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Kyoto University
Logic characterized by Boolean algebras with conjugate

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1 Introduction

In [1], Jarvinen and Kortelainen considered properties of lower (upper) approximation operators in rough set theory by use of the algebras with conjugate pair of maps. Let $B$ be any Boolean algebra. A pair $(f, g)$ of maps $f, g : B \to B$ is called conjugate ([2]) if, for all $x, y \in B$, the following condition is satisfied:

$$x \land f(y) = 0 \iff y \land g(x) = 0$$

Moreover if a pair $(f, f)$ is conjugate, then $f$ is called self-conjugate. If a Boolean algebra has a pair of conjugate maps, then we say simply a Boolean algebra with conjugate.

By $B$ we mean the class of all Boolean algebras with conjugate. In this short note we show that $B$ characterizes a certain kind of tense logic $K_t^*$, that is, for the class $\Phi$ of all formulas of $K_t^*$,

For any $B \in B$ and a map $\xi : \Phi \to B$, we have $\xi(A) = 1 \iff \vdash_{K_t^*} A$

2 tense logic $K_t^*$

We define a certain kind of tense logic named $K_t^*$ here. The logic is obtained from the minimal tense logic $K_t$ by removing the axioms $(\text{sym}) : A \to GPA, A \to HFA$ and $(\text{cl}) : GA \to GGA, HA \to HHA$.

Let $\Phi_0$ be a countable set $p_0, p_1, p_2, \ldots$ of propositional variables and $\land, \lor, \to, \neg, G, H$ are logical symbols. A formula of $K_t^*$ is defined as follows:

(1) Every propositional variable is a formula;

(2) If $A$ and $B$ are formulas, then so are $A \land B, A \lor B, A \to B, \neg A, GA, HA$.

Let $\Phi$ be the set of all formulas of $K_t^*$. We define symbols $F$ and $P$ respectively by

$$FA \equiv \neg \neg A, PA \equiv \neg H \neg A.$$

A logical system $K_t^*$ has the following axioms and rules of inference ([3]):

Axioms :

(1) $A \to (B \to A)$

(2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(3) $(\neg A \to \neg B) \to (B \to A)$

(4) $G(A \to B) \to (GA \to GB), H(A \to B) \to (HA \to HB)$

Rule of Inference :

(MP) Deduce $B$ from $A$ and $A \to B$;

(Nec) Deduce $GA$ and $HA$ from $A$.

We list typical axioms which characterize some properties of conjugate:
A well-known tense logic $K_t$ is an axiomatic extension of $K_t^*$, which has extra axioms $(sym)$ and $(cl)$, that is,

$$K_t = K_t^* + (sym) + (cl)$$

A formula $A$ is called provable when there is a finite sequence $A_1, A_2, \cdots, A_n (= A) \ (n \geq 1)$ of formulas such that, for every $i \ (1 \leq i \leq n)$,

1. $A_i$ is an axiom;
2. $A_i$ is deduced from $A_j, A_k \ (j, k < i)$ by (MP);
3. $A_i$ is done from $A_j \ (j < i)$ by (Nec).

We denote that $A$ is provable by

$$\vdash_{K_t} A \quad \text{(or simply } \vdash A).$$

A relational structure $(W, R)$ is called a Kripke frame, where $W$ is a non-empty set and $R$ is a binary relation on it. A valuation $v$ is a map from $\Phi_0$ to $\mathcal{P}(W)$, that is, $v : \Phi_0 \rightarrow \mathcal{P}(W)$. It is easy to show that a valuation $v$ can be extended uniquely to the set $\Phi$ of all formulas:

1. $v(A \land B) = v(A) \cap v(B)$
2. $v(A \lor B) = v(A) \cup v(B)$
3. $v(A \rightarrow B) = v(A)^c \cup v(B)$
4. $v(\neg A) = v(A)^c$
5. $v(GA) = \{x \in W | \forall y ((x, y) \in R \Rightarrow y \in v(A))\}$
6. $v(HA) = \{x \in W | \forall y ((y, x) \in R \Rightarrow y \in v(A))\}$

Thus we call the extended valuation above simply a valuation and denote it by the same symbol $v$.

Since, for all formulas $A$ and $B$,

$$\vdash_{K_t} A \land \neg A \rightarrow B \land \neg B, \quad \vdash_{K_t} A \lor \neg A \rightarrow B \lor \neg B,$$

We define symbols $\bot$ and $\top$ respectively by

$$\bot \equiv A \land \neg A, \quad \top \equiv A \lor \neg A.$$

Then for every formula $A \in \Phi$, we have

$$\vdash_{K_t} \bot \rightarrow A, \quad \vdash_{K_t} A \rightarrow \top.$$

A structure $M = (W, R, v)$ is called a Kripke model, where $(W, R)$ is a Kripke frame and $v$ is a valuation on it. Given a Kripke model $M = (W, R, v)$, we can interpret the formulas on it as follows: For $x \in W$, a formula $A$ is said to be true at $x$ on the Kripke model $M$ if

$$x \in v(A),$$

and denoted by

$$M \models_x A.$$

If $v(A) = W$, that is, $A$ is true at ever $x \in W$ on the Kripke model $M$, then $A$ is called true on $M$ and denoted by

$$M \models A.$$

Moreover $A$ is called valid if $A$ is true on every Kripke model $M$ and denoted by

$$\models A.$$

It is easy to show the next result ([3]):
Theorem 1. (Completeness Theorem) For every formula $A$, we have

\[ \vdash_{K_{\dot{t}}} A \Leftrightarrow A : \text{valid} \]

We can get the next result by use of filtration method ([3]):

Theorem 2. For every formula $A$, we have

\[ \vdash_{K_{\dot{t}}} A \Leftrightarrow A : \text{true for any finite Kripke model } \mathcal{M}. \]

3 Boolean algebra with conjugate pair

Let $B = (B, \wedge, \vee', 0, 1)$ be a Boolean algebra. A pair $(\varphi, \psi)$ of maps $\varphi, \psi : B \to B$ is called a conjugate pair if, for all $x, y \in B$,

\[ x \wedge \varphi(y) = 0 \iff y \wedge \psi(x) = 0. \]

We define some properties about a map $\varphi : B \to B$ as follows:

- $\varphi$ : extensive $\iff x \leq \varphi(x)$ (\forall x \in B)
- $\varphi$ : symmetric $\iff x \leq \varphi(y)$ implies $y \leq \varphi(x)$ (\forall x, y \in B)
- $\varphi$ : closed $\iff \varphi(y) \leq \varphi(x)$ (\forall x, y \in B)

It is clear that the following holds for a conjugate pair $(\varphi, \psi)$ ([1]):

- $\varphi$ : extensive $\iff \psi$ : extensive
- $\varphi$ : symmetric $\iff \varphi$ : self–conjugate
- $\varphi$ : closed $\iff \psi$ : closed

We introduce two operators $\varphi^\partial, \psi^\partial$ for the sake of simplicity

\[ \varphi^\partial(x) = (\varphi(x'))', \ \psi^\partial(x) = (\psi(x'))' \ (x \in B). \]

A conjugate pair $(\varphi, \psi)$ can be represented by

\[ \varphi(x) \leq y \iff x \leq \psi^\partial(y) \ (x, y \in B). \]

It is obvious from definition that

Proposition 1. For every $x \in B$ we have

- $\varphi$ : extensive $\iff \varphi^\partial(x) \leq x$
- $\varphi$ : symmetric $\iff x \leq \varphi^\partial(\varphi(x))$
- $\varphi$ : closed $\iff \varphi^\partial(x) \leq \psi^\partial(\varphi^\partial(x))$

Let $B$ be a Boolean algebra with conjugate and $\xi : \Phi \to B$ be a map. Each formula of $K_{\dot{t}}^*$ is interpreted on the algebra as follows:

1. $\xi(A \wedge B) = \xi(A) \wedge \xi(B)$
2. $\xi(A \vee B) = \xi(A) \vee \xi(B)$
3. $\xi(A \rightarrow B) = (\xi(A))' \vee \xi(B)$
4. $\xi(\neg A) = (\xi(A))'$
5. $\xi(GA) = (\varphi(\xi(A)))' = \varphi^\partial(\xi(A))$
6. $\xi(HA) = (\psi(\xi(A))')' = \psi^\partial(\xi(A))$
Lemma 1. For every formula $A$, we have
$$\vdash_{K_{t}} A \iff \xi(A) = 1 \text{ for all } \xi : \Phi \rightarrow B$$

Proof. It is sufficient to verify that each axiom $\alpha$ of $K_{t}^{*}$ has a value $\xi(\alpha) = 1$ and each rule of inference is preserved, that is, for the case of (MP),
$$\xi(A) = \xi(A \rightarrow B) = 1 \text{ imply } \xi(B) = 1$$
and for the case of (Nec)
$$\xi(A) = 1 \text{ implies } \xi(GA) = \xi(HA) = 1.$$ We omit their proof.

We can show the converse direction of the above. In order to do that we prepare some lemmas. At first we define a relation $\equiv$ on the set $\Phi$ of formulas of $K_{t}^{*}$: For $A, B \in \Phi$,
$$A \equiv B \iff \vdash_{K_{t}} A \rightarrow B \text{ and } \vdash_{K_{t}} B \rightarrow A$$

As to the relation $\equiv$ we can prove that

Lemma 2. $\equiv$ is a congruence on $\Phi$, that is, it is an equivalence relation and satisfies the compatible property: If $A \equiv B$ and $C \equiv D$, then
$$A \wedge C \equiv B \wedge D, \quad A \vee C \equiv B \vee D, \quad A \rightarrow D \equiv B \rightarrow D,$$
$$\neg A \equiv \neg B, \quad GA \equiv GB, \quad HA \equiv HB$$

Proof. We only prove that if $A \equiv B$ then $GA \equiv GB$. It follows from assumption that $\vdash A \rightarrow B$. From (Nec) we get
$$\vdash G(A \rightarrow B).$$
On the other hand, since $\vdash G(A \rightarrow B) \rightarrow (GA \rightarrow GB)$, we have from (MP)
$$\vdash GA \rightarrow GB.$$ Similarly, by $\vdash B \rightarrow A$, we get
$$\vdash GB \rightarrow GA.$$ This means that
$$GA \equiv GB.$$ Since $\equiv$ is the congruence, we can define operations on $\Phi/\equiv$: For $A, B \in \Phi$, we define
$$[A] \cap [B] = [A \wedge B],$$
$$[A] \cup [B] = [A \vee B],$$
$$[A]^* = [\neg A],$$
$$\varphi([A]) = [\neg G \neg A] = [FA],$$
$$\psi([A]) = [\neg H \neg A] = [PA],$$
$$0 = [\perp], \quad 1 = [T].$$

Lemma 3. $(\Phi/\equiv, \cap, \cup, ^*, 0, 1)$ is a Boolean algebra with $(\varphi, \psi)$ as a conjugate pair.
Proof. We show that $(\varphi, \psi)$ is the conjugate pair. Let $[A], [B] \in \Phi / \equiv$. We have to prove

$$[A \cap \varphi([B]) = 0 \iff [B \cap \psi([A]) = 0,$$

that is,

$$[A \wedge FB] = 0 \iff [B \wedge PA] = 0.$$

Suppose that $[A \wedge FB] = 0$. Since $\vdash A \wedge FB \rightarrow \bot$, we have $\vdash FB \rightarrow \neg A$. From (Nec) we get $\vdash HFB \rightarrow H\neg A$. Since $\vdash B \rightarrow HFB$, we also have $\vdash B \rightarrow H\neg A$. Thus we obtain

$$\vdash \neg(B \wedge PA),$$

that is,

$$[B \wedge PA] = 0.$$

It is similar the converse. \[ \square \]

Lemma 4. For any formula $A \in \Phi$,

$$\vdash_{K_{t}^{*}} A \iff [A] = 1$$

Proof. $\vdash_{K_{t}^{*}} A \iff \vdash_{K_{t}} A \rightarrow T$ and $\vdash_{K_{t}} T \rightarrow A$

$$\iff [A] = [T] = 1$$

From the above, we can prove the next theorem.

Theorem 3. Let $A \in \Phi$.

For any Boolean algebra $B$ with conjugate and a map $\xi : \Phi \rightarrow B$, we have $\xi(A) = 1$ $\iff$ $\vdash_{K_{t}^{*}} A$

Proof. We have already proved if part. To show the only if part, we assume that $\vdash_{K_{t}^{*}} A$. Since $\Phi / \equiv$ is the Boolean algebra with conjugate, if we take a map

$$\xi : \Phi \rightarrow \Phi / \equiv, \xi(A) = [A],$$

then on $\Phi / \equiv$ we get

$$\xi(A) \neq 1$$

by $\vdash_{K^{*}} A$. \[ \square \]

We can characterize some logics by Boolean algebras with conjugate.

Theorem 4. Logical systems $K_{t}^{*} + (ext)$, $K_{t}^{*} + (sym)$, $K_{t}^{*} + (cl)$ are characterized respectively by the Boolean algebras with extensive, symmetric, closed conjugate, that is, for any formula $A \in \Phi$

1. for any Boolean algebra $B$ with extensive conjugate and a map $\xi : \Phi \rightarrow B$, we have $\xi(A) = 1$ $\iff$ $\vdash_{K_{t}^{*} + (ext)} A$

2. for any Boolean algebra $B$ with symmetric conjugate and a map $\xi : \Phi \rightarrow B$, we have $\xi(A) = 1$ $\iff$ $\vdash_{K_{t}^{*} + (sym)} A$

3. for any Boolean algebra $B$ with closed conjugate and a map $\xi : \Phi \rightarrow B$, we have $\xi(A) = 1$ $\iff$ $\vdash_{K_{t}^{*} + (cl)} A$
Proof. We only show that, in any Boolean algebra with typical property, $\xi(A) = 1$ for the correspondent typical axioms $A$ in respective cases. Suppose that $\xi(A) = x \in B$.

(1) For an extensive conjugate $(\varphi, \psi)$, we have to prove that $\xi(GA \rightarrow A) = 1$. Since

$$\xi(GA \rightarrow A) = 1 \iff \xi(GA) \leq \xi(A)$$

$$\iff (\varphi(x'))' \leq x$$

$$\iff x' \leq \varphi(x')$$

and $\varphi$ is extensive, we have $\xi(GA \rightarrow A) = 1$.

(2) Let $(\varphi, \psi)$ be a symmetric conjugate. Since $\varphi = \psi$ by assumption, we have

$$\xi(A \rightarrow GPA) = 1 \iff \xi(A) \leq \xi(GPA)$$

$$\iff x \leq \varphi^0(\psi(x))$$

$$\iff \varphi(x) \leq \psi(x)$$

$$\iff \varphi(x) \leq \varphi(x).$$

Thus, $\xi(A \rightarrow GPA) = 1$.

(3) Suppose that $(\varphi, \psi)$ is a closed conjugate. It follows from the assumption that $\varphi^0(x) \leq \varphi^0(\varphi^0(x))$ ($x \in B$) and hence that

$$\xi(GA \rightarrow GGA) = 1 \iff \xi(GA) \leq \xi(GGA)$$

$$\iff \varphi^0(x) \leq \varphi^0(\varphi^0(x)).$$

This means that $\xi(A \rightarrow GPA) = 1$. \qed

4 Decidability

It is well-known that the minimal tense logic $K_t$ can be characterized by the class of finite Kripke models. Similarly we can show that $K_{t}^{*}$ is characterized by the class $B^{*}$ of finite Boolean algebras with conjugate.

Suppose that $\gamma_{K_{t}^{*}} A$. There is a finite Kripke model $M^{*} = (W, R, v)$ such that $x \notin v(A)$ for some $x \in W$, that is, $v(A) \neq W$. We construct a finite Boolean algebra $B^{*}$ with conjugate from the finite Kripke model $M^{*}$ as follows:

$$B^{*} = \mathcal{P}(W)$$

$$\varphi, \psi : B \rightarrow B$$

are defined respectively by

$$\varphi(X) = \{x \in B \mid R(x) \cap X \neq \emptyset \}$$

$$\psi(X) = \{x \in B \mid R^{-1}(x) \cap X \neq \emptyset \},$$

where $R(x), \ R^{-1}(x)$ are defined by

$$R(x) = \{y \in B \mid (x, y) \in R\}, \ R^{-1}(x) = \{y \in B \mid (y, x) \in R\}$$

We can prove the fundamental result.

Lemma 5. $B^{*}$ is a finite Boolean algebra with a conjugate pair $\varphi, \psi : B^{*} \rightarrow B^{*}$.

Proof. It is sufficient to prove that $\varphi, \psi : B^{*} \rightarrow B^{*}$ are conjugate. That is, we have to prove that for $X, Y \subseteq W$ (i.e., $X, Y \in B^{*}$),

$$X \cap \varphi(Y) = \emptyset \iff Y \cap \psi(X) = \emptyset.$$

Suppose that $Y \cap \psi(X) \neq \emptyset$. Since $y \in \psi(X)$ for some $y \in Y$, it follows from definition of $\psi(X)$ that

$$\exists x \in X \ \text{s.t.} \ (x, y) \in R.$$
We also have \((x, y) \in R\) and \(y \in Y\). This implies that
\[
R(x) \cap Y \neq \emptyset
\]
and \(x \in \varphi(Y)\). The fact that \(x \in X\) means
\[
x \in X \cap \varphi(Y), \text{ that is, } X \cap \varphi(Y) \neq \emptyset.
\]
The converse can be proved similarly. Thus \(B^*\) is the finite Boolean algebra with the conjugate pair \(\varphi, \psi : B^* \to B^*\). \(\square\)

Moreover if we take \(\xi^* : \Phi \to B^*\) as
\[
\xi^*(A) = v(A),
\]
then we have \(\xi^*(A) \neq 1\) from \(v(A) \neq W\). This means that \(\not\vdash_{K_t^*} A\) implies \(\xi^*(A) \neq 1\) for some finite Boolean algebra with conjugate and \(\xi^* : B^* \to B^*\). It is obvious the converse statement. We thus obtain the next result.

**Theorem 5.** The logic \(K_t^*\) can be characterized by the finite Boolean algebras with conjugate.

We can show the following similarly.

**Theorem 6.** The logics \(K_t^* + (ext), K_t^* + (sym), K_t^* + (cl)\) are characterized by the class of all finite Boolean algebras with extensive, symmetric, closed conjugate pair, respectively.

Thus we can conclude that our logical systems \(K_t^*(+\text{ext}, +\text{sym}, +\text{cl})\) are decidable, that is, we can determine whether a given formula is provable or not by finite steps.

**References**


