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# To BCI from Subtractive Algebra

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After introducing the cocepts BCK, BCI(1966), many mathematicians have tried the generalizations of both systems. The final one(1994) is due to A.Ursini(Univ. of Siena). It is a **subtractive algebra**[32]. This is an algebra with a constant 0 and a binary operation * satisfying the following two axioms:

$$x * x = 0. \qquad x * 0 = x.$$

In our discussion, there is another important identity, which is very often used. Its identity is

$$0 * x = 0.$$

On the other hand, in BCI, the following facts hold.

1. $x * 0 = x, \qquad x * x = 0,$
2. $x \leq y \Rightarrow z * y \leq z * x,\ x * z \leq y * z,$
3. $(x * y) * z = (x * z) * y,$ (permutation rule)
4. $(0 * x) * (0 * y) = 0 * (x * y),$
5. The minimal element of a branch with $x$ is given by $0 * (0 * x)$.
6. If $x$ is a minimal element, then $0 * (0 * x) = x$.
7. $x * y = y * x = 0 \Rightarrow x = y$. (quasi-identity)
8. $0 \leq x$ for all $x$ in BCK, but not in BCI.

We have the atlas of proper BCI of order $n \leq 5$(J.Hao[8], B.Hu[10]). J.Meng, Y.B.Jun and E.H.Roh[21] gave all proper BCI of order 6.

From now, we omit *, so $x * y$ is denoted by $xy$

The first generalization **BCH** of both systems is due to Q.Hu. BCH is defined by the following conditions.

(1) $xx = 0$,
(2) $(xy)z = (xz)y$,
(3) $xy = yx = 0 \Rightarrow x = y$.

The class of BCH does not make a variety, bacuse it has the quasi-identity (3). The exact proof is not simple. Many useful and important properties of BCH are published in his book [11, 369-456]. Q,Hu pointed out that proper BCH of order $\leq 3$ do not exist. J.Hao[9] proved that there exist exactly five proper BCH of order 4.

**Remark** In this paper, the class of each system with the quasi-identity (3) is never a variety. Moreover, many identities appear in this paper. An ordinary identity has same letters(variables) in both sides. For example, see the identity (2). On the other hand, some variables in one side disappear in another side. For example identity (1). A.Kolmogorov pointed out the importance of the later case in his course of algorithm(Yu.M.Movsisyan [22]). For example, we have Kolmogorov identity $(xy)(yz) = xz$ in semigroup theory.

Y.Komori(1984) introduced the concept of **BCC-algebra**:

1. $0x = 0$,
2. $x0 = x$,
3. $((xy)(zy))(xz) = 0$.
4. $xy = yx = 0 \Rightarrow x = y$.

This syestem was extensively studied by W.A.Dudek([3],[4],[5] and [6]).

**Remark** $((xy)(xz))(zy) = 0$ and $x0 = x \Rightarrow (x(xy))y = 0$.

From the first identity, $((x0)(xz))(z0) = 0$, Hence $x0 = x \Rightarrow (x(xz))z = 0$. The identity obtained is one of the axioms of BCK, BCI.

Y.B.Jun, E,H,Roh and H.S.Kim (1998) introduced the concept of **BH-algebra**([17],[32]):

1 $xx = 0$,
2. $x0 = x$,
3. $xy = yx = 0 \Rightarrow x = y$.

J.Neggers and H.S.Kim(2001) gave interesting examples of a BH-algebra. Let $\mathbb{R}$ be the set of real numbers.

$$xy = \begin{cases} 0, & if\ \mathrm{x} = 0, \\ \frac{(x-y)^2}{x}, & otherwise. \end{cases}$$

It is easy check that $(\mathbb{R}, ., 0)$ is a BH-algebra satisfying $0x = 0$, but not a BCH-algebra. We know an example of BH in which $0\mathrm{x} = 0$ does not hold.

.Neggers and H.S.Kim(1999) gave an algebra which is called a **d-algebra**[25].

1. $xx = 0$,
2. $0x = 0$.
3. $xy = 0 = yx \Rightarrow x = y$.

The system includes a quasi-identity.

There is a simple example of a d-algebra. Let X be the set of non-negative real numbers. Define

$$xy = max\{0, x - y\}.$$

Then $X$ is a d-algebra with $x0 = x$.

**Remark** If a d-algebra is associative, it is trivial. $0 = 0x = (xx)x = x(xx) = x0$ Hence $x = 0$. So the d-algebra is trivial. J.Negger and H.S.Kim [25].

They(2002) introduce a new concept of **B-algebra**[30]. The axioms are

1. $xx = 0$,
2. $x0 = x$,
3. $(xy)z = x(z(0y))$.

There is an interesting example of B-algebra. Let $X$ be the set of all real numbers except for nonnegative $-n$, Define $xy$ by

$$xy = \frac{n(x-y)}{n+y}$$

Then $X$ is a B-algebra. On the other hand, define $xy = xy^{-1}$ in a given group $G$. Then $G$ is a B-algebra. M.Kondo and Y.B.Jun[21] proved that the class of B-algebra is same with the class of groups.

However, J.Neggers and H.S.Kim work is never worthless(For example, A,Walendziak [34]).

C.B.Kim and H.S.Kim defined **BM algebra** with a constant 0 amd a binary operation * satisfying

1. $00 = 0$,
2. $x0 = x$,
3. $(zx)(zy) = yx$.

**Coxeter algabra** is introduced by H,S, Kim, Y.H.Kim and J.Neggers(2004)

1. $xx = 0$,
2. $x0 = x$,
3. $(xy)z = (xz)y$.

K.Iseki, J.Neggers and H.S.Kim defined ***J*-algebra**.

1. $x * 0 = x$,
2. $x * (x * (y * (y * x))) = y * (y * (x * (x * y))$.

**Remark.** In set theory, there is an identity:

$$A - (A - (B - (B - C))) = A \cap B \cap C.$$

Axiom 2 is a special case of the above identity.

S.M.Hong, Y.B.Jun and M.Alioztürk(2003) defined an algebra which is called **gBCK-algebra**.

1. $x0 = x$,
2. $xx = 0$,
3. $(xy)z = (xz)y$,
4. $(xy)z = (xz)(yz)$,

A positive implicative BCK is gBCK-akgebra, The following relations are true:

a) $0x = 0$,

b) $(xy)x = 0$,

c) $xy = 0 \Rightarrow (xz)(yz) = 0$.

Proof,

$$0 = 00 = (xx)(xx) = (xx)x = 0x.$$

$$(xy)x = (xx)y = 0y = 0.$$

$$xy = 0 \Rightarrow (xz)(yz) = (xy)z = 0z = 0.$$

In 2005, Y.B.Jun and H.S.Kim introduced the concept of a **substraction algebra**. This is defined by the following axioms:

1. $x(yx) = x$,
2. $x(xy) = y(yx)$,
3. $(xy)z = (xz)y$.

The algebra does not include a constant 0. However, $xx$ acts as a constant 0.

I introduced two systems in our early works. One of them is **I-algebra** which is formulated as follows:

1. $x0 = x$,
2. $0x = 0$,
3. $((xy)(xz))(zy) = 0$,
4. $(xy)(x(yx)) = 0$,
5. $xy = yx = 0 \Rightarrow x = y$.

Until now, I mentioned algebras and systems with only one binary operation. On the hand, various algebras and systens with several binary operations are in the literature.

I introduced a system in our researches(1967). It is **Griss-algebra**. The system has two operations * and +. A constant 0 is included in Griss system. $x \leq y$ means $xy = 0$. Griss system is formulated as follows:

1. $0x = 0$,
2. $x + x = x$,
3. $x; = y + x$,
4, $(x + y)(x + z) \leq zy$,
5. $x \leq x + y$,
6. $xy = yx = 0 \Rightarrow x = y$.

J.MNeggers and H.S.Kim(2002) defined an algerbra with a constant 0 and two binary operations satisfying the following three axioms.

1. $x0 = x$,
2. $(0x) + x = 0$,
3. $(xy)z = x(y + z)$.

The algebra is called $\beta$**-algebra**[32].

There are other important generalizations of BCK and BCI due to Bucarest research group(under G.Georgescu). They defined the concepts of pseudo-MV, pseudo-BL, pseudo-BCK, Iseki algebra, and pseudo-Iseki algebras and a pseudo t-norm. A.Iorgulescu(2003) defined many systems([12],[13]), for example, left-Iseki(RP) algebra,..., left Hájek(R)-algebra. Among them, I mentione only definiton of pseudo-BCK is given. A **pseudo-BCK** is a sytem $\mathbb{X} = (X, \leq, *, \diamond, 0)$, where $\leq$ is a binary relation, $*, \diamond$ are binary operations, and 0 is a constant,and is given by the following axiom sysytem:

1. $(x*y)\diamond(x*z) \leq z*y$, $(x\diamond y)*(x\diamond z) \leq z\diamond y$,
2. $x*(x\diamond y) \leq y$, $x\diamond(x*y) \leq y$,
3. $x \leq x$,
4. $0 \leq x$.
5. $x \leq y$ $y \leq x \Rightarrow x = y$,
6. $x \leq y \Leftrightarrow x*y = 0 \Leftrightarrow x\diamond y = 0$.

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