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T_2

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KING KONG. I shall tell thee of a problem as dreadful as the three-tongued *Würgerin von Nizhnij* [5]; a thousand nights I have dreamt of it, but to solve it I have failed.

1. Notions and setting of the problem

GODWIN ZILLA. Speak then, King, of what weighs so heavy on thy mind.

KONG. Imagine, God, a countably infinite base set X, the set $\mathscr{O}$ of all finitary operations on X, and for all $n \geq 1$ let the set $\mathscr{O}^{(n)}$ of n-ary operations on X. For simplicity of some formulations we assume X to be equipped with the order of the natural numbers (e.g. when we talk about maximum or minimum functions). Thou attend'st not!

ZILLA. O, good sir, I do! Thou intend'st, I reckon, a tale of the clone lattice.

KONG. I pray thee mark me. Indeed, we are interested in the structure of the interval $[\langle\mathscr{O}^{(1)}\rangle, \mathscr{O}]$ of the clone lattice ($\langle\mathscr{O}^{(1)}\rangle$ is the clone generated by $\mathscr{O}^{(1)}$ and therefore the clone of all essentially unary functions, i.e. functions which depend on at most one of their variables). More specifically, we are interested in the "upper part" of this interval. By a result of Gavrilov's [1], there exist only two precomplete clones above $\mathscr{O}^{(1)}$. Every clone of the interval in contained in a precomplete one, as $\mathscr{O}$ is generated by $\mathscr{O}^{(1)}$ plus only finitely many functions (e.g. $\mathscr{O}^{(1)}$ together with any binary injection generate $\mathscr{O}$). The interval is as large as the whole clone lattice, which has been shown recently by Goldstern and Shelah [4]. Dost thou hear?

ZILLA. Your tale, sir, would cure deafness. Canst thou describe the precomplete clones of the interval?

KONG. Be of comfort. They can be described using the following concept: For $n \geq 1$ and a set $\mathscr{G} \subseteq \mathscr{O}^{(n)}$, define $\mathrm{Pol}(\mathscr{G})$ to consist of all $f \in \mathscr{O}$ satisfying: Whenever $g_1, \ldots, g_m \in \mathscr{G}$, then the composite $f(g_1, \ldots, g_m) \in \mathscr{G}$ (m is the arity of f). This definition is identical with the usual definition of the Pol-operator in clone theory (preservation of a relation), if $\mathscr{G} \subseteq \mathscr{O}^{(2)}$ is considered an infinitary relation (of arity X^2, since $\mathscr{O}^{(2)} = X^{X^2}$).

ZILLA. I prithee, define the first precomplete clone.

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KONG. A function $f \in \mathscr{O}^{(n)}$ is called *almost unary* iff there exists $1 \leq k \leq n$ such that for all $c \in X$ we have that

$$\{f(x_1, \dots, x_{k-1}, c, x_{k+1}, \dots, x_n) : x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n \in X\}$$

is finite. In words, there is a variable of f such that the value of this variable determines the function value up to a finite set. Using the order of X we may equivalently define f to be almost unary iff there exist $1 \leq k \leq n$ and $F \in \mathscr{O}^{(1)}$ such that $f(x_1, \dots, x_n) \leq F(x_k)$ for all $x_1, \dots, x_n \in X$.

ZILLA. I assume, King, that there exist functions of such noble kind.

KONG. God, there are. An example of an almost unary function is $\min(x_1, \dots, x_n)$ (note that X has the order of the natural numbers); also, if $p(x_1, x_2)$ is any binary function, then

$$p_\Delta(x_1, x_2) = \begin{cases} p(x_1, x_2) & , x_1 > x_2 \\ 0 & , \text{otherwise} \end{cases}$$

is almost unary. If p is an injection, then p_Δ is a "canonical" almost unary function:

Fact 1 (Pinsker [7]). *Let $p \in \mathscr{O}^{(2)}$ be injective. Then $\langle \{p_\Delta\} \cup \mathscr{O}^{(1)} \rangle = \{f \in \mathscr{O} : f$ almost unary$\}$.*

ZILLA. Tell me thus, I prithee, are all functions almost unary?

KONG. Examples of functions which are *not* almost unary: $\max(x_1, \dots, x_n)$, any binary injection p, and the median of three $\mathrm{med}(x_1, x_2, x_3)$.

ZILLA. But wherefore hast thou introduced the notion of almost unary?

KONG. The set of all almost unary functions is a clone which we denote by $\mathscr{U}$. Write

$$T_1 = \mathscr{U}^{(2)} = \{f \in \mathscr{O}^{(2)} : f \text{ almost unary}\}.$$

Then $\mathrm{Pol}(T_1)$ is a maximal clone above $\mathscr{O}^{(1)}$ (Gavrilov [1]). An example of a function in $\mathrm{Pol}(T_1)$ but not in $\langle T_1 \rangle$ is the median function. Observe that $\langle T_1 \rangle = \mathscr{U}$ by Fact 1.

ZILLA. More to know did never meddle with my thoughts.

KONG. 'Tis time I should inform thee farther. Let $\Delta = \{(x_1, x_2) \in X^2 : x_2 < x_1\}$ and $\nabla = \{(x_1, x_2) \in X^2 : x_1 < x_2\}$. For $S_1, S_2 \subseteq X$ we set $\Delta_{S_1,S_2} = \Delta \cap (S_1 \times S_2)$ and $\nabla_{S_1,S_2} = \nabla \cap (S_1 \times S_2)$. Denote by $[X]^\omega$ the set of all infinite subsets of X. Now define

$$T_2 = \{f \in \mathscr{O}^{(2)} : \forall S_1, S_2 \in [X]^\omega \text{ neither } f \restriction_{\Delta_{S_1,S_2}} \text{ nor } f \restriction_{\nabla_{S_1,S_2}} \text{ are injective}\}.$$

ZILLA. I might call it a thing divine, for nothing natural I ever saw so noble. Canst thou show me but one function in T_2?

KONG. Indeed I can: $\max(x_1, x_2), \min(x_1, x_2)$. Examples of functions *not* in T_2: Any injection $p \in \mathscr{O}^{(2)}$, and for any such injection the corresponding p_Δ as defined before. Now mind the words of Gavrilov [1]:

Theorem 2. $\mathrm{Pol}(T_1)$ *and* $\mathrm{Pol}(T_2)$ *are maximal clones which contain* $\mathscr{O}^{(1)}$*, and there exist no other maximal clones above* $\mathscr{O}^{(1)}$*.*

ZILLA. I see a beauteous theorem, but not hast thou shown me a problem.

KONG. Know thus far forth: An example of an essentially ternary function in $\mathrm{Pol}(T_2)$ is $\mathrm{med}(x_1, x_2, x_3)$. In fact, $\mathrm{med}(x_1, x_2, x_3) \in \langle T_2 \rangle$ since T_2 contains the maximum and minimum functions which clearly generate the median.

The definition of T_2 can be understood better with an application of the infinite Ramsey's theorem. This theorem says that the partition relation $\aleph_0 \to (\aleph_0)^2_2$ holds; in words this means that whenever G is a countably infinite undirected complete graph and its edges are coloured with two colours, then there is a (countably) infinite complete subgraph of G on which the coloring is constant.

ZILLA. The connection with T_2...

KONG. ... is the following: Using Ramsey's theorem, one can prove that if $f(x_1, x_2) \in \mathscr{O}^{(2)}$ is arbitrary, and $S_1, S_2 \subseteq X$ are infinite, then these sets S_1, S_2 can be "thinned out" to infinite $S_1' \subseteq S_1$ and $S_2' \subseteq S_2$ such that $f \restriction_{\Delta_{S_1', S_2'}}$ is one of the following:

(1) Constant.
(2) A unary injective function of x_1.
(3) A unary injective function of x_2.
(4) Injective.

Of course, the same can be achieved for $f \restriction_{\nabla_{S_1', S_2'}}$. A function $f \in \mathscr{O}^{(2)}$ is in T_2 iff f is not of type (4) (injective) on any $\Delta_{S_1', S_2'}$ or $\nabla_{S_1', S_2'}$. This application of Ramsey's theorem is due to Goldstern and Shelah [3].

ZILLA. Dost thou not want to speak of a problem?

KONG. Hear a little further, and then I'll bring thee to the present business which now is upon us. In general, if $\mathscr{C}$ is a clone, then

$$\mathrm{Pol}(\mathscr{C}^{(1)}) \supseteq \mathrm{Pol}(\mathscr{C}^{(2)}) \supseteq \ldots \supseteq \mathrm{Pol}(\mathscr{C}^{(n)}) \supseteq \ldots$$

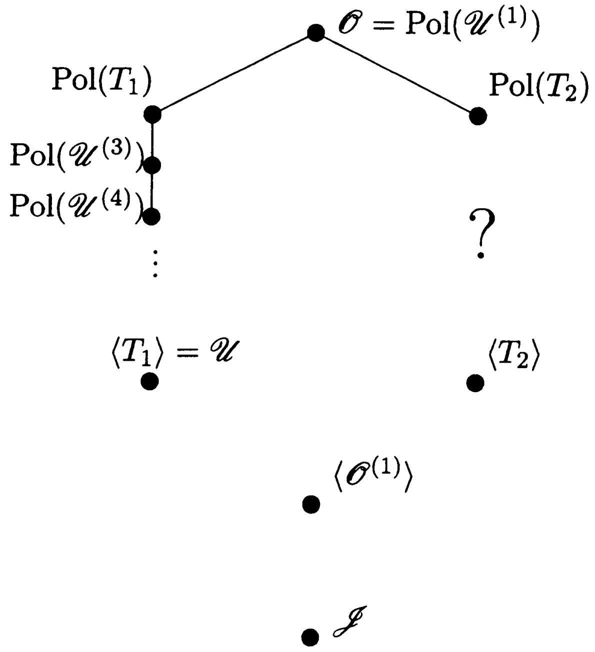
Moreover,

$$\mathrm{Pol}(\mathscr{C}^{(n)})^{(n)} = \mathscr{C}^{(n)} \quad \text{and} \quad \bigcap_{n \geq 1} \mathrm{Pol}(\mathscr{C}^{(n)}) = \mathscr{C}.$$

In the case of $\mathscr{C} = \mathscr{U}$, in [7] it has been shown that the clones obtained this way are distinct and the only ones containing $\mathscr{U}$:

$$\mathrm{Pol}(\mathscr{U}^{(1)}) = \mathscr{O} \supsetneq \mathrm{Pol}(\mathscr{U}^{(2)}) = \mathrm{Pol}(T_1) \supsetneq \ldots \supsetneq \mathrm{Pol}(\mathscr{U}^{(n)}) \supsetneq \ldots$$

and there exist no more clones containing T_1. Also, it has been shown there that all clones above T_1 are finitely generated over $\mathscr{O}^{(1)}$, and a generating system has been given for all those clones. This puts us into the following situation.



$$[\mathscr{U}, \mathscr{O}] = \{\mathscr{U}, \ldots, \mathrm{Pol}(\mathscr{U}^{(3)}), \mathrm{Pol}(\mathscr{U}^{(2)}), \mathrm{Pol}(\mathscr{U}^{(1)})\}$$

ZILLA. What dost thou mean by the "upper part" of the interval?

KONG. Machida [6] has introduced a natural metric d on the clone lattice: For two clones $\mathscr{C}, \mathscr{D}$ set $d(\mathscr{C}, \mathscr{D}) = 0$ if $\mathscr{C} = \mathscr{D}$, and $d(\mathscr{C}, \mathscr{D}) = \frac{1}{2^{n-1}}$ if $\mathscr{C} \neq \mathscr{D}$ and $n = \min\{k \geq 1 : \mathscr{C}^{(k)} \neq \mathscr{D}^{(k)}\}$. In the interval $[\langle \mathscr{O}^{(1)} \rangle, \mathscr{O}]$ we are considering, all clones have the same unary part $\mathscr{O}^{(1)}$. The two maximal clones have distinct binary parts, and the interval $[\langle T_1 \rangle, \mathrm{Pol}(T_1)]$ consists exactly of the clones $\mathscr{C}$ for which $d(\mathrm{Pol}(T_1), \mathscr{C}) < \frac{1}{2}$, and the interval $[\langle T_2 \rangle, \mathrm{Pol}(T_2)]$ of those clones $\mathscr{C}$ for which $d(\mathrm{Pol}(T_2), \mathscr{C}) < \frac{1}{2}$. Therefore, it can be argued that the missing step in describing the "upper part" of the interval $[\langle \mathscr{O}^{(1)} \rangle, \mathscr{O}]$, or more precisely the clones "closest" to the two precomplete clones of the interval, is to determine the interval $[\langle T_2 \rangle, \mathrm{Pol}(T_2)]$.

ZILLA. Wherefore hast thou not yet described the interval?

KONG. It can be expected that describing clones above T_2 is more difficult than describing clones above T_1: Goldstern [2] has shown that none of the nontrivial clones containing T_2 is countably generated over $\mathscr{O}^{(1)}$ (whereas all clones containing T_1 are finitely generated over $\mathscr{O}^{(1)}$, see Pinsker [7]). There is also a reason involving descriptive set theory supporting that conjecture, see [2] and [7].

2. Approaches to solving the problem

ZILLA. Methinks the interval is rather large?

KONG. Surprisingly, so far we even failed to find out whether or not this interval has more than one element, i.e., whether or not $\langle T_2 \rangle \neq \mathrm{Pol}(T_2)$.

Question 3. *Is* $\langle T_2 \rangle = \mathrm{Pol}(T_2)$, *or* $\langle T_2 \rangle \subsetneq \mathrm{Pol}(T_2)$*?*

The major problem seems to be finding a "nice" description of $\langle T_2\rangle$; the elements of $\mathrm{Pol}(T_2)$ can be explicitly described as we will show in the following. We first give a number of equivalent definitions of T_2. We use the following abbreviations: "inj" stands for "injective", "const" for "constant", "s.m." for "strictly monotone", "s.m. in one var." for "strictly monotone in one variable" (for a binary essentially unary function), and "ess. unary" for "essentially unary". Note that $\langle \mathscr{O}^{(1)}\rangle^{(2)}$ is the set of binary essentially unary operations.

Lemma 4. *Let $f \in \mathscr{O}^{(2)}$. Then $f \in T_2$ iff one (or all) of the following hold:*

- $\forall S_1, S_2 \in [X]^\omega$ *(neither* $f \restriction_{\Delta_{S_1,S_2}}$ *nor* $f \restriction_{\nabla_{S_1,S_2}}$ *inj).*
- $\forall S_1, S_2 \in [X]^\omega \exists S_1' \in [S_1]^\omega\ \exists S_2' \in [S_2]^\omega$ *(*$f \restriction_{\Delta_{S_1',S_2'}}$ *and* $f \restriction_{\nabla_{S_1',S_2'}}$ *ess. unary).*
- $\forall g_1, g_2 \in \mathscr{O}^{(1)}$ *inj (neither* $f(g_1x_1, g_2x_2) \restriction_\Delta$ *nor* $f(g_1x_1, g_2x_2) \restriction_\nabla$ *inj).*
- $\forall g_1, g_2 \in \mathscr{O}^{(1)}$ *inj* $\exists S' \in [X]^\omega$ *(*$f(g_1x_1, g_2x_2) \restriction_{\Delta_{S'2}}$ *and* $f(g_1x_1, g_2x_2) \restriction_{\nabla_{S'2}}$ *ess. unary).*
- $\forall g_1, g_2 \in \mathscr{O}^{(1)}$ *inj* $\exists h \in \mathscr{O}^{(1)}$ *inj (*$f(g_1hx_1, g_2hx_2) \restriction_\Delta$ *and* $f(g_1hx_1, g_2hx_2) \restriction_\nabla$ *ess. unary).*
- $\forall g_1, g_2 \in \mathscr{O}^{(1)}$ *inj* $\exists h_1, h_2 \in \mathscr{O}^{(1)}$ *inj* *(*$f(g_1h_1x_1, g_2h_2x_2) \restriction_\Delta$ *and* $f(g_1h_1x_1, g_2h_2x_2) \restriction_\nabla$ *ess. unary).*

Proof. This is a straightforward verification using the application of Ramsey's theorem mentioned before. □

Lemma 5. *Let $f \in \mathscr{O}^{(n)}$. Then $f \in \mathrm{Pol}(T_2)$ iff one (or all) of the following hold:*

- $\forall g_1, \dots, g_n \in T_2\ (f(g_1, \dots, g_n) \in T_2)$
- $\forall g_1, \dots, g_n \in \langle \mathscr{O}^{(1)}\rangle^{(2)}\ (f(g_1, \dots, g_n) \in T_2)$
- $\forall g_1, \dots, g_n \in \langle \mathscr{O}^{(1)}\rangle^{(2)} \exists h \in \mathscr{O}^{(1)}$ *inj* $(f(g_1(hx_1, hx_2), \dots, g_n(hx_1, hx_2)) \restriction_\Delta$ *ess. unary)*
- $\forall g_1, \dots, g_n \in \langle \mathscr{O}^{(1)}\rangle^{(2)} \exists S' \in [X]^\omega\ (f(g_1, \dots, g_n) \restriction_{\Delta_{S'2}}$ *ess. unary)*
- $\forall g_1, \dots, g_n \in \langle \mathscr{O}^{(1)}\rangle^{(2)}$ *const or s.m. in one var.* $\exists h \in \mathscr{O}^{(1)}$ *s.m.* $(f(g_1(hx_1, hx_2), \dots, g_n(hx_1, hx_2)) \restriction_\Delta$ *ess. unary)*
- $\forall g_1, \dots, g_n \in \langle \mathscr{O}^{(1)}\rangle^{(2)}$ *const or s.m. in one var.* $\exists h_1, \dots, h_n \in \mathscr{O}^{(1)}$ *s.m.* $(f(g_1(h_1x_1, h_1x_2), \dots, g_n(h_nx_1, h_nx_2)) \restriction_\Delta$ *ess. unary)*
- $\forall g_1, \dots, g_n \in \langle \mathscr{O}^{(1)}\rangle^{(2)}$ *const or s.m. in one var.* $\exists S' \in [X]^\omega$ $(f(g_1, \dots, g_n) \restriction_{\Delta_{S'2}}$ *ess. unary)*
- $\forall g_1, \dots, g_n \in \langle \mathscr{O}^{(1)}\rangle^{(2)}$ *const or s.m. in one var.* $\forall S \in [X]^\omega$ $(f(g_1, \dots, g_n) \restriction_{\Delta_{S2}}$ *not inj)*
- $\forall g_1, \dots, g_n \in \langle \mathscr{O}^{(1)}\rangle^{(2)}$ *const or s.m. in one var.* $(f(g_1, \dots, g_n) \restriction_\Delta$ *not inj)*

Proof. To verify this, one again uses Ramsey's theorem as well as the fact (see [1]) that $f \in \mathrm{Pol}(T_2)$ iff for all $g_1, \dots, g_n \in \langle \mathscr{O}^{(1)}\rangle^{(2)}$ it is true that $f(g_1, \dots, g_n) \in T_2$. □

ZILLA. How, King, dost thou advise to attack the problem?

KONG. God, a first approach to Question 3 is to consider ternary functions: We know that $\langle T_2\rangle^{(1)} = \mathrm{Pol}(T_2)^{(1)} = \mathscr{O}^{(1)}$ and $\langle T_2\rangle^{(2)} = \mathrm{Pol}(T_2)^{(2)} = T_2$.

Question 6. *Is there a function $f \in \mathrm{Pol}(T_2)^{(3)}$ which is not generated by T_2?*

Clearly, if $f \in \mathscr{O}$ has finite range, then $f \in \mathrm{Pol}(T_2)$. Therefore, a positive answer to the following questions would solve Questions 3 and 6 respectively.

Question 7. *Does there exist a function with finite range which is not generated by T_2? Does there exist a* ternary *function with finite range which is not generated by T_2?*

ZILLA. Dost thou, King, know of a function with finite range wicked enough to make thee believe it be not generated by T_2?

KONG. Assume $0, 1 \in X$. We call an operation $f \in \mathscr{O}$ *boolean* iff the range of f is contained in $\{0,1\} = 2$. Let $f : X^3 \to 2$ be so that for all finite $A \subseteq X$ we have: For all $g : A^2 \to 2$ there exists $c \in X$ such that $f(x_1, x_2, c) \restriction_{A^2} = g(x_1, x_2)$, where $f(x_1, x_2, c) \restriction_{A^2}$ is considered a binary function from A^2 to 2. This is possible, since X has only countably many finite subsets A, and on all such subsets there are only finitely many functions from A^2 to 2.

Lemma 8. *f is not generated by binary boolean functions.*

The following lemma is a direct consequence of the application of Ramsey's theorem mentioned before.

Lemma 9. *Let $h \in \mathscr{O}^{(2)}$. If the range of h is finite, then there exists an infinite $S \subseteq X$ such that $h \restriction_{\Delta_{S^2}}$ is constant.*

Proof of Lemma 8. Assume to the contrary that f has a representation as a term t of binary boolean functions. Let $t_1, \ldots, t_k$ be all the functions which appear in t. Then f can be written as follows: $f = s(t_1, \ldots, t_k)$, where $s : 2^k \to 2$ and $t_i : X^3 \to 2$, and all t_i depend only on two variables. There are only 2^k possibilities for the arguments of s, since the t_i take only two values and there are k arguments. Therefore f can also be written as $f = s'(g_1(x_1, x_2), g_2(x_1, x_3), g_3(x_2, x_3))$, where $g_i : X^2 \to 2^k$, $i = 1, 2, 3$, and $s' : (2^k)^3 \to 2$. By Lemma 9, we can "thin out" X to an infinite subset S in such a way that the restriction of g_1 to Δ_{S^2} is constant. Therefore, on Δ_{S^2} we have $f = s''(g_2(x_1, x_3), g_3(x_2, x_3))$, where $s'' : (2^k)^2 \to 2$. Now choose any $A_1, A_2 \subseteq S$ of size $2^k + 1$ so that $A_1 \times A_2 \subseteq \Delta_{S^2}$, i.e., $\max A_2 < \min A_1$. Let $g : A_1 \times A_2 \to 2$ be so that if $a, b \in A_1$ are distinct, then there exists $c \in A_2$ such that $g(a, c) \neq g(b, c)$. This is possible since for every fixed $a \in A_1$ we have 2^{2^k+1} possibilities of defining the unary function $g(a, x_2) : A_2 \to 2$, and we only have to define it for $2^k + 1$ values of $a \in A_1$. Now let $d \in X$ be so that $f(x_1, x_2, d) \restriction_{A_1 \times A_2} = g(x_1, x_2)$; d exists by the construction of f. Since $|A_1| = 2^k + 1$ and g_2 takes only 2^k values, there exist distinct $a, b \in A_1$ such that $g_2(a,d) = g_2(b,d)$. There is $c \in A_2$ such that $g(a,c) \neq g(b,c)$, for we have chosen g that way; therefore, $f(a,c,d) \neq f(b,c,d)$. Thus, $s''(g_2(a,d), g_3(c,d)) \neq s''(g_2(b,d), g_3(c,d))$. But this is impossible since $g_2(a,d) = g_2(b,d)$, and we arrive at a contradiction. $\square$

Question 10. *Is the f as in Lemma 8 generated by T_2?*

ZILLA. The strangeness of your story put heavyness in me.
KONG. Shake it off; here, mind the references.

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