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Basic algebras*

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Abstract

Basic algebras (roughly speaking, bounded lattices having antitone involutions on principal filters) can be regarded as a wide generalization of orthomodular lattices and MV-algebras, also including lattice effect algebras. The paper surveys the recent results in this field.

Introduction

It is well-known that the algebraic axiomatization of the classical propositional calculus is given by Boolean algebras, i.e. bounded distributive lattices with (unique) complementation, but not all reasonings may be described by means of the classical two-valued logic. Let us mention two 'basic' examples:

In the logic of quantum mechanics the law of excluded middle fails owing to the nature of quantum physics. In the 1940's, G. Birkhoff and J. von Neumann found out that the appropriate tool for axiomatizing this logic are orthomodular lattices. For completeness we recall that an *orthomodular lattice* is a bounded complemented lattice $(A, \vee, \wedge, ^{\perp}, 0, 1)$ satisfying the orthomodular law

$$x \leq y \quad \Rightarrow \quad x \vee (x^{\perp} \wedge y) = y. \tag{1}$$

A typical example of an orthomodular lattice is in Fig. 1. (Notice that $x^{\perp}$ need not be the only complement of x, thus orthomodular lattices are non-distributive in general because of the lack of the principle of excluded middle.)

Another example of a non-classical calculus is Łukasiewicz's many-valued propositional logic originally introduced by J. Łukasiewicz in the 1920's, the algebraic

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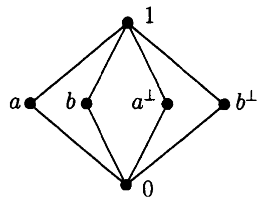

Figure 1:

semantics of which is given by MV-algebras that have been defined by C. C. Chang in the 1950's. The definition we present here is adopted from the monograph [10]: An *MV-algebra* is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ where $(A, \oplus, 0)$ is a commutative monoid, and the following identities are satisfied:

$$\neg\neg x = x,$$
$$x \oplus \neg 0 = \neg 0,$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

It is known that every MV-algebra is a bounded distributive lattice in which

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg x \vee \neg y); \tag{2}$$

denoting $1 := \neg 0$ the greatest element of the lattice, the underlying order is given by $x \leq y$ iff $\neg x \oplus y = 1$. It is also worth recalling that the whole variety of MV-algebras is generated by the *standard MV-algebra* $([0,1], \oplus, \neg, 0)$, where $[0,1]$ is the unit interval of reals and the operations are defined by $x \oplus y := \min\{1, x + y\}$ and $\neg x := 1 - x$.

From the algebraic point of view, the basic common feature of the logic of quantum mechanics and Łukasiewicz's logic is that their algebraic counterparts are bounded lattices with the property that every *section* (= principal order-filter) of the lattice is equipped with an *antitone involution*, i.e. an order-reversing involutive bijection from the section onto itself. Specifically, the antitone involutions on the sections $[a, 1]$ are given by $x \mapsto x^{\perp} \vee a$ in orthomodular lattices, and by $x \mapsto \neg x \oplus a$ in MV-algebras.

Starting from this observation, in [5] we have studied lattices with sectional antitone involutions and later in [7] we have introduced what we call *basic algebras*, roughly speaking, the variety of algebras $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ that correspond one-one to bounded lattices with sectional antitone involutions in the sense that (2) makes A into a bounded lattice, with 0 and $1 := \neg 0$ respectively as the least and the greatest element, and for every $a \in A$, the map $x \mapsto \neg x \oplus a$ is an antitone involution on $[a, 1]$. Therefore, basic algebras can be regarded as a common generalization of orthomodular lattices and MV-algebras.

1 Elements of basic algebras

DEFINITION 1. By a *bounded lattice with sectional antitone involutions* we mean a system $(A, \vee, \wedge, ({}^a)_{a \in A}, 0, 1)$ where $(A, \vee, \wedge, 0, 1)$ is a bounded lattice and, for every $a \in A$, the map $x \mapsto x^a$ is an antitone involution on the section $[a, 1]$.

It is worth observing that once we are given the antitone involutions on sections $[a, 1]$, we also have antitone involutions on 'lower sections' $[0, a]$; indeed, for each $a \in A$, the map $x \mapsto ((x^0)^{(a^0)})^0$ is an antitone involution on $[0, a]$. In particular, in orthomodular lattices these are given by $x \mapsto x^\perp \wedge a$, and in MV-algebras by $x \mapsto \neg(x \oplus \neg a)$.

DEFINITION 2. A *basic algebra* is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ that satisfies the identities

$$x \oplus 0 = x, \tag{BA1}$$

$$\neg\neg x = x, \tag{BA2}$$

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x, \tag{BA3}$$

$$\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus \neg(x \oplus z) = \neg 0. \tag{BA3}$$

The original axiomatization in [7] and [6] also contains $1 \oplus x = 1 = x \oplus 1$, where $1 := \neg 0$, but as shown in [9], these identities follow from (BA1)–(BA4), which are independent.

PROPOSITION 3. (Cf. [7])

1. *Let $(A, \vee, \wedge, ({}^a)_{a \in A}, 0, 1)$ be a bounded lattice with sectional antitone involutions. If we define*

$$x \oplus y := (x^0 \vee y)^y \quad \textit{and} \quad \neg x := x^0,$$

then the algebra $(A, \oplus, \neg, 0)$ is a basic algebra. We have $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$, and $x^a = \neg x \oplus a$ for $x \in [a, 1]$.

2. *Let $(A, \oplus, \neg, 0)$ be a basic algebra, and put*

$$x \vee y := \neg(\neg x \oplus y) \oplus y \quad \textit{and} \quad x \wedge y := \neg(\neg x \vee \neg y).$$

Then $(A, \vee, \wedge, 0, 1)$, where $1 := \neg 0$, is a bounded lattice whose underlying order is given by $x \leq y$ iff $\neg x \oplus y = 1$, and for each $a \in A$, the map

$$\gamma_a \colon x \mapsto \neg x \oplus a$$

is an antitone involution on $[a, 1]$. We have $\neg x = \gamma_0(x)$ and $x \oplus y = \gamma_y(\neg x \vee y)$.

3. *The correspondence between bounded lattices with sectional antitone involutions and basic algebras thus established is one-one.*

If we consider the property of having antitone involutions on all sections as the basic feature of the lattice-ordered structures we are interested in, then the previous proposition justifies the name 'basic algebra' used in [7]. We should warn the reader not to confuse these algebras with Hájek's basic fuzzy logic and BL-algebras (see [13]). As a matter of fact, the intersection of our basic algebras and BL-algebras are just MV-algebras.

It turns out to be useful to define another term operation, denoted by $\ominus$, as follows:

$$x \ominus y := \neg(y \oplus \neg x).$$

One can easily verify that:

$$\begin{gathered} x \oplus y = 1 \ominus ((1 \ominus y) \ominus x), \quad \neg x = 1 \ominus x, \\ x \leq y \quad \text{iff} \quad x \ominus y = 0, \\ x \vee y = (\neg y \ominus \neg x) \oplus y, \quad x \wedge y = x \ominus (x \ominus y), \\ (x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z), \quad x \ominus (y \wedge z) = (x \ominus y) \vee (x \ominus z). \end{gathered}$$

If the underlying lattice is distributive, then

$$(x \vee y) \oplus z = (x \oplus z) \vee (y \oplus z) \quad \text{and} \quad x \ominus (y \vee z) = (x \ominus y) \wedge (x \ominus z).$$

Moreover, in every basic algebra we have:

- for every $a \in A$, the map $x \mapsto a \ominus x$ is an antitone involution on $[0, a]$ (cf. the remark following Definition 1);
- $([0, a], \oplus_a, \neg_a, 0)$ is a basic algebra when equipped with the operations defined by
$$x \oplus_a y := a \ominus ((a \ominus y) \ominus x) \quad \text{and} \quad \neg_a x := a \ominus x.$$
We should observe that $x \ominus_a y := \neg_a(y \oplus_a \neg_a x) = x \ominus y$ for $x, y \in [0, a]$.

In the next theorem we characterize orthomodular lattices and MV-algebras within basic algebras. First we notice that according to Proposition 3.1, the basic algebra $(A, \oplus, \neg, 0)$ associated to an orthomodular lattice $(A, \vee, \wedge, ^{\perp}, 0, 1)$ is obtained by letting $\neg x := x^{\perp}$ and $x \oplus y := (x^{\perp} \vee y)^{\perp} \vee y = (x \wedge y^{\perp}) \vee y$.

THEOREM 4. *MV-algebras are precisely the associative basic algebras. Orthomodular lattices are equivalent to basic algebras satisfying the quasi-identity*

$$x \leq y \quad \Rightarrow \quad y \oplus x = y. \tag{3}$$

Proof. (i) Let $(A, \oplus, \neg, 0)$ be an associative basic algebra. Putting $z = 0$ in (BA4) and using associativity, we get

$$1 = \neg(\neg(\neg(x \oplus y) \oplus y) \oplus 0) \oplus (x \oplus 0) = \neg\neg(\neg(x \oplus y) \oplus y) \oplus x = \\ = (\neg(x \oplus y) \oplus y) \oplus x = \neg(x \oplus y) \oplus (y \oplus x),$$

which means that $x \oplus y \leq y \oplus x$ for all $x, y \in A$. Hence $\oplus$ is also commutative and $(A, \oplus, \neg, 0)$ is an MV-algebra.

(ii) Let $(A, \vee, \wedge, ^{\perp}, 0, 1)$ be an orthomodular lattice. Recalling how the corresponding basic algebra $(A, \oplus, \neg, 0)$ is defined, it is plain that (3) is just a reformulation of the orthomodular law (1) because $x \leq y$ implies $y \oplus x = (y \wedge x^{\perp}) \vee x = y$.

Conversely, assuming that $(A, \oplus, \neg, 0)$ is a basic algebra satisfying (3), we have to show that its underlying lattice is an orthomodular lattice. We observe that by (3) the addition $\oplus$ is idempotent, whence it follows that, for each $x \in A$, $\neg x$ is a complement of x. Indeed, $x \oplus x = (\neg x \vee x)^x = x$ iff $\neg x \vee x = 1$ (this also implies $\neg x \wedge x = \neg(\neg x \vee x) = \neg 1 = 0$). Now, if $(A, \vee, \wedge, \neg, 0, 1)$ were not an orthomodular lattice, then it would contain a subalgebra as shown in Fig. 2, where $y \oplus x = (\neg y \vee x)^x = 1^x = x$, a contradiction. $\square$

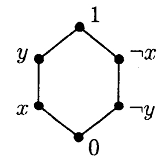

Figure 2:

Since basic algebras are defined by identities only, they form a variety. Theorem 4 then says that, relative to the variety of basic algebras, the variety of MV-algebras is axiomatized by the identity $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, and the variety of (basic algebras equivalent to) orthomodular lattices is axiomatized by the identity $y \oplus (x \wedge y) = y$, which is apparently equivalent to (3).

Let us recall that an algebra $(A, \mathfrak{F})$ is called *arithmetical* if its congruence lattice $\mathrm{Con}(A)$ is distributive and its congruences permute, in the sense that $\theta \circ \phi = \phi \circ \theta$ for all $\theta, \phi \in \mathrm{Con}(A)$. Further, an algebra $(A, \mathfrak{F})$ is *congruence regular* if every congruence $\theta \in \mathrm{Con}(A)$ is fully determined by every single of its classes, i.e., for any $a \in A$ and $\phi \in \mathrm{Con}(A)$, if $[a]_\theta = [a]_\phi$, then $\theta = \phi$.

A variety is called *arithmetical* or *congruence regular* provided that all its members have the property in question. Varieties having such congruence properties can be characterized by so-called Maltsev conditions, for details see e.g. [6].

For basic algebras we have:

THEOREM 5. (Cf. [7]) *The variety of basic algebras is congruence regular and arithmetical.*

In what follows we turn our attention to *commutative* basic algebras, i.e. basic algebras satisfying the identity $x \oplus y = y \oplus x$, which were investigated by M. Botur and R. Halaš (see [1], [2] and [3]).

THEOREM 6. (Cf. [7]) *The underlying lattices of commutative basic algebras are distributive.*

Proof. By way of contradiction we suppose that the lattice is not distributive. If it contains a copy of the pentagon, then it also contains two sublattices as in Fig. 3. Then $a = 1^a = (\neg c \vee a)^a = c \oplus a = a \oplus c = (\neg a \vee c)^c = (\neg u)^c = (\neg b \vee c)^c = b \oplus c = c \oplus b = (\neg c \vee b)^b = 1^b = b$. The case when the lattice contains a copy of the diamond leads to the same contradiction. $\square$

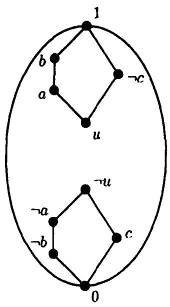

Figure 3:

Another analogy that can be made with MV-algebras is the following *Riesz decomposition property*:

THEOREM 7. *In every commutative basic algebra, if $x \leq a \oplus b$, then $x = a_1 \oplus b_1$ for some $a_1 \leq a$ and $b_1 \leq b$.*

Proof. Put $a_1 := x \ominus b = x \ominus (x \wedge b)$ and $b_1 := x \wedge b$. Then $a_1 \leq (a \oplus b) \ominus b = a \wedge \neg b \leq a$ and $a_1 \oplus b_1 = (x \ominus (x \wedge b)) \oplus (x \wedge b) = x$. $\square$

Our next intent is to prove that *finite* commutative basic algebras are actually finite MV-algebras. The proof below is based on that in [2].

Given a commutative basic algebra, we first define the non-negative multiples of an element x inductively:

$$\begin{aligned} 0 \otimes x &:= 0, \\ n \otimes x &:= ((n-1) \otimes x) \oplus x \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

LEMMA 8. (Cf. [2]) *Let $(A, \oplus, \neg, 0)$ be a finite commutative basic algebra. Every element of A is in the form*

$$\bigvee_{a \in M} n_a \otimes a, \tag{4}$$

where M is the set of the atoms of A, and $n_a \in \mathbb{N}_0$ for all $a \in M$.

Proof. It is not hard to show that if $x \wedge y = 0$, then $x \oplus y = x \vee y$, and $(m \otimes x) \wedge (n \otimes y) = 0$ for all $m, n \in \mathbb{N}_0$.

Now, suppose that there is $z \in A$ which cannot be written in the form (4). Then there exists $x \in A$ that is maximal among those elements which are of the form (4) and are less than or equal to z. Let $x = \bigvee_{a \in M} n_a \otimes a$. Further, there exists $y \in A$ such that $x \prec y \leq z$ (y covers x). Obviously, $b := y \ominus x$ is an atom and y is not in the form (4). Then

$$y = (y \ominus x) \oplus x = b \oplus \left(\bigvee_{a \in M} n_a \otimes a \right) = \bigvee_{a \in M} b \oplus (n_a \otimes a).$$

But for $a \neq b$ we have $b \oplus (n_a \otimes a) = b \vee (n_a \otimes a)$, so

$$y = (b \oplus (n_b \otimes b)) \vee b \vee \bigvee_{a \in M \setminus \{b\}} n_a \otimes a = ((n_b + 1) \otimes b) \vee \bigvee_{a \in M \setminus \{b\}} n_a \otimes a$$

which is an element of the form (4), a contradiction. □

Again, let $(A, \oplus, \neg, 0)$ be a finite commutative basic algebra and M the set of its atoms. For every atom $a \in M$,

$$N(a) = \{ n \otimes a \mid n \in \mathbb{N}_0 \}$$

is a finite chain $0 < a < \cdots < \hat{a}$. Moreover, the Riesz decomposition property entails that $N(a)$ agrees with the interval $[0, \hat{a}]$. Since $(N(a), \oplus_{\hat{a}}, \neg_{\hat{a}}, 0)$ is a finite linearly ordered basic algebra, it is an MV-algebra. We also know that $\ominus_{\hat{a}}$ in $N(a)$ is the restriction of $\ominus$.

THEOREM 9. (Cf. [2]) *Let $(A, \oplus, \neg, 0)$ be a finite commutative basic algebra and M the set of its atoms. Then the map*

$$(x_a)_{a \in M} \mapsto \bigvee_{a \in M} x_a$$

is an isomorphism of $\prod_{a \in M} N(a)$ onto A. Hence $(A, \oplus, \neg, 0)$ is an MV-algebra.

Seeing the aforementioned facts, the proof of Theorem 9 is just a straightforward calculation which we leave to the reader (see [2]).

Thus every finite commutative basic algebra is an MV-algebra because it is a direct product of finite linearly ordered MV-algebras. More generally, M. Botur and R. Halaš [3] proved that:

THEOREM 10. *Every complete (as a lattice) commutative basic algebra is a subdirect product of linearly ordered commutative basic algebras.*

This rises the question if every commutative basic algebra is an MV-algebra. The problem was solved by M. Botur [1] in the negative; an example of a proper commutative basic algebra can be obtained from the standard MV-algebra using the following fact:

THEOREM 11. (Cf. [1]) *Let* $([0,1], \oplus, \neg, 0)$ *be a commutative basic algebra. Then (up to isomorphism) the negation is given by*

$$\neg x := 1 - x.$$

The example itself is quite complicated, so we refer the reader to the original paper [1]. We only remark here that the addition in M. Botur's example looks as in Fig. 4.

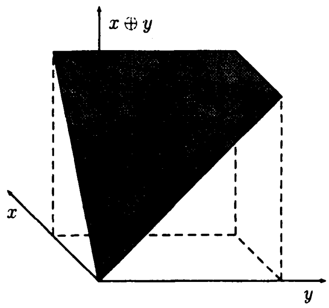

Figure 4:

2 Applications of basic algebras to effect algebras

Effect algebras were introduced by D. J. Foulis and M. K. Bennett [12] as algebraic structures representing unsharp events in quantum mechanics. They include orthomodulat lattices (and more generally, orthomodular posets) as well as MV-algebras as special subclasses.

DEFINITION 12. An *effect algebra* is a structure $(E, +, 0, 1)$ where $0, 1$ are elements of E and $+$ is a partial binary operation on E, satisfying the following conditions:

(EA1) $x + y = y + x$ if one side is defined,

(EA2) $x + (y + z) = (x + y) + z$ if one side is defined,

(EA3) for every x there exists a unique x' such that $x' + x = 1$, (EA4) $x + 1$ is defined only for $x = 0$.

Independently, F. Kôpka and F. Chovanec [14] defined the so-called difference posets (D-posets) that are equivalent to effect algebras:

DEFINITION 13. A *D-poset* is a structure $(D, \leq, -, 0, 1)$ where $(D, \leq, 0, 1)$ is a bounded poset and $-$ is a partial binary operation such that $x - y$ is defined iff $x \geq y$, satisfying the conditions

(DP1) $x - 0 = x$,

(DP2) if $x \leq y \leq z$, then $z - y \leq z - x$ and $(z - x) - (z - y) = y - x$.

The relationships between effect algebras and D-posets are as follows: Every effect algebra $(E, +, 0, 1)$ is naturally ordered by the partial order relation $\leq$ given by

$$x \leq y \quad \text{iff} \quad y = x + z \text{ for some } z.$$

The bounds of this underlying poset are 0 and 1. If $x \leq y$, then the element z such that $y = x + z$ is uniquely determined and is denoted by $y - x$. The structure $(E, \leq, -, 0, 1)$ obtained in this way is a D-poset. On the other hand, to every D-poset $(D, \leq, -, 0, 1)$ there corresponds the effect algebra $(D, +, 0, 1)$ obtained by letting

$$x + y := z \quad \text{iff} \quad z \geq y \text{ and } z - y = x.$$

DEFINITION 14. A *lattice effect algebra* is an effect algebra whose underlying poset is a lattice. A D-poset which is a lattice is called a *D-lattice*.

For comprehensive information on effect algebras and D-posets see [11].

Although effect algebras or D-posets describe quantum effects from the point of view of quantum logic, their disadvantage is that they are partial algebras which causes problems even when doing the most common algebraic constructions, such as substructures, congruences and quotients, because these concepts can be defined in several different ways. Therefore our aim has been to make effect algebras into total algebras, i.e., to extend partial $+$ to a total operation $\oplus$ which determines $+$. Here we focus on lattice effect algebras (see [7]), but in [8] we proved that similar results can be obtained for effect algebras in general.

THEOREM 15. (Cf. [7]) *Let $(E, +, 0, 1)$ be a lattice effect algebra. If we set*

$$x \oplus y := (x \wedge y') + y \quad \textit{and} \quad \neg x := x',$$

then $(E, \oplus, \neg, 0)$ is a basic algebra.

Proof. It is easy to show that for each $a \in E$, the map $\gamma_a \colon x \mapsto x' + a$ is an antitone involution on $[a, 1]$ (and at the same time $x \mapsto a - x$ is an antitone involution on $[0, a]$). Thus $(E, \vee, \wedge, (\gamma_a)_{a\in E}, 0, 1)$ is a lattice with sectional antitone involutions and, according to Proposition 3, the corresponding basic algebra is defined by $x \oplus y := (x^0 \vee y)^y = (x' \vee y)' + y = (x \wedge y') + y$ and $\neg x := x'$. □

We should observe that in the basic algebra $(E, \oplus, \neg, 0)$ associated to $(E, +, 0, 1)$ we have:

$$\begin{aligned} x \ominus y &:= \neg(y \oplus \neg x) = x - (x \wedge y), \\ x - y &= x \ominus y \quad \text{for } x \geq y, \\ x + y &= x \oplus y \quad \text{for } x \leq \neg y, \end{aligned}$$

thus the new total operations indeed extend the original partial operations.

We can easily answer the question which basic algebras are derived from lattice effect algebras:

THEOREM 16. (Cf. [7]) *Let $(A, \oplus, \neg, 0)$ be a basic algebra, and define the partial operation $+$ as follows:*

$$x + y \textit{ is defined iff } x \leq \neg y, \textit{ in which case } x + y := x \oplus y.$$

Then $(A, +, 0, 1)$ is a lattice effect algebra if and only if $(A, \oplus, \neg, 0)$ satisfies the quasi-identity

$$x \leq \neg y \quad \& \quad x \oplus y \leq \neg z \quad \Rightarrow \quad (x \oplus y) \oplus z = x \oplus (z \oplus y). \tag{5}$$

Proof. It suffices to note that (5) captures both commutativity and associativity of the partial addition $+$. Indeed, for $x = 0$ we have: $y \leq \neg z \Rightarrow y \oplus z = z \oplus y$. □

Analogously we can prove

THEOREM 17. *Let $(A, \oplus, \neg, 0)$ be a basic algebra, and define the partial operation $-$ as follows:*

$$x - y \textit{ is defined iff } x \geq y, \textit{ in which case } x - y := x \ominus y.$$

Then $(A, \leq, -, 0, 1)$ is a D-lattice if and only if $(A, \oplus, \neg, 0)$ satisfies the quasi-identity

$$x \leq y \leq z \quad \Rightarrow \quad (z \ominus x) \ominus (z \ominus y) = y \ominus x. \tag{6}$$

This characterization of lattice effect algebras in the setting of basic algebras leads to the following

DEFINITION 18. We call a basic algebra an *effect basic algebra* if it satisfies (5) or, equivalently, (6).

The next concept plays an important role in theory of effect algebras.

DEFINITION 19. In a lattice effect algebra, two elements x, y are *compatible* if

$$(x \vee y) - y = x - (x \wedge y).$$

Since we deal with lattice effect algebras only, we restrict the definition to them, but compatibility between elements can be considered in general effect algebras (see e.g. [11]).

The proofs of the following results can be found in [7]:

THEOREM 20. *Let $(E, \oplus, \neg, 0)$ be an effect basic algebra and $(E, +, 0, 1)$ the associated lattice effect algebra. Then $x, y \in E$ are compatible iff $x \oplus y = y \oplus x$.*

THEOREM 21. *For every basic algebra E, the following are equivalent:*

1. *E is an effect basic algebra;*
2. *every block of E (i.e., a maximal subset whose elements commute) is a subalgebra which itself is an MV-algebra.*

It is also possible to prove the following characterization of MV-algebras as certain effect basic algebras:

THEOREM 22. *For every effect basic algebra E, the following are equivalent:*

1. *E is an MV-algebra;*
2. *E is commutative;*
3. *E satisfies the Riesz decomposition property.*

EXAMPLE 23. In conclusion, we present the smallest effect basic algebra which is neither an orthomodular lattice nor an MV-algebra (see Fig. 5):

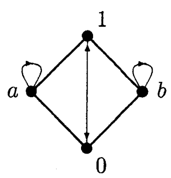

Figure 5: The algebra from Example 23

$\oplus$	0	a	b	1	$\neg$
0	0	a	b	1	1
a	a	1	b	1	a
b	b	a	1	1	b
1	1	1	1	1	0

$+$	0	a	b	1	$'$
0	0	a	b	1	1
a	a	1	$\times$	$\times$	a
b	b	$\times$	1	$\times$	b
1	1	$\times$	$\times$	$\times$	0

(The arrows in Fig. 5 represent the antitone involution $x \mapsto x^0$ in $[0,1]$; the antitone involutions in the remaining sections are trivial.)

THEOREM 24. (Cf. [4]) *The variety generated by the basic algebra from Example 23 is axiomatized, relative to the variety of distributive effect basic algebras, by the identity*

$$(x \ominus y) \ominus (z \oplus z) = (x \ominus (z \oplus z)) \ominus (y \ominus (z \oplus z)).$$

In the lattice of subvarieties of basic algebras, the configuration of the subvarieties mentioned in this paper is shown in Fig. 6. Let us note that effect basic algebras EBA do not form the join of orthomodular lattices OML and MV-algebras MVA in the lattice of subvarieties. Indeed, the algebra from Example 23 is a subdirectly irreducible member of EBA which belongs to neither OML not MVA, hence owing to congruence distributivity, EBA is not the join of OML and MVA, the description of which remains an open problem.

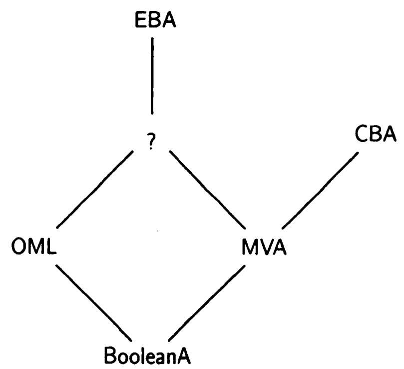

Figure 6: Some varieties

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