. Lastly, by Eq. (25), $f(X, Y)$ are expressed by the following formula.

Example 6 Consider the function $f$ define by Table 13, which is in (B-6). The formula expressing $f$ is given by Property 6. Since $f$ is in the case (2) of Property 6, $f$ is expressed by the formula given in Eq. (29).

First, consider the formula $G(X, Y)$. It follows by Eq. (17) and (18) that one-variable functions $f_{A}^{1}(X)$ and $f_{A}^{2}(Y)$ are obtained below.
Thus, we obtain \( g^1_0(X, Y) \) and \( g^1_2(X, Y) \) of Eq. (23) below.
\[
\begin{align*}
  g^1_0(X, Y) &= Y^{12} \\
  g^1_1(X, Y) &= \left( 1 \land X^{12} \right) \sqcup X^2 \\
  g^1_2(X, Y) &= X^{12} \land \left( 1 \sqcup X^{02} \right)
\end{align*}
\]

By Eq. (26), we have the function \( G(X, Y) \) expressed by \( \land \)-ing of all the above \( g^i_A(X, Y) \)'s. Table 19 is the truth table of \( G \).

Next, consider \( w(X, Y) \) of Eq. (28). Because
\[
Q_{02} = \{(2, 01), (2, 02), (2, 012), (02, 02)\},
\]

it follow by Eq. (28) that the formula \( w \) is obtained below.
\[
w(X, Y) = 1 \sqcup X^2 \sqcup Y^2
\]

Table 20 is the truth table of \( w \). Lastly, it follows by Eq. (29) that the following formula expresses the function \( f \).
\[
f(X, Y) = G(X, Y) \land w(X, Y)
\]
\[
f(X, Y) = g^1_0(X, Y) \land g^1_1(X, Y) \land g^2_0(X, Y) \land g^2_1(X, Y) \land g^2_2(X, Y) \land w(X, Y).
\]

(End of Example)

7 Conclusions

This paper discussed functions over \( P_r \) that preserves the set inclusion relation \( \subseteq \). We referred the three kinds of operations Min, Max, and Literals over \( E_r \), because they are functionally complete on the \( r \)-valued set \( E_r \). This paper then proved some of the mathematical properties of functions over \( P_r \) that can be expressed by formulas. It is one of the open problems that which set of operations \( \hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m \) over \( P_r \) can realize any function over \( P_r \) preserving the set inclusion relation \( \subseteq \).
References


