Degree of Nondeterminism
for Pushdown Automata

IBARAKI Tatsuya* and KUBOSAWA Shumpei†

1 Introduction

While time complexity and space complexity are commonly studied in computer science, complexity with respect to nondeterminism for pushdown automata (PDA) is not known widely. We thought that we can measure nondeterminism of a PDA by a function which maps an input length $n$ of the PDA to the maximum number of nondeterministic transitions required to accept the input.

In this paper, we first define the measure of nondeterminism for pushdown automata, and then consider the measure of nondeterminism for certain context-free languages (CFL).

While we were preparing materials of this presentation, we found similar definitions and applications of this kind of complexity in the literature. They are Salomaa et al. [1] and Goldstine et al. [2]. But, since the languages they consider are different from ours, we think it worth to present our original ones in this paper.

2 Preliminaries

The definition of PDA which we use here is similar to the definition given by M. Sipser [3]. In this paper we consider only non-deterministic PDA’s.

Definition 2.1
For any alphabet $A$ we denote $A \cup \{ \varepsilon \}$ by $A_\varepsilon$. We define non-deterministic PDA as a 6-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, F)$$

where $Q$ is a finite set of states, $\Sigma$ is an input alphabet, $\Gamma$ is a stack alphabet, $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow \wp(Q \times \Gamma_\varepsilon)$ is a transition function, $q_0 \in Q$ is an initial state and $F \subseteq Q$ is a set of final states.

For these pushdown automata, we define acceptance by final state acceptance rather than by empty-stack acceptance.

*Faculty of Commerce and Management, Hitotsubashi University
†Faculty of Social Sciences, Hitotsubashi University
Definition 2.2
Let $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a PDA. We define a configuration of $M$ by a tuple $c = (r, w, s)$ where $r \in Q$, $w \in \Sigma^*$ and $s \in \Gamma^*$. A configuration $c = (r, w, s)$ is an accepting configuration when $r \in F$ and $w = \epsilon$. For any two configurations $c_1 = (r_1, w_1, s_1)$ and $c_2 = (r_2, w_2, s_2)$, we say that there is a transition from $c_1$ to $c_2$, $c_1 \vdash c_2$, if $(r_2, b) \in \delta(r_1, x, a)$ where $w_1 = xw_2$ for $x \in \Sigma_\epsilon$ and $s_1 = at, s_2 = bt$ for $a, b \in \Gamma_\epsilon$ and $t \in \Gamma^*$. A sequence of configuration $C = (c_0, c_1, \ldots, c_m)$ for $m \in \mathbb{N}$ is a computation of $M$ when $c_i \vdash c_{i+1}$ for all $0 \leq i \leq m - 1$. A computation $C = (c_0, c_1, \ldots, c_m)$ of $M$ is an accepting computation for $w \in \Sigma^*$ if $c_0 = (q_0, w, \epsilon)$ and $c_m$ is an accepting configuration. When there is at least one accepting computation for $w$, $M$ accepts $w$. The language accepted by $M$ is $L(M) = \{w \in \Sigma^* | w$ is accepted by $M\}$.

3 Degree of nondeterminism

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a PDA. For each configuration $c$ on $M$, we define $\nu_M(c)$ by

\[
\nu_M(c) = \begin{cases} 
0 & \text{if } \#\{c'|c \vdash c'\} < 2 \\
1 & \text{if } \#\{c'|c \vdash c'\} \geq 2
\end{cases}.
\]

For each computation $C = (c_0, c_1, \ldots, c_m)$ of $M$, we define $\nu_M(C)$ by

\[
\nu_M(C) = \sum_{i=0}^{m-1} \nu_M(c_i).
\]

For all $w \in L(M)$, we define $\nu_M(w)$ by

\[
\nu_M(w) = \min\{\nu_M(C)|C$ is an accepting computation of $M for $w\}.
\]

For all $n \in \mathbb{N}$, we define $\nu_M(n)$ by

\[
\nu_M(n) = \begin{cases} 
\max\{\nu_M(w)|w \in L(M)^{(n)}\} & \text{if } L(M)^{(n)} \neq \phi \\
0 & \text{if } L(M)^{(n)} = \phi
\end{cases}.
\]

For a function $f : \mathbb{N} \to \mathbb{N}$, we say that $M$ runs at nondeterministic degree of $f(n)$, if $\nu_M(n) \leq f(n)$ for all $n \in \mathbb{N}$. Furthermore, for a function $t : \mathbb{N} \to \mathbb{N}$, we define the class $ND(t(n))$ of languages by

$ND(t(n)) = \{L|L = L(M) and M runs at nondeterministic degree of \mathcal{O}(t(n)) for some PDA M\}$.

4 A language in $ND(\log n)$

Definition 4.1 (language)
Let a sequence $\{c_t\}_{t=0}^\infty$ of strings on $\Sigma = \{0, \#\}$ be defined as follows:
Let \( L_1 \) and \( L_2 \) be defined by
\[
L_1 = \{c_t|t \geq 1\}
\]
and
\[
L_2 = \overline{L_1} (= \Sigma^* \setminus L_1).
\]

Here are some examples of strings contained in \( L_1 \).
0#00
0#00#0000
0#00#0000#00000000

**Theorem 4.1**
\( L_2 \) has nondeterministic degree of \( O(\log n) \), i.e.,
\[
L_2 \in ND(\log n).
\]

The idea of following proof is that if \( M \), a recognizer of \( L_2 \), read a string \( w \) with nondeterministic transitions only when \( M \) read a #, the number of nondeterministic transitions never exceed \( \log |w| \). This is because if a string \( w \) which contains more #’s than the string which is in \( L_1 \) with the same length, \( M \) accepts \( w \) before arriving at the terminal.

**Proof of Theorem 4.1**
We prove this theorem by showing the existence of PDA \( M \) that accepts \( L_2 \) and runs by nondeterministic degree of \( O(\log n) \). Here we define that the stack alphabet of \( M \) is \( \{X, $\} \). And we denote a sequence of 0’s by a “block” and blank symbol on the input tape as \( B \).

We construct \( M \) as follows.
\( M=\) “On input string \( w \):

\begin{align*}
\text{Step 1} & \quad \text{Push $ onto the stack.} \\
\text{Step 2} & \quad \text{If the input head reads 0, push X onto the stack.} \\
& \quad \text{If the input head reads #, ACCEPT.} \\
\text{Step 3} & \quad \langle \langle \text{Nondeterministic step } \rangle \rangle \\
& \quad \text{If the input head reads #, nondeterminisitically either go to Step 4 or Step 5.} \\
& \quad \text{If the input head reads 0, push XX onto the stack and go to Step 3.} \\
\text{Step 4} & \quad \text{Clear the stack and go to step 3.} \\
\text{Step 5} & \quad \text{(Check the number of 0’s - main process)} \\
& \quad \text{If the input head reads 0 and the stack head reads X, pop X up from the stack and go to Step 5.} \\
& \quad \text{If the input head reads 0 and the stack head reads $ up from the stack, go to step 6. If the input head reads B, ACCEPT.}
\end{align*}
Step 6  If the input head reads 0, ACCEPT.

First, it is clear that $M$ accepts $L$.

Next, we show that $M$ runs at nondeterministic degree of $O(\log n)$.

By the definition of $M$, the number of $M$'s guessing steps is less than or equal to the number of #’s in Step 3.

Therefore, it suffices to show that number of #’s in $c_t$ is less than a constant multiple of $\log |c_t|$.

$$|c_t| = \sum_{k=0}^{t} (2^k + 1) - 1 = 2^{t+1} + t - 1$$

$$\log(|c_t| - t + 1) - 1 = t$$

Therefore we have

$$t \leq \log |c_t|.$$  

5  A language in $ND(\sqrt{n})$

In order to describe the next example, we use directed graphs expressed by adjacency matrices. A source is a node with at least one outgoing arc and no incoming arc. A sink is a node with no outgoing arcs and at least one incoming arc.

Now, we introduce another type of node which we call a quasi-sink.

![Figure 1: Quasi-sink and sink](image)

Definition 5.1

A quasi-sink is a node of a directed graph with exactly one outgoing arc which goes to some sink and no other outgoing arcs. (It may have any number, including 0, of incoming arcs.)

In this paper, an adjacency matrix is converted into a string in the following way.

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0
\end{pmatrix} \rightarrow \#10101\#00000\#00111\#01000\#10110$$
Each row is serially aligned from left to right and separated by #’s.

**Definition 5.2**
For an alphabet $\Sigma = \{0, 1, \#\}$ and $l \geq 0$ let $\Sigma^l$ denote the set of strings over $\Sigma$ whose length is $l$. Let $L_{AM} = \{w | 2l \in \mathbb{N}, w \in (\{0, 1\}^l)^l\}$ which consists of adjacency matrices expressed by strings as described above. Now, let

$$L_1 = \Sigma^* \setminus L_{AM}$$

and $L_2$ be the language consisting of strings which correspond to adjacency matrices of directed graphs with at least one quasi-sink. Finally we define that

$$L_{qs} = L_1 \cup L_2.$$  

**Theorem 5.1**
$L_{qs}$ has nondeterministic degree of $O(\sqrt{n})$, i.e.,

$$L_{qs} \in ND(\sqrt{n}).$$

Before proving the theorem, we explain how our recognizer of $L_{qs}$ works. $L_{qs}$ is the union of two languages $L_1$ and $L_2$ as above.

$L_1$ can further be classified in two cases.

One is the case where there exist at least one pair of adjacent rows with different length. In this case, the recognizer only guesses at each #’s till it reads rows whose length is equal to the next row. Here is an example of those strings.

```plaintext
#01000
#00101
#10000
#00110
#00000
```

In the above case, the recognizer nondeterministically transit only four times. At the first three #’s, the machine guesses whether compare the following row and the next row, or not. But at the fourth #, the recognizer has a chance to accept the input because the length of the fourth and the fifth row are different. So $\nu_M(n) \leq \sqrt{n}$ holds. In this case, $n = 29$ and $4 < \sqrt{29}$ where $M$ is the recognizer.

The other case is the case where a string forms a matrix, but it is not a square matrix. The followings are examples of those strings.

```plaintext
#01010
#00101
#00000
#00100
```

```plaintext
#010000101010
#0011111101111
#010111010101
```
To accept these strings, the recognizer is required to guess only at the first #. In other words, the machine needs to compare the length of the first row and the number of columns.

$L_2$ can be accepted with less than or equal to $\sqrt{n}$ nondeterministic transitions if the stack is efficiently used. The recognizer needs to transit its state nondeterministically at each # before arriving at the quasi-sink or sink row. So the number of guessing does not exceed the square root of the input length.

$$L_{2s}$$ can be accepted with less than or equal to $\sqrt{n}$ nondeterministic transitions if the stack is efficiently used. The recognizer needs to transit its state nondeterministically at each # before arriving at the quasi-sink or sink row. So the number of guessing does not exceed the square root of the input length.

$$\mathcal{O}(\sqrt{n})$$

Proof of Theorem 5.1
We prove it by construction of the PDA $M$ which runs by nondeterministic degree of $\mathcal{O}(\sqrt{n})$ and recognizes $L_{qs}$ as follows. Here the stack alphabet of $M$ is $\{\$, X\}.

$M$ = "On input string $w$:

Step 1  If the input head reads #, push $\$ onto the stack.
Otherwise, ACCEPT.

Step 2  \{\text{Nondeterministic step}\}
\text{Nondeterministically} go to Step 3, go to Step 4, go to Step 5 or go to Step 6.

Step 3  \{\text{Nondeterministic step}\}
\text{Nondeterministically} process either following (a) or (b).
(a) Check the length of current row and that of the next row.
    If these rows are different, ACCEPT.
(b) Do nothing while the head read current row.
    And go to Step 3.

**Step 4** (Check if the number of rows and columns are different.)
Pop two letters up from the stack.
While the input head reads 0 or 1, push X onto the stack.
If the input head read # and the stack head reads X, pop X
up from the stack and do nothing while reading following 0 or 1.
If the input head read # and the stack head reads $, ACCEPT.
If the input head read B and the stack head reads $, REJECT.
Otherwise, ACCEPT.

**Step 5** (Check if a quasi-sink exist prior to the sink.)
In this case, M can decide in the way we described above with
Figure 2.

**Step 6** \( \langle \langle \text{Nondeterministic step} \rangle \rangle \)

**Nondeterministically** process either following (a) or (b).
(a) If the input head reads 0, repeat this step.
    If the input head reads 1, REJECT.
    If the input head reads # and the stack head reads X, pop
    X from the stack and go to Step 7.
(b) If the input head reads 0 or 1, repeat this step.
    If the input head reads #, push X onto the stack and go
to Step 6.

**Step 7** \( \langle \langle \text{Nondeterministic step} \rangle \rangle \)

**Nondeterministically** process either following (a-1) or (b).
(a-1) If the input head reads 0 and the stack head reads X,
pop X from the stack and repeat this step.
    If the input head reads 1 and the stack head reads $, go
to (a-2).
(a-2) If the input head reads 0, continue (a-2).
(a-3) If the input head reads # or B, ACCEPT.
(b) If the input head reads 0 or 1, repeat this step.
    If the input head reads #, go to Step 7.

On this PDA, the number of nondeterministic transitions required to \( L_1 \)
and a half part of \( L_2 \) is stated above. So here we describe why the other
part of \( L_2 \) can be recognized by nondeterministic degree of \( \mathcal{O}(\sqrt{n}) \). In step 6,
nondeterministic transitions only occur at each #’s. And it is clear that the input
forms adjacency matrix in this case. Otherwise such input string is accepted
by step 3 or step 4. The worst case of this part is that a quasi-sink is located
at the last row. In this case, nondeterministic transitions occur \( r \) times if the
input matrix consists of \( r \) rows. Now this string contains \( r \) #’s and \( r^2 \) numbers,
so the length is \( r + r^2 \). Hence the number of nondeterministic transitions \( r \) is
always less than \( \sqrt{|w|} = \sqrt{r + r^2} \).
6 Conclusion

We have defined the degree of nondeterminism for pushdown automata, and obtained an upper bound of the degree of nondeterminism for two languages. The first language can be recognized by a pushdown automaton which runs by nondeterministic degree of $\mathcal{O}(\log(n))$, so it is in $ND(\log(n))$. And the second one is in $ND(\sqrt{n})$.

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References

