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Homomorphism on Triple-semilattice

Kiyomitsu Horiuchi
(堀内 清光)

Faculty of Science and Engineering, Konan University
(甲南大学理工学部)
Okamoto, Higashinada, Kobe 658-8501, Japan

Homomorphism is a fundamental concept in the algebra system with the operation. In case of the algebra system with two or more operations, the concept of individual homomorphism can be considered respectively for each operation. For instance, there are two operations, meet $\wedge$ and join $\vee$, on the lattice. Hence, there are two homomorphism, $\wedge$-homomorphism, and $\vee$-homomorphism. These two homomorphism is independent for a general lattice. That is, there are mappings that is not $\wedge$-homomorphism but $\vee$-homomorphism, and vice versa. In case of the totally ordered set, $\wedge$-homomorphism and $\vee$-homomorphism are corresponding. In another algebra system with two operations or more, there are some cases where the second operation is sure to become homomorphism if the first operation is homomorphism. Moreover, another operation might become homomorphism from the homomorphism of two operations or more.

We would like to introduce the order on sets of the operations of the algebra system by using this. In this paper, it study the order on the mathematics object known well "set operations" and "addition and multiplication on the set real number or rational number field". And it reports on the condition that the homomorphism of one operation of the remainder is approved by homomorphism in two operations on triple-semilattice with three operations.

1 Homomorphic order

We adopt the notation in [1] for universal algebra. Suppose A and B are two algebras of the same type F. A mapping $\alpha : A \rightarrow B$ is called a homomorphism from A to B if

$$\alpha f^{A}(a_{1}..., a_{n}) = f^{B}(\alpha a_{1}..., \alpha a_{n}) \tag{1}$$

for each n-ary operation f in F and each sequence $a_{1}..., a_{n}$ from A.

For individual operation f in F, a mapping $\alpha : A \rightarrow B$ is called a f-homomorphism from A to B if

$$\alpha f^{A}(a_{1}..., a_{n}) = f^{B}(\alpha a_{1}..., \alpha a_{n}). \tag{2}$$

Definition 1 Let f_1 and f_2 be operations in F. We say that f_1 is homomorphic stronger than f_2, and we write $f_1 \gg f_2$, if every f_1-homomorphism mapping is f_2-homomorphism, i.e., for every two algebras A and B of type F and for every $\alpha : A \to B$ if α is f_1-homomorphism then α is f_2-homomorphism. And we write $f_1 \equiv f_2$, if $f_1 \gg f_2$ and $f_2 \gg f_1$. This order is called **homomorphic order**.

In the definition, the set of mappings that operation f_2 becomes homomorphism contain the set of mappings that operation f_1 becomes homomorphism. The weakest operations are homomorphism on all mappings. Actually, projection ($f(a_1, a_2) = a_1$) is one of the weakest operations.

Example 1 *On the algebra of group, there are oprations* $\cdot$ *and* $^{-1}$. *Then,* $\cdot \gg {}^{-1}$. *That is, for group* $< A, \cdot, ^{-1}, 1 >$ *and* $< B, \cdot, ^{-1}, 1 >$, *if* $\alpha : A \to B$ *is* $\cdot$*-homomorphism then* α *is* $^{-1}$*-homomorphism.*

Example 2 *On the algebra of totally ordered set, we consider operations* $\wedge$ *and* $\vee$ *of lattice. Then,* $\wedge \equiv \vee$. *That is, for totally ordered set* $< A, \wedge, \vee >$ *and* $< B, \wedge, \vee >$, *if* $\alpha : A \to B$ *is* $\wedge$*-homomorphism then* α *is* $\vee$*-homomorphism and if* α *is* $\vee$*-homomorphism then* α *is* $\wedge$*-homomorphism.*

In case of a general lattice, it is natural that $\wedge$-homomorphism and $\vee$-homomorphism are independence. This order is more interesting when thinking adding the operation made by the combination.

Example 3 *On the Boolean algebra, there are operations* $\wedge$, $\vee$ *and* $'$. *In addition to, we consider Sheffer operation* $|$, $a|b = (a \wedge b)'$. *Then,* $\wedge$, $\vee$ *and* $'$ *can be represented by* $|$. *That is,* $a \wedge b = (a|b)|(a|b)$, $a \vee b = (a|a)|(b|b)$ *and* $a' = a|a$. *Hence,* $| \gg \vee$, $| \gg \wedge$ *and* $| \gg {}'$.

Definition 2 Let f_0, f_1, ..., f_n be operations in F. We write $\{f_1, f_2, ..., f_n\} \gg f_0$ if for every two algebras A and B of type F and for every $\alpha : A \to B$ if α is f_k-homomorphism for $k \in \{1, ...n\}$ then α is f_0-homomorphism.

In Example 3, it is clear that $\{\vee, {}'\} \gg |$.

2 Observation of set operations

In this section, we study the order on set operation. This is a typical model of Boolean algebra. Let X and Y be sets. The collection of all subsets of a set X denoted by $\mathcal{P}(X)$. The set operation usually used is enumerated as follows.

nullary operation	ϕ (emptyset)	X (whole set)	
unary operation	$^{-}$ (complement)		
binary operation	$\cup$ (union)	$\cap$ (intersection)	$\setminus$ (difference)
	$\to$ (implication)	$\vert$ (Sheffer operation)	

$\bar{A} = \{x \notin A\}$, $A \cup B = \{x \in A \text{ or } x \in B\}$, $A \cap B = \{x \in A \text{ and } x \in B\}$,
$A \setminus B = \{x \in A \text{ and } x \notin B\}$, $A \to B = \{x \notin A \text{ or } x \in B\}$,
$A \mid B = \{x \notin A \text{ or } x \notin B\}$.

It is possible to compose the other binary operations. However, it is enough in what enumerated in the above. (Is it possible to create essential and substantial ternary operation on $\mathcal{P}(X)$?)
It thinks about the condition whose mapping $\alpha : \mathcal{P}(X) \to \mathcal{P}(Y)$ is homomorphism for these operations. And, the homomorphic order by the combination of these operations is shown in Figure 1.

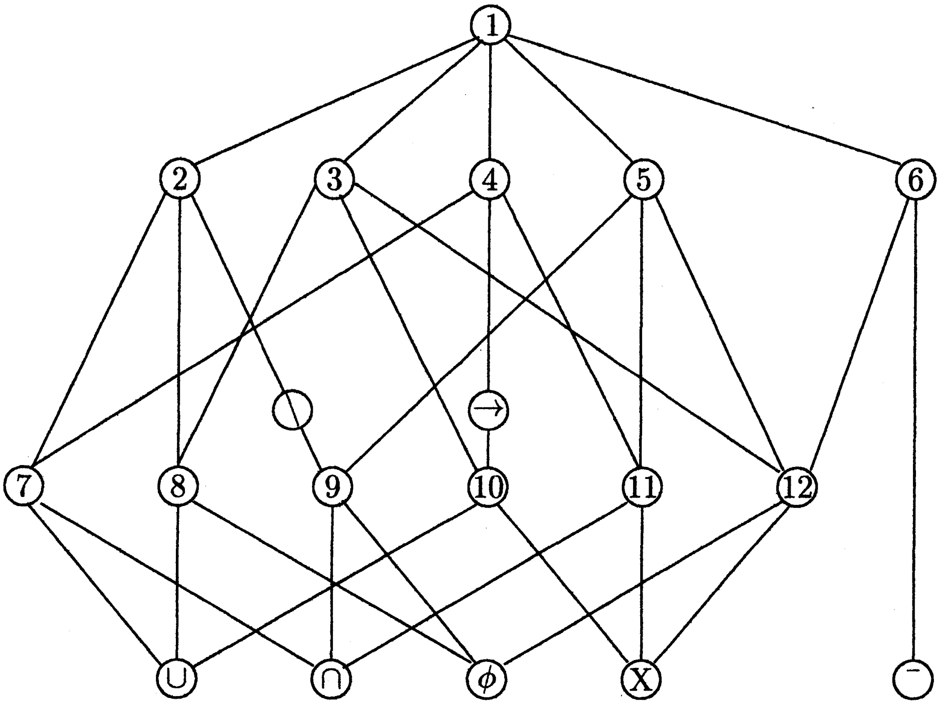

Figure 1: The homomorphic order $\gg$

$1 = \{\cup, \cap, \phi, X, \setminus, \to, \ ^{-}\} \equiv \{\cup, \cap, \phi, X\} \equiv \{\cap, \ ^{-}\} \equiv \{X, \setminus\} \equiv \mid$ etc.
$2 = \{\cup, \cap, \phi\} \equiv \{\cup, \setminus\}$
$3 = \{\cup, \phi, X\}$
$4 = \{\cup, \cap, X\} \equiv \{\cap, \to\}$
$5 = \{\cap, \phi, X\}$
$6 = \{\phi, X, \ ^{-}\} \equiv \{\phi, \ ^{-}\} \equiv \{X, \ ^{-}\}$
$7 = \{\cup, \cap\}$ $8 = \{\cup, \phi\}$ $9 = \{\cap, \phi\}$
$10 = \{\cup, X\}$ $11 = \{\cap, X\}$ $12 = \{\phi, X\}$
In this figure, we will pay attention to $\{\cup, \cap, \phi\} \gg \setminus$. It is because that $\setminus$ can not represented by $\cup$, $\cap$ and ϕ. To similar, $^{-}$ can not represented by $\cup$, $\cap$, ϕ and X. It

is understood that representable and the homomorphic order are different concepts even if in Boolean algebra. In Example 1 of the previous section, the $^{-1}$ can not represented by $\cdot$.

3 Observation of operations on R

Let $\mathbf{R}$ be the set of real numbers and $\mathbf{Q}$ be the set of rational numbers. In this section we discuss the condition that a mapping α from $\mathbf{R}$ to $\mathbf{R}$ becomes homomorphism for individual operation f in $\mathbf{R}$.
First, $\mathbf{R}$ is a totally ordered set, we have the operations $\vee$ and $\wedge$. If $\alpha : \mathbf{R} \to \mathbf{R}$ is $\vee$-homomorphism (that is, $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$), then α is a monotone increase ($x \leq y$ imply $\alpha(x) \leq \alpha(y)$).
The constant function on $\mathbf{R}$ will be written as $c_r(x) = r(r \in R)$. Although these $c_r(x)$ are unary operations in $\mathbf{R}$ seemingly, it is actually these are nullary operations. If α is c_r-homomorphism (that is, $\alpha(c_r(x)) = c_r(\alpha(x))$), then $\alpha(r) = r$.
We define f_r ($r \in \mathbf{R}$) by $f_r(x) = rx$. When $r = 0$, $f_0 = c_0$. For any mapping α, f_1 is homomorphism. If $\alpha : \mathbf{R} \to \mathbf{R}$ is f_r-homomorphism (that is, $\alpha(f_r(x)) = f_r(\alpha(x))$), then $\alpha(r^m k) = r^m \alpha(k)$ for ever $m \in \mathbf{Z}$ (the set of all integers) for every $k \in \mathbf{R}$. The following consists of this.

Proposition 1 Let $s, r \in \mathbf{R}$ such that $s = r^n (n \in \mathbf{Z})$. If $\alpha : \mathbf{R} \to \mathbf{R}$ is f_r-homomorphism then α isf_s-homomorphism. That is, $f_r \gg f_s$.

For example, $f_{\sqrt{2}} \gg f_2 \gg f_4 \gg f_8 \gg \ldots \gg f_0 \gg f_1$, $f_{-2} \gg f_4$, $f_2 \equiv f_{2^{-1}}$. However $f_4 \not\gg f_2$. In addition, if $\alpha : \mathbf{R} \to \mathbf{R}$ satisfies both f_{-2}-homomorphism and f_3-homomorphism, then $\alpha((-2)^m * 3^n\ k) = (-2)^m * 3^n \alpha(k)$ for ever $m, n \in \mathbf{Z}$ for every $k \in \mathbf{R}$. Hence, $\{f_{-2}, f_3\} \gg f_{(-2)^m * 3^n} (m, n \in \mathbf{Z})$. It is noted that the $\{(-2)^m * 3^n | m, n \in \mathbf{Z}\}$ is dense in $\mathbf{R}$.

Next, we consider binary operation $+$ in $\mathbf{R}$. The equation $\alpha(x+y) = \alpha(x) + \alpha(y)$ is certainly Cauchy equation! This imply $\alpha(qx) = q\alpha(x)$ ($q \in \mathbf{Q}$), hence $+ \gg \{f_q | q \in \mathbf{Q}\}$. But $\{f_r | r \in \mathbf{Q}\} \not\gg +$. It is easy that $\{f_r | r \in \mathbf{R}\} \gg +$. However, from the existence of Hamel base that relates to the axiom of choice, $+ \not\gg \{f_r | r \in \mathbf{R}\}$.

Proposition 2 For a mapping $\alpha : \mathbf{R} \to \mathbf{R}$,

$$\{f_r | r \in \mathbf{R}\} \gg + \gg \{f_q | q \in \mathbf{Q}\} \tag{3}$$

If it thinks $+$ to be an operation in $\mathbf{Q}$ and α to be a mapping from $\mathbf{Q}$ to $\mathbf{Q}$, then $+ \equiv \{f_q | q \in \mathbf{Q}\}$.
The situation changes completely if the condition of $\vee$-homomorpism is added to $\alpha : \mathbf{R} \to \mathbf{R}$.

Proposition 3 For a mapping $\alpha : \mathbf{R} \to \mathbf{R}$,

$$\{f_{-2}, f_3, \vee\} \equiv + \tag{4}$$

Moreover, we consider binary operation $*$. If it thinks $*$ to be an operation in $\mathbf{Q}$ and α to be a mapping from $\mathbf{Q}$ to $\mathbf{Q}$, c_1-homomorpism and $+$-homomorpism imply $c_q(x) = q$ for all $q \in \mathbf{Q}$. That is, $\{c_1, +\} \gg *$. If $*$ is an operation in $\mathbf{R}$ and α is a mapping $\mathbf{R} \to \mathbf{R}$, then $\{c_1, +\} \not\gg *$. However, the following are derived from the proposition above.

Proposition 4 For a mapping $\alpha : \mathbf{R} \to \mathbf{R}$,

$$\{c_1, +, \vee\} \equiv \{c_1, f_{-2}, f_3, \vee\} \gg * \tag{5}$$

Let $n \in \mathbf{N}$ and $n \neq 1$. We define t_n n-ary operation on $\mathbf{R}$ by $t_n(x_1, x_2, ..., x_n) = x_1 + x_2 + ... + x_n$. Then,

Proposition 5 For a mapping $\alpha : \mathbf{R} \to \mathbf{R}$, $n \in \mathbf{N}$ and $n \neq 1$,

$$+ \equiv t_n \tag{6}$$

It thinks about the operation of another type (average). Let $n \in \mathbf{N}$ and $n \neq 1$. We define h_n n-ary operation on $\mathbf{R}$ by $h_n(x_1, x_2, ..., x_n) = (x_1+x_2+...+x_n)/n$. Let $\alpha : \mathbf{R} \to \mathbf{R}$ be h_n-homomorphism, i.e., $\alpha(h_n(x_1, x_2, ..., x_n)) = h_n(\alpha(x_1), \alpha(x_2), ..., \alpha(x_n))$. We define $\beta(x) = \alpha(x) - \alpha(0)$, then $\beta(x_1+x_2+...+x_n) = \beta(x_1)+\beta(x_2)+...+\beta(x_n)$. As α is t_n-homomorphism for $\alpha(0) = 0$, $+ \equiv t_n \gg h_n$. This fact is clear also in the another way. From $h_n(x_1, x_2, ..., x_n) = f_{1/n}(t_n(x_1, x_2, ..., x_n))$, $\{f_{1/n}, t_n\} \gg h_n$. Since the Proposition 1 and 5, $t_n \equiv + \gg f_n \equiv f_{1/n}$. Hence $t_n \equiv \{f_{1/n}, t_n\} \gg h_n$. But, when $\alpha(x) = 1$, α is not $+$-homomorpism but h_n-homomorpism.

Proposition 6 For a mapping $\alpha : \mathbf{R} \to \mathbf{R}$, $n \in \mathbf{N}$ and $n \neq 1$,

$$+ \gg h_n \tag{7}$$

We remark that the construction of the h_n operation cannot be done from $+$ operation alone.
The following proposition can be easily shown.

Proposition 7 For a mapping $\alpha : \mathbf{R} \to \mathbf{R}$, $n, n' \in \mathbf{N}$ and $n \neq 1 \neq n'$,

$$h_n \equiv h_{n'} \tag{8}$$

It is clear that $\{f_n, h_n\} \gg t_n$. Hence,

Proposition 8 For a mapping $\alpha : \mathbf{R} \to \mathbf{R}$, if there exists $n_1 \in \mathbf{N}$ ($n_1 \neq 1$) such that α is f_{n_1}-homomorpism and there exists $n_2 \in \mathbf{N}$ ($n_2 \neq 1$) such that α is h_{n_2}-homomorpism, then α is $+$-homomorpism.

4 Observation of triple-semilattice

We proposed the algebra system with three operations in [4], [5]. It is a kind of dimensional expansion of the lattice theory. The definition is written again here.

A semilattice $(S, *)$ is a set S with a single binary, idempotent, commutative and associative operation $*$.

$$a * a = a \qquad (idempotent) \tag{9}$$
$$a * b = b * a \qquad (commutative) \tag{10}$$
$$a * (b * c) = (a * b) * c \qquad (associative) \tag{11}$$

Definition 3 Let T be a set. Let $*_1$, $*_2$ and $*_3$ be three binary operations on T. If $(T, *_1)$, $(T, *_2)$ and $(T, *_3)$ are semilattices respectively, the quartet $(T, *_1, *_2, *_3)$ is called a **triple-semilattice**.

The role that looks like the lattice in putting the following relational expression is done though three operations are various the way things are going.

Definition 4 Let T be a triple-semilattice. We call next six equality **roundabout-absorption laws**.

$$((a *_1 b) *_2 b) *_3 b = b \tag{12}$$
$$((a *_1 b) *_3 b) *_2 b = b \tag{13}$$
$$((a *_2 b) *_1 b) *_3 b = b \tag{14}$$
$$((a *_2 b) *_3 b) *_1 b = b \tag{15}$$
$$((a *_3 b) *_1 b) *_2 b = b \tag{16}$$
$$((a *_3 b) *_2 b) *_1 b = b \tag{17}$$

for every $a, b \in T$. We say that T have the roundabout-absorption laws if and only if these six identity holds in it. A triple-semilattice is called a **trice** if it satisfies the roundabout-absorption laws.

Let $(T, *_1, *_2, *_3)$ be a triple-semilattice and $a, b, c \in T$ $(a \neq b \neq c \neq a)$.

Definition 5 We say that an ordered triplex (a, b, c) is in a **triangular situation** if (a, b, c) have the following properties:

$$b *_1 c = a \tag{18}$$
$$c *_2 a = b \tag{19}$$
$$a *_3 b = c \tag{20}$$

Strictly speaking, a correspond to $*_1$, b correspond to $*_2$ and c correspond to $*_3$. Therefore, if necessary, we must say that $(a,\ b,\ c)$ is in a $*_1 *_2 *_3$-triangular situation.

Definition 6 Let T be a triple-semilattice. We say that T has the **triangle constructive laws** if T has the following properties:

$$(d *_1 e) *_3 (d *_2 e) = d *_3 e \tag{21}$$
$$(d *_1 e) *_2 (d *_3 e) = d *_2 e \tag{22}$$
$$(d *_3 e) *_1 (d *_2 e) = d *_1 e \tag{23}$$

for all $d, e \in T$.

Suppose that a triple-semilattice T satisfies the identities ((21) $\sim$ (23)) for $d, e \in T$. Let $a = d *_1 e,\ \ b = d *_2 e,\ \ c = d *_3 e$. If $a \neq b \neq c \neq a$, then $(a,\ b,\ c)$ is in a triangular situation. That is why we named the identities "the triangle constructive laws".

Definition 7 Let T be a triple-semilattice. We say that T has the **triangle natural laws** if T has the following properties:

$$\text{if } x *_1 y = z \text{ and } x *_2 z = y, \text{ then } y *_3 z = x \tag{24}$$
$$\text{if } x *_3 y = z \text{ and } x *_2 z = y, \text{ then } y *_1 z = x \tag{25}$$
$$\text{if } x *_1 y = z \text{ and } x *_3 z = y, \text{ then } y *_2 z = x \tag{26}$$

for all $x, y, z \in T$.

Theorem 1 Suppose $(T,\ *_1, *_2, *_3)$ and $(T',\ *_1, *_2, *_3)$ are triple-semilattices with triangle constructive and triangle natural laws. For a mapping $\alpha : T \to T'$,

$$\{*_2,\ *_3\} \gg *_1 \tag{27}$$
$$\{*_3,\ *_1\} \gg *_2 \tag{28}$$
$$\{*_1,\ *_2\} \gg *_3 \tag{29}$$

That is, if α is $*_1$-homomorphism and $*_2$-homomorphism, then α is $*_3$-homomorphism.

Let T be a trice (triple-semilattice with roundabout-absorption laws). If T has the triangle constructive laws, then T has the triangle natural laws. Hence, in a certain sense, trice with triangle constructive laws corresponds to chain (totally ordered set) of lattice theory.

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References

[1] S. Burris, H.P.Sankappanavar. *A Course in Universal Algebra* GTM 78 Springer-Verlag, 1981.

[2] G. Birkhoff. *Lattice Theory (third ed.)* Amer. Math. Soc. Colloq. Publ., 1967.

[3] G. Grätzer. *General Lattice Theory (Second ed.)* Birkhäuser, 1998.

[4] K. Horiuchi. Trice and Two delegates operation. Scientiae Mathematicae, 2-3 (1999)373–384.

[5] K. Horiuchi, A. Tepavčević. On distributive trices. Discussiones Mathematicae General Algebra and Applications, 21(2001)21–29.

[6] K. Horiuchi. Some weak laws on bisemilattice and triple-semilattice. Scientiae Mathematicae, 59, No.1 (2004): 41–61.