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Asymptotic behavior of eigenvalues of the Laplacian with the mixed boundary condition and its application

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1 Introduction and Main Results

In this paper, based on a recent work [5], we present our study on an asymptotic behavior of eigenvalues of the Laplacian on a thin domain under the mixed boundary condition. Let $\Omega\subset\mathbf{R}^{n}$ $(n\geq 2)$ be a bounded domain with smooth boundary $\Gamma=\partial\Omega$. For a sufficiently small $\epsilon>0$, define $\Omega(\epsilon)=\{x\in\Omega\mid d(x,\Gamma)<\epsilon\}$, $\Gamma(\epsilon)=\{x\in\Omega\mid d(x,\Gamma)=\epsilon\}$. Consider the eigenvalue problem:

$$-\Delta\Phi=\lambda\Phi \quad \text{in} \quad \Omega(\epsilon), \quad \Phi=0 \quad \text{on} \quad \Gamma(\epsilon), \quad \frac{\partial\Phi}{\partial\nu}=0 \quad \text{on} \quad \Gamma, \tag{1}$$

where $\nu(x)$ is the outward unit normal vector on Γ.

Let $\{\lambda_{k}(\epsilon)\}_{k=1}^{\infty}$ be the eigenvalues satisfying $0<\lambda_{1}(\epsilon)<\lambda_{2}(\epsilon)\leq\lambda_{3}(\epsilon)\leq\cdots\rightarrow+\infty$ and $\{\Phi_{k,\epsilon}(x)\}_{k=1}^{\infty}$ be the associated eigenfunctions. We may assume $\Phi_{1,\epsilon}(x)>0$ $(x\in\Omega(\epsilon))$ and $\Phi_{1,\epsilon}$ can be obtained as the minimizer of $\lambda_{1}(\epsilon)=\inf\{R_{\epsilon}(\Phi)\mid\Phi\in H^{1}(\Omega(\epsilon)),\ \Phi=0 \text{ on } \Gamma(\epsilon)\}$, where

$$R_{\epsilon}(\Phi)=\frac{\int_{\Omega(\epsilon)}|\nabla_{x}\Phi|^{2}\,dx}{\int_{\Omega(\epsilon)}|\Phi|^{2}\,dx}.$$

In general k-th eigenvalue $\lambda_{k}(\epsilon)$ can be characterized by using the min-max principle.

$$\lambda_{k}(\epsilon)=\sup_{E\subset L^{2}(\Omega(\epsilon)),\dim E\leq k-1}\inf\{R_{\epsilon}(\Phi)\mid\Phi\in H^{1}(\Omega(\epsilon)),\Phi=0 \text{ on } \Gamma(\epsilon),\ \Phi\perp E\}.$$

Here E is a linear subspace of $L^{2}(\Omega(\epsilon))$ and $\Phi\perp E$ means $(\Phi,\Psi)_{L^{2}(\Omega(\epsilon))}=0$ for every $\Psi\in E$. We denote by $H(\xi)$ the mean curvature of Γ at $\xi\in\Gamma$. Then we have the following asymptotic behavior of $\lambda_{k}(\epsilon)$ as $\epsilon\rightarrow 0$.

Theorem 1 *Let $k \geq 1$. Then, as $\epsilon \to 0$, we have*

$$\epsilon^2 \lambda_k(\epsilon) = \overline{\lambda}_1 - (\max_{\xi \in \Gamma} H(\xi))\ \epsilon + O(\epsilon^{3/2}).$$

Here, $\overline{\lambda}_1 = \frac{\pi^2}{4}$ and $\overline{\lambda}_1$ is the first eigenvalue of the eigenvalue problem:

$$-\phi''(s) = \lambda \phi(s),\ s \in (0,1), \quad \phi'(0) = 0,\ \phi(1) = 0.$$

Theorem 1 also suggests that the eigenfunctions $\Phi_{k,\epsilon}(x)$ concentrates on a certain point $\xi^* \in \Gamma$ which attains the maximum of the mean curvature $H(\xi)$.

Remark 1 *A closely related result has been obtained by Krejcirik [6] for $n = 2, 3$ with a rough remainder order term $o(\epsilon)$ instead of $O(\epsilon^{3/2})$. The method is quite different from ours. His result is motivated on a quantum wave guide problem, especially on the work of Dittrich and Kriz [3], which studied existence and non-existence of bound-states on a bent strip under Dirichlet-Neumann boundary condition. For a quantum wave guide problem, see [2], [7] and the references therein. Moreover, concentration phenomena of eigenfunctions also have been studied by S.A Nazarov et. al. [1] on a thin cylindrical domain with Neumann boundary condition on the lateral boundary and Dirichlet boundary condition on other boundaries.*

If we assume the maximum point $\xi^* \in \Gamma$ of $H(\xi)$, i.e. $H(\xi^*) = \max_{\xi \in \Gamma} H(\xi)$, is **non-degenerate**, we can obtain more precise asymptotic behavior of $\lambda_k(\epsilon)$.

Suppose there exists a unique maximum point $\xi^* \in \Gamma$ of $H(\xi)$. We may assume ξ^* is the origin by a suitable transformation. Furthermore, we assume this maximum point is non-degenerate, namely there exist positive constants $\gamma_j > 0, j = 1, 2, \cdots, n-1$, such that $H(\xi)$ can be written by

$$H(\xi) = H(O) - \sum_{j=1}^{n-1} \gamma_j \xi_j^2 + O(|\xi|^3)$$

by using a suitable normal local coordinate $\xi = (\xi_1, \xi_2, \cdots, \xi_{n-1})$ near the origin $O \in \Gamma$.

We denote by $\mathbf{Z}_+ = \{0\} \cup \mathbf{N} = \{0, 1, 2, \cdots\}$ and consider the set

$$\{\Lambda_k\}_{k=1}^{\infty} = \{\sum_{l=1}^{n-1} (2m_l + 1)\sqrt{\gamma_l} \mid (m_1, m_2, \cdots, m_{n-1}) \in \mathbf{Z}_+^{n-1}\}$$

with $\Lambda_1 < \Lambda_2 \leq \cdots \Lambda_k \leq \Lambda_{k+1} \leq \cdots$. Then we have the following sharp asymptotics.

Theorem 2 *Suppose that the mean curvature function $H(\xi)$ has a unique maximum point $\xi^* \in \Gamma$ of $H(\xi)$, which is non-degenerate. Let $k \geq 1$. Then we have*

$$\epsilon^2 \lambda_k(\epsilon) = \overline{\lambda}_1 - (\max_{\xi \in \Gamma} H(\xi))\ \epsilon + \Lambda_k \epsilon^{3/2} + o(\epsilon^{3/2}) \quad \text{as } \epsilon \to 0.$$

Remark 2 *When $\Omega = \{x \in \mathbf{R}^n \mid R - \epsilon < |x| < R\}$, by using a direct computation we have*

$$\epsilon^2 \lambda_k(\epsilon) = (\pi/2)^2 - \frac{(n-1)}{R}\epsilon + (\frac{\Lambda_k}{R^2} - \frac{n^2-1}{4R^2} - \frac{(n-1)^2}{R^2\pi^2})\epsilon^2 + o(\epsilon^2),$$

where Λ_k is the k-th eigenvalue of the Laplacian on S^{n-1}. When $H(\xi)$ is constant near its maximum point, then the following formula would hold in general:

$$\epsilon^2 \lambda_k(\epsilon) = (\pi/2)^2 - c_1\epsilon + O(\epsilon^2), \; c_1 = \max H(\xi).$$

Although Theorem 1 and 2 has its own interest, our another motivation is to solve the question raised by K. Umezu [8] in his study of a certain bifurcation problem arising in population dynamics. As an application of Theorem 1 we give a partial result to that question. Let $\Omega \subset \mathbf{R}^2$ be a bounded smooth domain with smooth boundary $\partial\Omega$. Let $m \in L^\infty(\Omega)$ be a sign changing function satisfying $\int_\Omega m\, dx < 0$. Then it is well-known that the problem:

$$\lambda_1(m) = \inf\Big\{ \frac{\int_\Omega |\nabla\phi|^2\, dx}{\int_\Omega m(x)\phi^2\, dx} \mid \phi \in H^1(\Omega), \int_\Omega m(x)\phi^2\, dx > 0 \Big\} \tag{2}$$

is attained by $\phi(x;m) > 0$ $(x \in \Omega)$ and $\lambda_1(m) > 0$. Then the question is the following: find the condition on $m(x)$ which implies the inequality:

$$\frac{\int_\Omega \phi(x;m)^3\, dx}{\int_{\partial\Omega} \phi(x;m)^3\, dS} < \frac{|\Omega|}{|\partial\Omega|}. \tag{3}$$

We can give a sufficient condition for general domains Ω.

Theorem 3 *Let $n = 2$, $\Omega(\epsilon) = \{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$ and consider the function $m(x)$ satisfying $m(x) = 1$ on $\Omega(\epsilon)$, $m(x) = -s$ on $\Omega \setminus \Omega(\epsilon)$ for $s > 0$. Then there exist a sufficiently small $\epsilon_0 > 0$ and sufficiently large $s_0 > 0$ such that the inequality (3) holds for $0 < \epsilon < \epsilon_0$ and $s > s_0$.*

Let us briefly explain the relation between the question above and the bifurcation problem studied by K.Umezu. Consider the problem

$$-\Delta u = \lambda(m(x)u - u^2), \quad x \in \Omega,$$

$$\frac{\partial u}{\partial \nu} = \lambda b u^2, \quad x \in \partial\Omega,$$

where $m(x)$ is a sign-changing function satisying $\int_\Omega m(x)\, dx < 0$. If the inequality (3) is satisfied, take b such that

$$\frac{\int_\Omega \phi(x;m)^3\, dx}{\int_{\partial\Omega} \phi(x;m)^3\, dS} < b < \frac{|\Omega|}{|\partial\Omega|}.$$

Then Umezu proved that there exists a (subcritical) bifurcation curve $(\lambda, u(x,\lambda))$ which bifurcates at $(\lambda_1(m), 0)$ with $0 < \lambda < \lambda_1(m)$ and $u(x,\lambda) \to +\infty$ as $\lambda \to 0$. So the inequality (3) is a sufficient condition to determine a structure of the bifurcation curve.

2 Outline of the Proof of Theorem 1 and 2

2.1 Preliminaries

First, using a local coordinate $(\xi_1, \xi_2, \cdots, \xi_{n-1})$ for $\xi \in \Gamma = \partial\Omega$, every point $x \in \Omega(\epsilon)$ in the neighborhood of Γ can be expressed by $x = \xi - t\nu(\xi)$ with $x \in \Gamma, 0 < t < \epsilon$.

So let $(\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n) = (\xi_1, \xi_2, \cdots, \xi_{n-1}, t)$ be a local coordinate of $\Gamma \times (-\epsilon, \epsilon)$ and by (g_{ij}) the metric tensor associated with this local coordinate. Then we have $g_{in} = g_{ni} = 0$ $(1 \le i \le n-1)$ and $g_{nn} = 1$. Let $(g^{ij}) = (g_{ij})^{-1}$ and $G = \det\,(g_{ij})_{1\le i,j\le n} = \det(g_{ij})_{1\le i,j\le n-1}$. Then we can write the norm of the gradient and the Laplacian of Φ by using this local coordinate as follows:

$$|\nabla_x \Phi|^2 = \sum_{i,j=1}^{n} g^{ij} \frac{\partial\Phi}{\partial\xi_i}\frac{\partial\Phi}{\partial\xi_j} = \sum_{i,j=1}^{n-1} g^{ij} \frac{\partial\Phi}{\partial\xi_i}\frac{\partial\Phi}{\partial\xi_j} + (\frac{\partial\Phi}{\partial t})^2,$$

$$\Delta\Phi = \sum_{i,j=1}^{n-1} \frac{1}{\sqrt{G}}\frac{\partial}{\partial\xi_i}\Big(g^{ij}\sqrt{G}\frac{\partial\Phi}{\partial\xi_j}\Big) + \frac{1}{\sqrt{G}}\frac{\partial}{\partial t}\Big(\sqrt{G}\frac{\partial\Phi}{\partial t}\Big).$$

Taking $\Phi(\xi, t) = t$, we have

$$\frac{1}{\sqrt{G}}\frac{\partial}{\partial t}\Big(\sqrt{G}\Big) = \Delta t = \mathrm{div}(\nabla t) = -\mathrm{div}(\overline{\nu}) = -H(\xi, t),$$

where $\overline{\nu}(\xi, t)$ is the extended unit normal such that $\overline{\nu}(\xi, 0) = \nu(\xi)$. Now, we obtain the following formula:

$$\sqrt{G(\xi, t)} = \sqrt{G(\xi, 0)}(1 - H(\xi)t) + O(t^2),$$

as $t \to 0$, where $H(\xi)$ is the mean curvature function of Γ at $\xi \in \Gamma$. Note that, when $\Gamma = \partial B(0, R)$, then $H(\xi) = \frac{n-1}{R}$ for $\xi \in \Gamma$. Using a local coordinate and the transformation $\tilde{\Phi}(\xi, \tau) = \Phi(\xi, \epsilon\tau), \xi \in \Gamma, 0 < \tau < 1$, we can rewrite the Rayleigh quotient as follows:

$$R_\epsilon(\Phi) = \frac{\int_{\Omega(\epsilon)} |\nabla_x\Phi|^2\,dx}{\int_{\Omega(\epsilon)} \Phi^2\,dx} = \frac{\int_{\Gamma\times(0,\epsilon)}(|\nabla_\xi\Phi|^2 + (\frac{\partial\Phi}{\partial t})^2)\sqrt{G(\xi,t)}\,d\xi\,dt}{\int_{\Gamma\times(0,\epsilon)} \Phi^2\sqrt{G(\xi,t)}\,d\xi\,dt}$$

$$= \frac{1}{\epsilon^2}\frac{\int_{\Gamma\times(0,1)}(\epsilon^2|\nabla_\xi\tilde{\Phi}|^2 + (\frac{\partial\tilde{\Phi}}{\partial\tau})^2)\sqrt{G(\xi,\epsilon\tau)}\,d\xi\,d\tau}{\int_{\Gamma\times(0,1)} \tilde{\Phi}^2\sqrt{G(\xi,\epsilon\tau)}\,d\xi\,d\tau} = \frac{1}{\epsilon^2}\tilde{R}_\epsilon(\tilde{\Phi}). \tag{4}$$

Now, we recall the definition of the Hermite polynomials $H_m(s)$: for $m \in \mathbf{Z}_+$ and $s \in \mathbf{R}$ define

$$H_m(s) = (-1)^m \exp(\frac{s^2}{2})\frac{d^m}{ds^m}\Big(\exp(-\frac{s^2}{2})\Big).$$

Let $\phi_m(t) = H_m(\sqrt{2}t)\exp(-\frac{t^2}{2})$, $t \in \mathbf{R}$. Then one can see $\phi_m(t)$ satisfies

$$-\frac{d^2}{dt^2}\phi_m(t) + t^2\phi_m(t) = (2m+1)\phi_m(t).$$

Now for $k > 0, \epsilon > 0$ and $m \in \mathbf{Z}_+$, we put

$$\rho_{k,m,\epsilon}(t) = \frac{1}{\pi^{\frac{1}{4}}} \frac{1}{(m!)^{\frac{1}{2}}} k^{\frac{1}{4}} \epsilon^{-\frac{1}{8}} \phi_m \left(\frac{\sqrt{k}t}{\epsilon^{\frac{1}{4}}} \right), \ (t \in \mathbf{R}).$$

Basic properties of the function $\rho_{k,m,\epsilon}$ are as follows:

Lemma 1 *$\rho_{k,m,\epsilon}$ satisfies the following:*

$$\int_{\mathbf{R}} \rho_{k,m,\epsilon}(t) \rho_{k,l,\epsilon}(t)\, dt = \delta(m,l), \quad (m, l \in \mathbf{Z}_+, k, \epsilon > 0),$$

$$-\epsilon \frac{d^2}{dt^2} \rho_{k,m,\epsilon}(t) + k^2 t^2 \rho_{k,m,\epsilon}(t) = k(2m+1)\epsilon^{\frac{1}{2}} \rho_{k,m,\epsilon}(t), \ t \in \mathbf{R}.$$

For the proof of Lemma 1 and other useful properties of $\rho_{k,m,\epsilon}$, see [5].

We will explain how to choose a test function for the case $k = 1$. As a test function we want to choose $\tilde{\Phi}(\xi, \tau) = \psi_{p,\epsilon}(\xi)\phi_1(\tau)$, where $\phi_1(\tau) = \sqrt{2}\cos(\frac{\pi}{2}\tau)$ and a suitably chosen $\psi_{p,\epsilon}(\xi) \in H^1(\Gamma)$. Now take any $k_j > 0$ ($j = 1, 2, \cdots, n-1$ and any $\mathbf{p} = (m_1, m_2, \cdots, m_{n-1}) \in \mathbf{Z}_+^{n-1}$. Let $0 < a < b$ be small numbers and let $\eta(t) \in C_0^\infty(\mathbf{R})$ is a suitable cut-off function. Then we can take our test functions as follows:

$$\psi_{\mathbf{p},\epsilon}(\xi) = \eta(\xi_1)\rho_{k_1,m_1,\epsilon}(\xi_1)\eta(\xi_2)\rho_{k_2,m_2,\epsilon}(\xi_2) \cdots \eta(\xi_{n-1})\rho_{k_{n-1},m_{n-1},\epsilon}(\xi_{n-1})$$

by using a local normal coordinate. To construct a test function for $k > 1$, we must choose k different pair of $\mathbf{p}$ which assures the orthogonality condition.

2.2 Proof of Theorem 1(upper bound for $k = 1$):

For simplicity, we only explain the case $k = 1$. As a test function, using a notation $\psi_\epsilon(\xi) = \psi_{\mathbf{p},\epsilon}(\xi)$ for simplicity, we consider

$$\tilde{\Phi}_\epsilon(\xi, \tau) = \psi_\epsilon(\xi)\phi_1(\tau), \quad \phi_1(\tau) = \sqrt{2}\cos(\frac{\pi}{2}\tau)$$

with normalization $\int_\Gamma \psi_\epsilon(\xi)^2 \sqrt{G(\xi,0)}\, d\xi = 1$. Then the rescaled Rayleigh quotient is expressed by

$$\tilde{R}_\epsilon(\tilde{\Phi}_\epsilon) = \frac{N_1(\epsilon) + N_2(\epsilon)}{M(\epsilon)},$$

$$\begin{aligned} M(\epsilon) &= \int_{\Gamma \times (0,1)} \psi_\epsilon(\xi)^2 \phi_1(\tau)^2 \sqrt{G(\xi,0)}(1 - H(\xi)\epsilon\tau + O(\epsilon^2))\, d\xi\, d\tau \\ &= 1 - \int_\Gamma \psi_\epsilon^2 H(\xi)\sqrt{G(\xi,0)}\, d\xi \times (\frac{1}{2} - \frac{2}{\pi^2})\epsilon + O(\epsilon^2), \\ N_1(\epsilon) &= \int_{\Gamma \times (0,1)} \psi_\epsilon(\xi)^2 (\phi_1'(\tau))^2 \sqrt{G(\xi,0)}(1 - H(\xi)\epsilon\tau + O(\epsilon^2))\, d\xi\, d\tau \end{aligned}$$

$$= \overline{\lambda}_1 - \overline{\lambda}_1(\frac{1}{2} + \frac{2}{\pi^2}) \int_\Gamma \psi_\epsilon^2 H(\xi)\sqrt{G(\xi,0)}\, d\xi \times \epsilon + O(\epsilon^2),$$

$$N_2(\epsilon) = \epsilon^2 \int_{\Gamma\times(0,1)} |\nabla\psi_\epsilon(\xi)|^2(\phi_1(\tau))^2\sqrt{G(\xi,0)}(1 - H(\xi)\epsilon\tau + O(\epsilon^2))\, d\xi\, d\tau$$

$$= \epsilon^2 \int_\Gamma |\nabla\psi_\epsilon(\xi)|^2\sqrt{G(\xi,0)}\, d\xi + O(\epsilon^{\frac{5}{2}}),$$

$$= O(\epsilon^{\frac{3}{2}}),$$

since our test function $\psi_\epsilon(\xi)$ satisfies the following estimate(see [5]):

$$\int_\Gamma |\nabla\psi_\epsilon(\xi)|^2\sqrt{G(\xi,0)}\, d\xi = O(\epsilon^{-\frac{1}{2}}).$$

Therefore, we obtain

$$\tilde{R}_\epsilon(\tilde{\Phi}_\epsilon) = \left\{\overline{\lambda}_1 - \overline{\lambda}_1(\frac{1}{2} + \frac{2}{\pi^2}) \int_\Gamma \psi_\epsilon^2 H(\xi)\sqrt{G(\xi,0)}\, d\xi \times \epsilon + O(\epsilon^{\frac{3}{2}})\right\}$$

$$\times\left\{1 - \int_\Gamma \psi_\epsilon^2 H(\xi)\sqrt{G(\xi,0)}\, d\xi \times (\frac{1}{2} - \frac{2}{\pi^2})\epsilon + O(\epsilon^2)\right\}^{-1}$$

$$= \overline{\lambda}_1 - \overline{\lambda}_1\Big((\frac{1}{2} + \frac{2}{\pi^2}) - (\frac{1}{2} - \frac{2}{\pi^2})\Big) \int_\Gamma \psi_\epsilon^2 H(\xi)\sqrt{G(\xi,0)}\, d\xi \times \epsilon + (\epsilon^{\frac{3}{2}})$$

$$= \overline{\lambda}_1 - c_1\epsilon + \int_\Gamma \psi_\epsilon^2 \hat{H}(\xi)\sqrt{G(\xi,0)}\, d\xi \times \epsilon + (\epsilon^{\frac{3}{2}}),$$

where $H(\xi) = c_1 - \hat{H}(\xi)$ with $c_1 = \max H, \hat{H}(\xi) \geq 0$. These yields the desired upper bound.

2.3 Proof of Theorem 1(lower bound for $k = 1$):

Let $\tilde{\Phi}_\epsilon(\xi,\tau)$ be the 1st eigenfunction. Then

$$\epsilon^2\lambda_1(\epsilon) = \frac{\int_{\Gamma\times(0,1)}\Big(\epsilon^2|\nabla\tilde{\Phi}_\epsilon(\xi,\tau)|^2 + (\partial_\tau\tilde{\Phi}_\epsilon)^2\Big)\sqrt{G(\xi,\epsilon\tau)}\, d\xi\, d\tau}{\int_{\Gamma\times(0,1)} |\tilde{\Phi}_\epsilon(\xi,\tau)|^2\sqrt{G(\xi,\epsilon\tau)}\, d\xi\, d\tau}$$

with normalization

$$\int_{\Gamma\times(0,1)} |\tilde{\Phi}_\epsilon(\xi,\tau)|^2\sqrt{G(\xi,0)}\, d\xi\, d\tau = 1.$$

Let $\phi_l(\tau) = \sqrt{2}\cos((l - \frac{1}{2})\pi\tau)$, $(l \geq 1)$, $\overline{\lambda}_l = (l - \frac{1}{2})^2\pi^2$ and

$$\alpha^{(l)}(\xi,\epsilon) = \int_0^1 \tilde{\Phi}_\epsilon(\xi,s)\phi_l(s)\, ds.$$

By using the Fourier expansion, we can decompose as follows:

$$\tilde{\Phi}_\epsilon(\xi,\tau) = \tilde{\Phi}_\epsilon^{(1)}(\xi,\tau) + \tilde{\Phi}_\epsilon^{(2)}(\xi,\tau),$$

where

$$\tilde{\Phi}^{(1)}_\epsilon(\xi,\tau)=\alpha^{(1)}(\xi,\epsilon)\phi_1(\tau),$$
$$\tilde{\Phi}^{(2)}_\epsilon(\xi,\tau)=\sum_{l=2}^{\infty}\alpha^{(l)}(\xi,\epsilon)\phi_l(\tau).$$

Our normalization implies

$$\sum_{l=1}^{\infty}\int_\Gamma(\alpha^{(l)}(\xi,\epsilon))^2\sqrt{G(\xi,0)}\,d\xi=1.$$

Moreover, we have

$$\int_{\Gamma\times(0,1)}(\partial_\tau\tilde{\Phi}_\epsilon)^2\sqrt{G(\xi,0)}\,d\xi\,d\tau=\sum_{l=1}^{\infty}\int_\Gamma\overline{\lambda}_l(\alpha^{(l)}(\xi,\epsilon))^2\sqrt{G(\xi,0)}\,d\xi$$
$$=\overline{\lambda}_1+\sum_{l=1}^{\infty}(\overline{\lambda}_l-\overline{\lambda}_1)\int_\Gamma(\alpha^{(l)}(\xi,\epsilon))^2\sqrt{G(\xi,0)}\,d\xi.$$

Note that there exists a constant $\delta_1=\delta_1(\epsilon)=O(\epsilon)$ such that

$$1-\delta_1(\epsilon)\le\frac{\sqrt{G(\xi,\epsilon\tau)}}{\sqrt{G(\xi,0)}}\le 1+\delta_1(\epsilon).$$

This yields

$$\epsilon^2\lambda_1(\epsilon)\ge\frac{\int_{\Gamma\times(0,1)}(\partial_\tau\tilde{\Phi}_\epsilon)^2\sqrt{G(\xi,\epsilon\tau)}\,d\xi\,d\tau}{\int_{\Gamma\times(0,1)}(\tilde{\Phi}_\epsilon)^2\sqrt{G(\xi,\epsilon\tau)}\,d\xi\,d\tau}$$
$$\ge\frac{1-\delta_1(\epsilon)}{1+\delta_1(\epsilon)}\frac{\int_{\Gamma\times(0,1)}(\partial_\tau\tilde{\Phi}_\epsilon)^2\sqrt{G(\xi,0)}\,d\xi\,d\tau}{\int_{\Gamma\times(0,1)}(\tilde{\Phi}_\epsilon)^2\sqrt{G(\xi,0)}\,d\xi\,d\tau}$$
$$=\frac{1-\delta_1(\epsilon)}{1+\delta_1(\epsilon)}\int_{\Gamma\times(0,1)}(\partial_\tau\tilde{\Phi}_\epsilon)^2\sqrt{G(\xi,0)}\,d\xi\,d\tau.$$

Now, first we will establish a rough estimate. Thus we obtain

$$\frac{1-\delta_1(\epsilon)}{1+\delta_1(\epsilon)}\Big(\overline{\lambda}_1+\sum_{l=2}^{\infty}\overline{\lambda}_l(\alpha^{(l)}(\xi,\epsilon))^2\sqrt{G(\xi,0)}\,d\xi\Big)$$
$$\le\epsilon^2\lambda_1(\epsilon)\le\overline{\lambda}_1-c_1\epsilon+O(\epsilon^{3/2}).$$

Then we have

$$\sum_{l=2}^{\infty}\int_\Gamma\overline{\lambda}_l(\alpha^{(l)}(\xi,\epsilon))^2\sqrt{G(\xi,0)}\,d\xi=O(\epsilon).$$

By this estimate, we can get

$$\int_{\Gamma\times(0,1)}(\partial_\tau\tilde{\Phi}^{(1)})^2\sqrt{G(\xi,0)}\,d\xi\,d\tau=\overline{\lambda}_1+O(\epsilon),$$

$$\int_{\Gamma\times(0,1)}(\partial_\tau\tilde{\Phi}^{(2)})^2\sqrt{G(\xi,0)}\,d\xi\,d\tau = O(\epsilon),$$
$$\int_{\Gamma\times(0,1)}(\tilde{\Phi}^{(1)})^2\sqrt{G(\xi,0)}\,d\xi\,d\tau = 1+O(\epsilon),$$
$$\int_{\Gamma\times(0,1)}(\tilde{\Phi}^{(2)})^2\sqrt{G(\xi,0)}\,d\xi\,d\tau = O(\epsilon).$$

Now,

$$\int_{\Gamma\times(0,1)}(\tilde{\Phi})^2\sqrt{G(\xi,\epsilon\tau)}\,d\xi\,d\tau$$
$$= 1-\int_{\Gamma\times(0,1)}(\tilde{\Phi}^{(1)}+\tilde{\Phi}^{(2)})^2\sqrt{G(\xi,0)}H(\xi)\tau\,d\xi\,d\tau\times\epsilon+O(\epsilon^2)$$
$$= 1-(\frac{1}{2}-\frac{2}{\pi^2})\int_\Gamma(\alpha^{(1)})^2\sqrt{G(\xi,0)}H(\xi)\,d\xi\times\epsilon+Q_1(\xi),\ Q_1(\xi)=O(\epsilon^{\frac{3}{2}}).$$

Similarly, we obtain

$$\int_{\Gamma\times(0,1)}(\partial_\tau\tilde{\Phi})^2\sqrt{G(\xi,\epsilon\tau)}\,d\xi\,d\tau$$
$$= \overline{\lambda}_1+\sum_{l=2}^{\infty}(\overline{\lambda}_l-\overline{\lambda}_1)\int_\Gamma(\alpha^{(l)})^2\sqrt{G(\xi,0)}\,d\xi$$
$$-(\frac{1}{2}+\frac{2}{\pi^2})\overline{\lambda}_1\int_\Gamma(\alpha^{(1)})^2H(\xi)\sqrt{G(\xi,0)}\,d\xi\times\epsilon+Q_2(\epsilon)$$

with $Q_2(\epsilon)=O(\epsilon^{\frac{3}{2}})$. Combining these estimates, we obtain

$$\overline{\lambda}_1-c_1\epsilon+O(\epsilon^{\frac{3}{2}})\geq\epsilon^2\lambda_1(\epsilon)$$
$$\geq\overline{\lambda}_1-\overline{\lambda}_1\Big((\frac{1}{2}+\frac{2}{\pi^2})-(\frac{1}{2}-\frac{2}{\pi^2})\Big)\int_\Gamma(\alpha^{(l)})^2H(\xi)\sqrt{G(\xi,0)}\,d\xi\times\epsilon$$
$$+\sum_{l=2}^{\infty}(\overline{\lambda}_l-\overline{\lambda}_1)\int_\Gamma(\alpha^{(l)})^2\sqrt{G(\xi,0)}\,d\xi$$
$$+\epsilon^2\int_{\Gamma\times(0,1)}|\nabla\tilde{\Phi}_\epsilon|^2\sqrt{G(\xi,\epsilon\tau)}\,d\xi\,d\tau(1+O(\epsilon))+O(\epsilon^{\frac{3}{2}}).$$
$$= \overline{\lambda}_1-\int_\Gamma(\alpha^{(l)})^2(c_1-\hat{H}(\xi))\sqrt{G(\xi,0)}\,d\xi\times\epsilon$$
$$+\sum_{l=2}^{\infty}(\overline{\lambda}_l-\overline{\lambda}_1)\int_\Gamma(\alpha^{(l)})^2\sqrt{G(\xi,0)}\,d\xi$$
$$+\epsilon^2\int_{\Gamma\times(0,1)}|\nabla\tilde{\Phi}_\epsilon|^2\sqrt{G(\xi,\epsilon\tau)}\,d\xi\,d\tau(1+O(\epsilon))+O(\epsilon^{\frac{3}{2}}).$$

Now we have

$$\sum_{l=2}^{\infty}(\overline{\lambda}_l-\overline{\lambda}_1)\int_\Gamma(\alpha^{(l)})^2\sqrt{G(\xi,0)}\,d\xi=o(\epsilon)$$

and this improves the estimate of $Q_j(\xi), j=1,2$ as follows: $Q_j(\epsilon)=o(\epsilon^{\frac{3}{2}})$. Therefore, we can conclude

$$-c_1+C_2\epsilon^{1/2}\geq(\epsilon^2\lambda_1(\epsilon)-\overline{\lambda}_1)\epsilon^{-1}$$

$$\geq \left\{ -c_1(1+O(\epsilon)) + \int_\Gamma (\alpha^{(1)})^2 \hat{H}(\xi)\sqrt{G(\xi,0)}\,d\xi \right.$$
$$+\frac{1}{\epsilon}\sum_{l=2}^{\infty}(\overline{\lambda}_l - \overline{\lambda}_1)\int_\Gamma (\alpha^{(l)})^2\sqrt{G(\xi,0)}\,d\xi$$
$$\left. +\epsilon\int_{\Gamma\times(0,1)} |\nabla\tilde{\Phi}_\epsilon|^2\sqrt{G(\xi,\epsilon\tau)}\,d\xi\,d\tau + (Q_2(\xi) - \overline{\lambda}_1 Q_1(\xi))\epsilon^{-1}\right\} \times (1+O(\epsilon))^{-1}$$

Now, we are ready to obtain an improved estimate. Thus we obtain

$$\int_\Gamma (\alpha^{(1)})^2 \hat{H}(\xi)\sqrt{G(\xi,0)}\,d\xi = O(\epsilon^{\frac{1}{2}}),$$
$$\sum_{l=2}^{\infty}(\overline{\lambda}_l - \overline{\lambda}_1)\int_\Gamma (\alpha^{(l)})^2\sqrt{G(\xi,0)}\,d\xi = O(\epsilon^{\frac{3}{2}}),$$
$$\epsilon^2\int_{\Gamma\times(0,1)} |\nabla\tilde{\Phi}_\epsilon|^2\sqrt{G(\xi,\epsilon\tau)}\,d\xi\,d\tau = O(\epsilon^{\frac{3}{2}}),$$

and hence we get the desired lower bound:

$$(\epsilon^2\lambda_1(\epsilon) - \overline{\lambda}_1)\epsilon^{-1} \geq -c_1 + O(\epsilon^{\frac{1}{2}}).$$

2.4 Comments on the proof of Theorem 2

To obtain a sharp upper bound, we choose the precise vector $\mathbf{p}$ and $\{k_i\}$ for the test functions to match the coefficients appear in the Taylor expansion of the mean curvature function. Once we obtain the desired sharp upper bound, noting the concentration of L^2 norm near the unique maximum point of $H(\xi)$, we can arrive at the desired lower bound. For the details, see [5].

3 Proof of Theorem 3

3.1 limiting problem and an interpolation inequality

First, the following proposition connects the problem of Umezu and our problem. Take any sequence $\{s_j\}$ such that $s_j \to +\infty$ $(j \to +\infty)$. Then let $m_j(x)$ be a function satisfying $m_j(x) = 1$ on $\Omega(\epsilon)$, $m_j(x) = -s_j$ on $\Omega \setminus \Omega(\epsilon)$ and let $\lambda(m_j(x)$ and $\phi^{(j)}(x) = \phi(x; m_j)$ be the associated eigenvalue and eigenfunction, respectively.

Proposition 1 *$\phi^{(j)}$ converges weakly to $\Phi_{1,\epsilon}$ in $H^1(\Omega)$ and $\lambda(m_j(x)) \to \lambda_1(\epsilon)$ as $j \to +\infty$. Here $\Phi_{1,\epsilon}(x)$ is the zero extention to Ω and can be seen as an element of $H^1(\Omega)$. Moreover, when $n = 2$, we have*

$$\frac{\int_\Omega (\phi^{(j)}(x))^3\,dx}{\int_{\partial\Omega}(\phi^{(j)}(x))^3\,dS} \to \frac{\int_{\Omega(\epsilon)}(\Phi_{1,\epsilon}(x))^3\,dx}{\int_{\partial\Omega}(\Phi_{1,\epsilon}(x))^3\,dS}$$

as $j \to +\infty$.

We can prove Proposition 1 easily by using a standard argument. We also need the following interpolation inequality.

Proposition 2 *Let $n = 2$ and $\phi \in H^1(\Gamma \times (0,1))$ with $\phi(\xi,1) = 0 \quad (\xi \in \Gamma)$. Then there exists constants $C_1 > 0$ and $C_2 > 0$ such that the following inequalities hold: as $U = \Gamma \times (0,1)$,*

$$\sup_{0\le s\le 1} \int_\Gamma |\phi(\xi,s)|^3 \sqrt{G(\xi,0)}\, d\xi \le C_1 \Big(\int_U |\phi(\xi,\tau)|^4 \sqrt{G(\xi,0)}\, d\xi d\tau \Big)^{1/2}$$

$$\times \Big(\int_U |\frac{\partial \phi(\xi,\tau)}{\partial \tau}(\xi,\tau)|^2 \sqrt{G(\xi,0)}\, d\xi\, d\tau \Big)^{1/2},$$

$$\int_U |\phi(\xi,\tau)|^4 \sqrt{G(\xi,0)}\, d\xi d\tau \le C_2 \Big(\int_U |\phi(\xi,\tau)|^2 \sqrt{G(\xi,0)}\, d\xi d\tau \Big)^{1/2}$$

$$\times \Big(\int_U (|\nabla_\xi \phi(\xi,\tau)|^2 + |\nabla_\tau \phi(\xi,\tau)|^2 + |\phi(\xi,\tau)|^2) \sqrt{G(\xi,0)}\, d\xi d\tau \Big)^{3/2}.$$

For the proof of Proposition 2, see [5].

3.2 Outline of the proof of Theorem 3

First by $\tilde{\Phi}(\xi,\tau) = \Phi(\xi,\epsilon\tau)$ we have

$$\frac{\int_{\Omega(\epsilon)} (\Phi_{1,\epsilon}(x))^3\, dx}{\int_\Gamma (\Phi_{1,\epsilon}(x))^3\, dS} = \epsilon \Big(\frac{\int_{\Gamma\times(0,1)} \tilde{\Phi}(\xi,\tau)^3 \sqrt{G(\xi,\epsilon\tau)}\, d\xi\, d\tau}{\int_\Gamma \tilde{\Phi}(\xi,0)^3 \sqrt{G(\xi,0)}\, d\xi} \Big).$$

Since $\sqrt{G(\xi,\epsilon\tau)} = \sqrt{G(\xi,0)} + O(\epsilon)$, it is enough to estimate the quantity:

$$\frac{\int_{\Gamma\times(0,1)} \tilde{\Phi}(\xi,\tau)^3 \sqrt{G(\xi,0)}\, d\xi\, d\tau}{\int_\Gamma \tilde{\Phi}(\xi,0)^3 \sqrt{G(\xi,0)}\, d\xi}.$$

Now we use the Fourier decomposition used in the proof of Theorem 1:

$$\tilde{\Phi}(\xi,\tau) = \tilde{\Phi}^{(1)}(\xi,\tau) + \tilde{\Phi}^{(2)}(\xi,\tau),\ \tilde{\Phi}^{(1)}(\xi,\tau) = \alpha_1(\xi,\epsilon)\phi_1(\tau),$$

where

$$\alpha_1(\xi,\epsilon) = \int_0^1 \tilde{\Phi}(\xi,s)\phi_1(s)\, ds > 0$$

with $\phi_1(s) = \sqrt{2}\cos(\frac{\pi}{2}s)$. On the other hand, from Theorem 1 and its proof, we note that

$$\epsilon^2 \int_{\Gamma\times(0,1)} |\nabla_\xi \tilde{\Phi}|^2 \sqrt{G(\xi,\epsilon\tau)}\, d\xi\, d\tau = O(\epsilon^{\frac{3}{2}})$$

holds. By using this key estimate and Proposition 2, we can obtain the desired estimate. For the details, see [5].

4 Future problems

We give several comments on open questions in this field.
(1) The computation of the coefficient of the fourth order term $O(\epsilon^2)$ would be rather difficult.
(2) Dirichlet-Robin or Robin-Neumann mixed boundary condition would be interesting.
(3) Similar asymptotics would hold for an eigenvalue problem with Dirichlet boundary condition with Neumann window (cf. [4]).
(4) Asymptotic behavior of the least energy of a nonlinear eigenvalue problem $-\Delta u = u^p$ in Ω, $u = 0$ on $\partial\Omega$ for $p > 1$, for example, on a thin domain would be interesting.

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