Logic characterized by Boolean algebras with conjugate

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1 Introduction

In [1], Jarvinen and Kortelainen considered properties of lower (upper) approximation operators in rough set theory by use of the algebras with conjugate pair of maps. Let $B$ be any Boolean algebra. A pair $(f, g)$ of maps $f, g : B \to B$ is called conjugate ([2]) if, for all $x, y \in B$, the following condition is satisfied:

$$x \wedge f(y) = 0 \iff y \wedge g(x) = 0$$

Moreover if a pair $(f, f)$ is conjugate, then $f$ is called self-conjugate. If a Boolean algebra has a pair of conjugate maps, then we say simply a Boolean algebra with conjugate.

By $B$ we mean the class of all Boolean algebras with conjugate. In this short note we show that $B$ characterizes a certain kind of tense logic $K^*_t$, that is, for the class $\Phi$ of all formulas of $K^*_t$,

For any $B \in B$ and a map $\xi : \Phi \to B$, we have $\xi(A) = 1 \iff \vdash_{K^*_t} A$

2 tense logic $K^*_t$

We define a certain kind of tense logic named $K^*_t$ here. The logic is obtained from the minimal tense logic $K^*_t$ by removing the axioms $(sym): A \to GPA, A \to HFA$ and $(cl): GA \to GGA, HA \to HHA$.

Let $\Phi_0$ be a countable set $p_0, p_1, p_2, \cdots$ of propositional variables and $\wedge, \lor, \to, \neg, G, H$ are logical symbols. A formula of $K^*_t$ is defined as follows:

(1) Every propositional variable is a formula;

(2) If $A$ and $B$ are formulas, then so are $A \wedge B, A \lor B, A \to B, \neg A, GA, HA$.

Let $\Phi$ be the set of all formulas of $K^*_t$. We define symbols $F$ and $P$ respectively by

$$FA \equiv \neg\neg A, \quad PA \equiv \neg H \neg A.$$  

A logical system $K^*_t$ has the following axioms and rules of inference ([3]):

Axioms :

(1) $A \to (B \to A)$

(2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(3) $(\neg A \to \neg B) \to (B \to A)$

(4) $G(A \to B) \to (GA \to GB), \quad H(A \to B) \to (HA \to HB)$

Rule of Inference :

(MP) Deduce $B$ from $A$ and $A \to B$;

(Nec) Deduce $GA$ and $HA$ from $A$.

We list typical axioms which characterize some properties of conjugate:
A well-known tense logic \( K_t \) is an axiomatic extension of \( K^*_t \), which has extra axioms \((sym)\) and \((cl)\), that is, \( K_t = K^*_t + (sym) + (cl) \).

A formula \( A \) is called **provable** when there is a finite sequence \( A_1, A_2, \ldots, A_n (= A) \) \( (n \geq 1) \) of formulas such that, for every \( i \) \( (1 \leq i \leq n) \),

1. \( A_i \) is an axiom;
2. \( A_i \) is deduced from \( A_j, A_k \) \( (j, k < i) \) by \((MP)\);
3. \( A_i \) is done from \( A_j \) \( (j < i) \) by \((Nec)\).

We denote that \( A \) is provable by \( \vdash_{K_t} A \) (or simply \( \vdash A \)).

A relational structure \((W, R)\) is called a **Kripke frame**, where \( W \) is a non-empty set and \( R \) is a binary relation on it. A valuation \( v \) is a map from \( \Phi_0 \) to \( \mathcal{P}(W) \), that is, \( v : \Phi_0 \rightarrow \mathcal{P}(W) \).

It is easy to show that a valuation \( v \) can be extended uniquely to the set \( \Phi \) of all formulas:

\[
\begin{align*}
1. & \quad v(A \land B) = v(A) \cap v(B) \\
2. & \quad v(A \lor B) = v(A) \cup v(B) \\
3. & \quad v(A \rightarrow B) = v(A)^c \cup v(B) \\
4. & \quad v(\neg A) = v(A)^c \\
5. & \quad v(GA) = \{x \in W \mid \forall y ((x, y) \in R \Rightarrow y \in v(A))\} \\
6. & \quad v(HA) = \{x \in W \mid \forall y ((y, x) \in R \Rightarrow y \in v(A))\}
\end{align*}
\]

Thus we call the extended valuation above simply a valuation and denote it by the same symbol \( v \).

Since, for all formulas \( A \) and \( B \)

\[
\vdash_{K^*_t} A \land \neg A \rightarrow B \land \neg B, \quad \vdash_{K^*_t} A \rightarrow \neg A \rightarrow B \lor \neg B,
\]

We define symbols \( \bot \) and \( \top \) respectively by

\[
\bot \equiv A \land \neg A, \quad \top \equiv A \lor \neg A.
\]

Then for every formula \( A \in \Phi \), we have

\[
\vdash_{K^*_t} \bot \rightarrow A, \quad \vdash_{K^*_t} A \rightarrow \top.
\]

A structure \( \mathcal{M} = (W, R, v) \) is called a **Kripke model**, where \( (W, R) \) is a Kripke frame and \( v \) is a valuation on it. Given a Kripke model \( \mathcal{M} = (W, R, v) \), we can interpret the formulas on it as follows: For \( x \in W \), a formula \( A \) is said to be **true** at \( x \) on the Kripke model \( \mathcal{M} \) if

\[
x \in v(A),
\]

and denoted by

\[
\mathcal{M} \models_x A.
\]

If \( v(A) = W \), that is, \( A \) is true at ever \( x \in W \) on the Kripke model \( \mathcal{M} \), then \( A \) is called **true** on \( \mathcal{M} \) and denoted by

\[
\mathcal{M} \models A.
\]

Moreover \( A \) is called **valid** if \( A \) is true on every Kripke model \( \mathcal{M} \) and denoted by

\[
\models A.
\]

It is easy to show the next result ([3]):
Theorem 1. (Completeness Theorem) For every formula $A$, we have

$$\vdash_{K_{\dot{t}}} A \iff A : \text{valid}$$

We can get the next result by use of filtration method ([3]):

Theorem 2. For every formula $A$, we have

$$\vdash_{K_{\dot{t}}} A \iff A : \text{true for any finite Kripke model } \mathcal{M}.$$

3 Boolean algebra with conjugate pair

Let $B = (B, \wedge, \vee', 0, 1)$ be a Boolean algebra. A pair $(\varphi, \psi)$ of maps $\varphi, \psi : B \to B$ is called a conjugate pair if, for all $x, y \in B$,

$$x \wedge \varphi(y) = 0 \iff y \wedge \psi(x) = 0.$$

We define some properties about a map $\varphi : B \to B$ as follows:

- $\varphi : \text{extensive} \iff x \leq \varphi(x)$ ($\forall x \in B$)
- $\varphi : \text{symmetric} \iff x \leq \varphi(y)$ implies $y \leq \varphi(x)$ ($\forall x, y \in B$)
- $\varphi : \text{closed} \iff y \leq \varphi(x)$ implies $\varphi(y) \leq \varphi(x)$ ($\forall x, y \in B$)

It is clear that the following holds for a conjugate pair $(\varphi, \psi)$ ([1]):

- $\varphi : \text{extensive} \iff \psi : \text{extensive}$
- $\varphi : \text{symmetric} \iff \varphi : \text{self–conjugate}$
- $\varphi : \text{closed} \iff \psi : \text{closed}$

We introduce two operators $\varphi^\partial, \psi^\partial$ for the sake of simplicity

$$\varphi^\partial(x) = (\varphi(x'))', \psi^\partial(x) = (\psi(x'))'(x \in B).$$

A conjugate pair $(\varphi, \psi)$ can be represented by

$$\varphi(x) \leq y \iff x \leq \psi^\partial(y) \ (x, y \in B).$$

It is obvious from definition that

**Proposition 1.** For every $x \in B$ we have

- $\varphi : \text{extensive} \iff \varphi^\partial(x) \leq x$
- $\varphi : \text{symmetric} \iff x \leq \varphi^\partial(\varphi(x))$
- $\varphi : \text{closed} \iff \varphi^\partial(x) \leq \varphi^\partial(\varphi^\partial(x))$

Let $B$ be a Boolean algebra with conjugate and $\xi : \Phi \to B$ be a map. Each formula of $K^*_t$ is interpreted on the algebra as follows:

1. $\xi(A \land B) = \xi(A) \land \xi(B)$
2. $\xi(A \lor B) = \xi(A) \lor \xi(B)$
3. $\xi(A \rightarrow B) = (\xi(A))' \lor \xi(B)$
4. $\xi(\neg A) = (\xi(A))'$
5. $\xi(GA) = (\varphi((\xi(A))'))' = \varphi^\partial(\xi(A))$
6. $\xi(HA) = (\psi((\xi(A))'))' = \psi^\partial(\xi(A))$
**Lemma 1.** For every formula $A$, we have

$$\vdash_{K_{t}} A \Rightarrow \xi(A) = 1$$

for all $\xi : \Phi \rightarrow B$

*Proof.* It is sufficient to verify that each axiom $\alpha$ of $K_{t}^{*}$ has a value $\xi(\alpha) = 1$ and each rule of inference is preserved, that is, for the case of (MP),

$$\xi(A) = \xi(A \rightarrow B) = 1 \text{ imply } \xi(B) = 1$$

and for the case of (Nec)

$$\xi(A) = 1 \text{ implies } \xi(GA) = \xi(HA) = 1.$$  

We omit their proof. \hfill $\square$

We can show the converse direction of the above. In order to do that we prepare some lemmas. At first we define a relation $\equiv$ on the set $\Phi$ of formulas of $K_{t}^{*}$: For $A, B \in \Phi$,

$$A \equiv B \iff \vdash_{K_{t}} A \rightarrow B \text{ and } \vdash_{K_{t}} B \rightarrow A$$

As to the relation $\equiv$ we can prove that

**Lemma 2.** $\equiv$ is a congruence on $\Phi$, that is, it is an equivalence relation and satisfies the compatible property: If $A \equiv B$ and $C \equiv D$, then

$$A \land C \equiv B \land D, \quad A \lor C \equiv B \lor D,$$

$$A \rightarrow D \equiv B \rightarrow D,$$

$$\neg A \equiv \neg B,$$

$$GA \equiv GB, \quad HA \equiv HB$$

*Proof.* We only prove that if $A \equiv B$ then $GA \equiv GB$. It follows from assumption that $\vdash A \rightarrow B$. From (Nec) we get

$$\vdash G(A \rightarrow B).$$

On the other hand, since $\vdash G(A \rightarrow B) \rightarrow (GA \rightarrow GB)$, we have from (MP)

$$\vdash GA \rightarrow GB.$$  

Similarly, by $\vdash B \rightarrow A$, we get

$$\vdash GB \rightarrow GA.$$  

This means that

$$GA \equiv GB.$$  

$\square$

Since $\equiv$ is the congruence, we can define operations on $\Phi/\equiv$: For $A, B \in \Phi$, we define

$$[A] \cap [B] = [A \land B],$$

$$[A] \cup [B] = [A \lor B],$$

$$[A]^* = [\neg A],$$

$$\varphi([A]) = [\neg G \neg A] = [FA],$$

$$\psi([A]) = [\neg H \neg A] = [PA],$$

$$0 = [\perp], \quad 1 = [T].$$

**Lemma 3.** $(\Phi/\equiv, \cap, \cup, ^*, 0, 1)$ is a Boolean algebra with $(\varphi, \psi)$ as a conjugate pair.
Proof. We show that \((\varphi, \psi)\) is the conjugate pair. Let \([A], [B] \in \Phi/\equiv\). We have to prove

\[ [A] \cap \varphi([B]) = 0 \iff [B] \cap \psi([A]) = 0, \]

that is,

\[ [A \land FB] = 0 \iff [B \land PA] = 0. \]

Suppose that \([A \land FB] = 0\). Since \(\vdash A \land FB \rightarrow \bot\), we have \(\vdash FB \rightarrow \neg A\). From (Nec) we get \(\vdash HFB \rightarrow H\neg A\). Since \(\vdash B \rightarrow HFB\), we also have \(\vdash B \rightarrow H\neg A\). Thus we obtain

\(\vdash \neg(B \land PA),\)

that is,

\([B \land PA] = 0.\)

It is similar the converse.

\[ \square \]

Lemma 4. For any formula \(A \in \Phi\),

\(\vdash_{K_t^*} A \iff [A] = 1 \) in \(\Phi/\equiv\)

Proof.

\[ \vdash_{K_t^*} A \iff \vdash_{K_t^*} A \rightarrow \top \text{ and } \vdash_{K_t^*} \top \rightarrow A \]

\[ \iff [A] = [\top] = 1 \]

\[ \square \]

From the above, we can prove the next theorem.

Theorem 3. Let \(A \in \Phi\).

For any Boolean algebra \(B\) with conjugate and a map \(\xi : \Phi \rightarrow B\), we have \(\xi(A) = 1\)

\(\iff \vdash_{K_t^*} A\)

Proof. We have already proved if part. To show the only if part, we assume that \(\not\vdash_{K_t^*} A\). Since \(\Phi/\equiv\) is the Boolean algebra with conjugate, if we take a map

\(\xi : \Phi \rightarrow \Phi/\equiv, \xi(A) = [A],\)

then on \(\Phi/\equiv\) we get

\(\xi(A) \neq 1\)

by \(\not\vdash_{K_t^*} A\).

\[ \square \]

We can characterize some logics by Boolean algebras with conjugate.

Theorem 4. Logical systems \(K_t^* + (ext), K_t^* + (sym), K_t^* + (cl)\) are characterized respectively by the Boolean algebras with extensive, symmetric, closed conjugate, that is, for any formula \(A \in \Phi\)

1. for any Boolean algebra \(B\) with extensive conjugate and a map \(\xi : \Phi \rightarrow B\), we have \(\xi(A) = 1 \iff \vdash_{K_t^* + (ext)} A\)

2. for any Boolean algebra \(B\) with symmetric conjugate and a map \(\xi : \Phi \rightarrow B\), we have \(\xi(A) = 1 \iff \vdash_{K_t^* + (sym)} A\)

3. for any Boolean algebra \(B\) with closed conjugate and a map \(\xi : \Phi \rightarrow B\), we have \(\xi(A) = 1 \iff \vdash_{K_t^* + (cl)} A\)
Proof. We only show that, in any Boolean algebra with typical property, $\xi(A) = 1$ for the correspondent typical axioms $A$ in respective cases. Suppose that $\xi(A) = x \in B$.

1) For an extensive conjugate $(\varphi, \psi)$, we have to prove that $\xi(GA \rightarrow A) = 1$. Since

$$\xi(GA \rightarrow A) = 1 \iff \xi(GA) \leq \xi(A)$$
$$\iff (\varphi(x'))' \leq x$$
$$\iff x' \leq \varphi(x')$$

and $\varphi$ is extensive, we have $\xi(GA \rightarrow A) = 1$.

2) Let $(\varphi, \psi)$ be a symmetric conjugate. Since $\varphi = \psi$ by assumption, we have

$$\xi(AGPA) = 1 \iff \xi(A) \leq \xi(GPA)$$
$$\iff x \leq \varphi^\partial(\psi(x))$$
$$\iff x \leq \psi^\partial(\psi(x))$$
$$\iff \varphi(x) \leq \psi(x)$$
$$\iff \varphi(x) \leq \varphi(x).$$

Thus, $\xi(A \rightarrow GPA) = 1$.

3) Suppose that $(\varphi, \psi)$ is a closed conjugate. It follows from the assumption that $\varphi^\partial(x) \leq \varphi^\partial(\varphi^\partial(x))$ $(x \in B)$ and hence that

$$\xi(GA \rightarrow GGA) = 1 \iff \xi(GA) \leq \xi(GGA)$$
$$\iff \varphi^\partial(x) \leq \varphi^\partial(\varphi^\partial(x)).$$

This means that $\xi(A \rightarrow GPA) = 1$. $\square$

4 Decidability

It is well-known that the minimal tense logic $K_t$ can be characterized by the class of finite Kripke models. Similarly we can show that $K^*_t$ is characterized by the class $B^*$ of finite Boolean algebras with conjugate.

Suppose that $\nu_{K^*_t} A$. There is a finite Kripke model $\mathcal{M}^* = (W, R, v)$ such that $x \notin v(A)$ for some $x \in W$, that is, $v(A) \neq W$. We construct a finite Boolean algebra $B^*$ with conjugate from the finite Kripke model $\mathcal{M}^*$ as follows:

$$B^* = \mathcal{P}(W)$$
$$\varphi, \psi : B \rightarrow B \text{ are defined respectively by}$$
$$\varphi(X) = \{x \in B \mid R(x) \cap X \neq \emptyset\}$$
$$\psi(X) = \{x \in B \mid R^{-1}(x) \cap X \neq \emptyset\},$$

where $R(x), R^{-1}(x)$ are defined by

$$R(x) = \{y \in B \mid (x, y) \in R\}, \quad R^{-1}(x) = \{y \in B \mid (y, x) \in R\}$$

We can prove the fundamental result.

Lemma 5. $B^*$ is a finite Boolean algebra with a conjugate pair $\varphi, \psi : B^* \rightarrow B^*$.

Proof. It is sufficient to prove that $\varphi, \psi : B^* \rightarrow B^*$ are conjugate. That is, we have to prove that for $X, Y \subseteq W$ (i.e., $X, Y \in B^*$),

$$X \cap \varphi(Y) = \emptyset \iff Y \cap \psi(X) = \emptyset.$$

Suppose that $Y \cap \psi(X) \neq \emptyset$. Since $y \in \psi(X)$ for some $y \in Y$, it follows from definition of $\psi(X)$ that

$$\exists x \in X \text{ s.t. } (x, y) \in R.$$
We also have \((x, y) \in R\) and \(y \in Y\). This implies that
\[
R(x) \cap Y \neq \emptyset
\]
and \(x \in \varphi(Y)\). The fact that \(x \in X\) means
\[
x \in X \cap \varphi(Y), \text{ that is, } X \cap \varphi(Y) \neq \emptyset.
\]
The converse can be proved similarly. Thus \(B^*\) is the finite Boolean algebra with the conjugate pair
\(\varphi, \psi : B^* \to B^*\).

Moreover if we take \(\xi^* : \Phi \to B^*\) as
\[
\xi^*(A) = v(A),
\]
then we have \(\xi^*(A) \neq 1\) from \(v(A) \neq W\). This means that \(\not\vdash_{K_t^*} A\) implies \(\xi^*(A) \neq 1\) for some finite Boolean algebra with conjugate and \(\xi^* : B^* \to B^*\). It is obvious the converse statement. We thus obtain the next result.

**Theorem 5.** The logic \(K_t^*\) can be characterized by the finite Boolean algebras with conjugate.

We can show the following similarly.

**Theorem 6.** The logics \(K_t^* + (ext), K_t^* + (sym), K_t^* + (cl)\) are characterized by the class of all finite Boolean algebras with extensive, symmetric, closed conjugate pair, respectively.

Thus we can conclude that our logical systems \(K_t^* (+ (ext), + (sym), + (cl))\) are decidable, that is, we can determine whether a given formula is provable or not by finite steps.

**References**


