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A NOTE ON AN OVERDETERMINED PROBLEM WITH NON-CONSTANT NEUMANN BOUNDARY CONDITION

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ABSTRACT. We review some results about a variant of the Saint-Venant problem and about a related overdetermined problem. The latter is a generalization of the Serrin problem where the overdetermination reads $\nabla u(x)=g(x)$ on the boundary of the unknown domain, and $g:\mathbb{R}^{N}\rightarrow[0, \infty)$ is a given function. We analyze some geometric properties of the solution Ω in relation with g and we prove some new results about the continuity of Ω with respect to g, assuming g is an homogeneous function.

1. INTRODUCTION

An overdetermined problem usually consists in a Dirichlet problem given in an unknown domain, whose solution satisfies some extra condition (classically a Neumann boundary condition) which determines univocally the shape of the domain itself. Then the solution of the problem is given by a couple domain-function, where the former is the real object of research. The most famous overdetermined problem is probably the following:

$$\begin{cases}-\Delta u=1 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\\ |\nabla u|=c & \text{on } \partial\Omega,\end{cases} \tag{1}$$

where $c>0$ is a given constant. In a famous paper Serrin [28] proved that, under suitable regularity assumptions, if a solution to this problem exists, then Ω must be a ball $B(O, R)$ and $u(x)=(R^{2}-|x|^{2})/(2n)$. After Serrin, a large amount of literature has been produced about variants of (1), often dealing with similar problems where the Laplacian is substituted by some other operator and the overdetermination writes again as $u=0$ and $|\nabla u|=$ constant on $\partial\Omega$; but also other kinds of overdetermined conditions have been considered in literature, see for instance [4, 6, 14, 15, 17, 20, 27, 29, 30] and references therein. Moreover, related stability issues have been investigated, see for instance [3, 8, 16].

Here we consider the following generalization of (1): given a function $g:\mathbb{R}^{N}\rightarrow[0, +\infty)$, positive outside the origin, we investigate the problem

$$\begin{cases}-\Delta u=1 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\\ |\nabla u(x)|=g(x) & \text{on } \partial\Omega.\end{cases} \tag{2}$$

More precisely: for any bounded open set Ω, we denote by u_Ω the *stress function* of Ω, that is the solution of the *torsion problem*:

$$\begin{cases} -\Delta u_\Omega = 1 & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

or its weak form

$$u_\Omega \in H_0^1(\Omega) \ : \ \int_\Omega \nabla u_\Omega \nabla v = \int_\Omega u_\Omega v \quad \forall v \in H_0^1(\Omega). \tag{4}$$

Then we ask whether a domain Ω exists such that the solution of (3) satisfies

$$|\nabla u_\Omega(x)| = g(x) \qquad x \in \partial\Omega\,, \tag{5}$$

and we investigate some geometric properties of Ω in connection with the properties of the assigned function g.

Let us quote that the same overdetermined conditions (5) have been already considered for differential problems of the torsion and of the Bernoulli types in [1, 2, 5, 19, 22].

The overdetermined problem (2) is naturally linked to the shape optimization problem we describe here below. Let J be the functional defined as the opposite of the torsional rigidity:

$$J(\Omega) = \frac{1}{2}\int_\Omega |\nabla u_\Omega|^2\, dx - \int_\Omega u_\Omega\, dx = -\frac{1}{2}\int_\Omega u_\Omega\, dx = -\frac{1}{2}\int_\Omega |\nabla u_\Omega|^2\, dx. \tag{6}$$

and let

$$\phi(\Omega) := \int_\Omega g^2(x)\, dx. \tag{7}$$

The shape optimization problem consists in minimizing J with the constraint $\phi(\Omega) \le 1$, i.e.

$$\min\{J(\Omega)\ :\ \phi(\Omega) \le 1\}\,. \tag{8}$$

Notice that (8) is a variant of the famous Saint-Venant problem. This consists in looking for the domain with given area which has maximal torsional rigidity; the answer is the ball, as proved by G. Polyà [25]. Here we consider the same problem in the class of non-uniformly dense sets, whose density is driven by the function g.

We also recall that problem (8) is close to the one considered by B. Gustafsson and H. Shahgholian in [19]. Indeed they study the partial differential equation $-\Delta u = f$, where f is a function (or a measure) *whose positive part* $(f)^+ = \max\{0, f\}$ *has compact support*, which is not our case.

In [7], the authors proved that problem (11) admits a solution (in the class of quasi-open sets) under the assumptions

$$g(x) > 0 \ \text{ for } x \neq 0\,, \qquad \lim_{|x|\to+\infty} g(x) = +\infty\,. \tag{9}$$

The existence proof is based on the concentration-compactness argument by Bucur [11].

On the other hand, the simple existence of a solution to the shape optimization problem (8) does not guarantee the existence of a solution of the overdetermined problem (2). To encounter this problem, in [7] the following assumption on g is considered:

$$\begin{cases} g: \mathbb{R}^N \to \mathbb{R} \text{ positively homogeneous of degree } \alpha \\ \qquad \text{(i.e. } g(tx) = t^\alpha g(x)\ \forall t > 0, \forall x \in \mathbb{R}^N), \\ g \text{ Hölder continuous, } g > 0 \text{ outside } 0. \end{cases} \tag{10}$$

Thanks to (10), it is possible to find a solution Ω to problem (2) and to prove a series of properties of such a solution.

In Section 2 we recall some basic properties and results about the shape optimization problem (8) and its connection with the overdetermined problem (2). In Section 3 we review some basic properties of the solution of (2) and its geometric properties, like starshape, convexity, Steiner symmetry. In Section 4 we give some new results about the continuity of Ω with respect to g. Notice that most of the results recalled in this paper are taken from [7]. Precisely only Lemma 4.2, Corollary 4.3) and Theorem 4.4 contain original results, while Theorem 3.2 is a slight improvement of the corresponding result of [7].

2. The shape optimization problem

Let us consider the energy functional J defined in (6). It is easily seen by the maximum principle that J is decreasing with respect to set inclusion, that is

$$J(\Omega_1) \geq J(\Omega_2) \quad \text{if} \quad \Omega_1 \subset \Omega_2 .$$

Problem (8) consists in minimizing $J(\Omega)$ among open sets satisfying

$$\phi(\Omega) = \int_\Omega g^2 dx \leq 1 . \tag{11}$$

Notice that the measure of a set Ω satisfying (11) must be bounded if $g(x) \to +\infty$ as $|x| \to \infty$. Thanks to this, it is possible to prove the following.

Theorem 2.1. *Under assumption* (9), *there exists a quasi-open set* Ω *solving the shape optimization problem* (8).

The proof in [7] follows the lines of [18] (see also [21]) and uses a concentration-compactness argument as in [11] to prove the existence of a minimizer which is *quasi-open* and may be unbounded. We refer to [21] for a precise definition and discussion of the concept of quasi-open sets; here we just say that, roughly speaking, quay-open sets are super level sets of functions in $H^1(\mathbb{R}^N)$.

Apart from the proof of existence, in the rest of the paper [7] the function g is assumed to be homogeneous, precisely to satisfy assumption (10). This makes problem (8) easier and with a nice behavior with respect to homotheties. More precisely, for every $t > 0$, $\Omega \subset \mathbb{R}^N$, it holds

$$\begin{aligned} J(t\Omega) &= -\frac{1}{2} \int_{t\Omega} t^2 u_\Omega(x/t)\, dx = t^{2+N} J(\Omega), \\ \phi(t\Omega) &= \int_{t\Omega} g^2(x)\, dx = t^{2\alpha+N} \phi(\Omega), \end{aligned}$$

where the first equality follows from the fact that the stress function of $t\Omega$ is

$$u_{t\Omega}(x) = t^2 u_\Omega(x/t)\,. \tag{12}$$

Then using the notion of *local shape subsolution* introduced in [12], it is proved that the minimizer is in fact bounded and it is possible to obtain its regularity as in [9] (see also [10]). The main difficulty is to prove that the solution Ω is actually an open set. Once proved this, one can get higher regularity by classical techniques from free boundary problems like in [5] and [19].

Theorem 2.2. *Under assumption* (10) *the shape optimization problem* (8) *admits a solution* Ω. *If* $\alpha \neq 1$, Ω *is connected. Moreover, in dimension* $N = 2$ *the solution is* $C^{1,\beta}$ *for some* $\beta > 0$*; in dimension* $N \geq 3$, *the reduced boundary* $\partial_{red}\Omega$ *is* $C^{1,\beta}$ *and* $\partial\Omega \setminus \partial_{red}\Omega$ *has zero* $(N-1)$*-Hausdorff measure.*

Thanks to (12) the existence of a solution to the overdetermined problem (2) follows by choosing a suitable dilation of a solution of (8).

Corollary 2.3. *Let* g *satisfy* (10) *for some* $\alpha > 0$, $\alpha \neq 1$. *Then there exists a solution to the overdetermined Free Boundary Problem* (2).

Remark 2.4. *Notice that the case* $\alpha = 1$ *is special, as it can be seen by considering the radially symmetric situation, where it is possible to have either no solution or an infinite number of solutions.*

3. The overdetermined problem and the geometry of Ω

Under assumption (10) with

$$\alpha > 1$$

several geometric properties of the solutions of the overdetermined problem (2) are proved in [7]. The first one is the fact that the origin O must be inside the domain. This property may look rather technical, but it is fundamental to obtain many other properties of the solution.

Proposition 3.1. *Assume that* g *satisfied* (10) *with* $\alpha > 1$. *Let* Ω *be a solution of the minimization problem (8). Then the origin* O *is inside* Ω.

Then the monotonicity of Ω with respect to g is proved. Here we present a slight improvement of the corresponding result in [7].

Theorem 3.2. *Let* Ω_1, Ω_2 *be (regular enough) bounded solutions to Problem (2) related to* g_1 *and* g_2, *respectively, with* $O \in \Omega_i$ *for* $i = 1, 2$. *Assume that for some* $\alpha > 1$ *at least one between the following two assumptions hold:*

$$g_1(tx) \geq t^\alpha g_1(x) \quad \textit{for } t > 0\,,\ x \in \mathbb{R}^N\,, \tag{13}$$

or

$$g_2(tx) \leq t^\alpha g_2(x) \quad \textit{for } t > 0\,,\ x \in \mathbb{R}^N\,, \tag{14}$$

If $g_1(x) \geq g_2(x)$ *for every* $x \in \mathbb{R}^N$, *then* $\Omega_1 \subseteq \Omega_2$.

Proof. The proof is very similar to the proof of [7, Theorem 3.2], but we give it for completeness.

Assume that (13) holds and assume by contradiction $\Omega_1 \not\subset \Omega_2$. Then

$$t = \sup\{s > 0 \ : \ s\Omega_1 \subseteq \Omega_2\} < 1.$$

Furthermore $t > 0$ for $O \in \Omega_2$ and Ω_1 is bounded.

Consider the set $t\Omega_1$: $t\Omega_1 \subseteq \Omega_2$; there exists $\bar{x} \in \partial(t\Omega_1) \cap \partial\Omega_2$, with $\nu_{t\Omega_1}(\bar{x}) = \nu_{\Omega_2}(\bar{x}) = \nu$, where $\nu_\Omega(x)$ denotes the outer unit normal to $\partial\Omega$ at x.

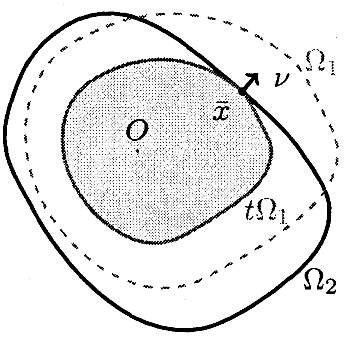

FIGURE 1. $t\Omega_1 \subseteq \Omega_2$ with $\bar{x} \in \partial(t\Omega_1) \cap \partial\Omega_2$

We want to compare $u_{t\Omega_1}$ and u_{Ω_2}, the stress functions of $t\Omega_1$ and Ω_2, respectively. Notice that $u_{t\Omega_1}(x) = t^2 u_{\Omega_1}(\frac{x}{t})$. Define $w = u_{\Omega_2} - u_{t\Omega_1}$; it satisfies

$$\begin{cases} \Delta w = 0 & \text{in } t\Omega_1, \\ w \geq 0 & \text{on } \partial t\Omega_1, \\ w(\bar{x}) = 0. \end{cases}$$

Hence by Hopf Lemma, it holds $\frac{\partial w}{\partial \nu}(\bar{x}) > 0$. On the other hand

$$0 < \frac{\partial w}{\partial \nu}(\bar{x}) = |\nabla u_{\Omega_2}(\bar{x})| - |\nabla u_{t\Omega_1}(\bar{x})| = g_2(\bar{x}) - t g_1(\frac{\bar{x}}{t}),$$

since ν is parallel to $\nabla u_{\Omega_2}(\bar{x}), \nabla u_{t\Omega_1}(\bar{x})$ and $|\nabla u_{t\Omega_1}(\bar{x})| = t|\nabla u_{\Omega_1}(\frac{\bar{x}}{t})|$, with $\frac{\bar{x}}{t} \in \partial\Omega_1$. Hence, by assumption (13) and since $t < 1$ and $\alpha > 1$, we get

$$g_2(\bar{x}) > t g_1(\frac{\bar{x}}{t}) \geq t^{1-\alpha} g_1(\bar{x}) > g_1(\bar{x}),$$

which contradicts the assumption $g_1 \geq g_2$.

Now assume that (14) holds and, by contradiction, assume $\Omega_1 \not\subset \Omega_2$. Set

$$\tau = \inf\{s > 0 \ : \ \Omega_1 \subseteq s\Omega_2\}$$

and notice that $\tau > 1$ (and $\tau < \infty$ since $O \in \Omega_2$ and Ω_1 is bounded).

Consider the set $\tau\Omega_2$. Then $\Omega_1 \subseteq \tau\Omega_2$ and there exists $\bar{x} \in \partial\Omega_1 \cap \partial\tau\Omega_2$, with $\nu_{\Omega_1}(\bar{x}) = \nu_{\tau\Omega_2}(\bar{x}) = \nu$. Now we compare u_{Ω_1} and $u_{\tau\Omega_2}$, the stress functions of Ω_1 and

$\tau\Omega_2$ respectively, and we set $w = u_{\tau\Omega_2} - u_{\Omega_1}$ in Ω_1. Arguing as before we get $w \geq 0$ in $\overline{\Omega}_1$ and $w(\bar{x}) = 0$ and the Hopf Lemma yields

$$tg_2(\frac{\bar{x}}{\tau}) > g_1(\bar{x}).$$

On the other hand by (14)

$$tg_2(\frac{\bar{x}}{\tau}) \leq \tau^{1-\alpha} g_2(\bar{x}) \leq g_2(\bar{x}),$$

since $\tau > 1$ and $\alpha > 1$. Combining the latter with the former contradicts again the assumption $g_1 \geq g_2$. □

As a natural straightforward corollary of the previous theorem, the uniqueness of the solution follows.

Corollary 3.3. *If g satisfies assumptions* (10) *for $\alpha > 1$, then the solution of* (2) *is unique.*

From now on, Ω will denote the solution of the overdetermined problem (2), unless otherwise explicitly specfied.

In [7] it is also investigated the geometry of Ω in connection with the properties of g. In particular, the following results about starshape and convexity hold.

Theorem 3.4. *If g satisfies assumption (10) for $\alpha > 1$, then Ω is starshaped (with respect to O).*

For possible reader's convenience, we recall that a set Ω is said *starshaped with respect to a point $x_0 \in \Omega$* if

$$t(x - x_0) + x_0 \in \Omega \quad \text{for every } x \in \Omega \text{ and every } t \in [0, 1].$$

When $x_0 = O$ we simply say that Ω is starshaped.

We also recall that a lower semicontinuous function $u : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$ is said *quasi-convex* if it has convex sublevel sets, or, equivalently, if

$$u\left((1-\lambda)x_0 + \lambda x_1\right) \leq \max\{u(x_0), u(x_1)\},$$

for every $\lambda \in [0, 1]$, and every $x_0, x_1 \in \mathbb{R}^N$. If u is defined only in a proper subset Ω of $\mathbb{R}^N$, we extend u as $+\infty$ in $\mathbb{R}^N \setminus \Omega$ and we say that u is quasi-convex in Ω if such an extension is quasi-convex in $\mathbb{R}^N$.

Theorem 3.5. *Let g be a quasi-convex and homogeneous function of degree $\alpha \geq 2$, with $g(x) > 0$ for $x \neq 0$. Then Ω is convex.*

Notice that, due to the α-homogeneity, the quasi-convexity of g is equivalent (for $\alpha > 0$) to the following apparently stronger property:

$$g^{1/\alpha} \text{ is convex.}$$

Finally, it is noticed in [7] that also Steiner symmetry is preserved by the solution of the shape optimization problem (8) (and then by the solution of (2) given by Corollary 2.3). Precisely, the following holds.

Theorem 3.6. *Consider a function g satisfying assumptions (10) and assume g to be Steiner symmetric with respect to the hyperplane $\{x_N = 0\}$, that is*

$$g(x', x_N) \geq g(x', y_N), \quad \textit{whenever } |x_N| \geq |y_N|,\ x' \in \mathbb{R}^{N-1}.$$

Then the solution Ω to Problem (8) is symmetric with respect to $\{x_N = 0\}$.

4. Continuity with respect to g

Always thanks to homogeneity, it is easily understood that g is completely determined by α and by the shape of one of it level sets. For this reason it is convenient to set

$$G_1 = \{x \in \mathbb{R}^N \, : \, g(x) < 1\}.$$

More generally for $t \in (0, \infty)$ we denote by G_t the (open) t-sublevel set of g, that is

$$G_t = \{x \in \mathbb{R}^N \, : \, g(x) < t\}.$$

Then G_1 and α fully characterize g and it is easily seen that all the level sets are dilations of G_1, precisely

$$G_t = t^{\frac{1}{\alpha}} G_1 . \tag{15}$$

Notice that, as level set of a homogeneous (not identically zero) function, G_1 must be starshaped.

If G_1 is a ball then g is radial and Ω is a ball. This may suggest some strong relation between the shape of G_1 and the shape of Ω, but in fact this is the only one case where Ω is a level set of g (i.e. it has the same shape as G_1), as the famous Serrin's result [28] implies. On the other hand, some estimate of Ω in term of G_1 is possible and we recall the following result from [7].

Theorem 4.1. *If G_1 is regular enough, there exist two positive constants A and B, depending only on G_1, such that if $\alpha > 1$ it holds*

$$A^{1/(\alpha-1)} G_1 \subseteq \Omega \subseteq B^{1/(\alpha-1)} G_1 .$$

The constants A and B can be explicitly computed as follows: let u_1 be the solution of

$$\begin{cases} -\Delta u_1 = 1 & \text{in } G_1 \\ u_1 = 0 & \text{on } \partial G_1 ; \end{cases} \tag{16}$$

then

$$A = \min_{\partial G_1} |\nabla u_1| , \quad B = \max_{\partial G_1} |\nabla u_1| . \tag{17}$$

Notice that $A \leq B$ and $A < B$ unless G_1 is a ball (see [28]), that is g is radial. In such a case, the solution Ω is a ball. An interesting question which naturally rises is the following: if g is close, in some sense, to be a radial function, is Ω close (in a suitable sense) to be a ball? And how does the distance of Ω from the round shape depend on the distance of g from the radial shape?

In fact it is possible to use Theorem 4.1 to estimate the stability of the radial symmetry, but in [7] better results are obtained in this sense by using Theorem 3.2. Here we prove that it is possible to improve the argument of Section 7 of [7] based on Theorem 3.2 to get the continuity of Ω with respect to g or, in other words, to get the stability of the solution of (8) or (2), even when radial symmetry is not involved, for $\alpha > 1$.

Precisely we consider two α-homogeneous functions g and h and we show that we can control the Hausdorff distance between the associated solutions Ω_g and Ω_h of the corresponding overdetermined problem (2) problem in terms of some distance between g and h, precisely in terms of the Hausdorff distance between G_1 and H_1, where

$$G_1 = \{x : g(x) \leq 1\}, \qquad H_1 = \{x : h(x) \leq 1\}.$$

We recall that the Hausdorff distance between two sets E and F is defined as

$$d_H(E,F) = \max\{\sup_{x\in F} d(x,E), \sup_{x\in E} d(x,F)\} = \min\{r \geq 0 : E \subseteq F+rB_1, F \subseteq E+rB_1\}.$$

First we give this easy lemma.

Lemma 4.2. *Let g and h satisfy* (10) *with the same $\alpha > 1$ and denote by Ω_g and Ω_h the solutions of problem (2) related to g and h respectively. Assume there exists $\epsilon > 0$ such that*

$$(1-\epsilon)g(x) \leq h(x) \leq (1+\epsilon)g(x) \quad x \in \mathbb{R}^N. \tag{18}$$

Then

$$(1+\epsilon)^{-1/(\alpha-1)}\Omega_g \subseteq \Omega_h \subseteq (1-\epsilon)^{-1/(\alpha-1)}\Omega_g.$$

Proof. First observe that, given $\lambda > 0$ and E solution of (2) associated to a given function k, satisfying (10), the solution of

$$\begin{cases} -\Delta u = 1 & \text{in } E \\ u = 0 & \text{on } \partial E, \\ |\nabla u(x)| = \lambda k(x) & \text{on } \partial E. \end{cases}$$

is $\lambda^{-1/(\alpha-1)}E$.

Then the conclusion follows from (18) thanks to Theorem 3.2. □

Corollary 4.3. *Let g and h satisfy* (10) *with the same $\alpha > 1$ and denote by Ω_g and Ω_h the solutions of problem (2) related to g and h respectively. Assume there exists $\epsilon > 0$ such that*

$$(1-\epsilon)G_1 \subseteq H_1 \subseteq (1+\epsilon)G_1. \tag{19}$$

Then

$$(1-\epsilon)^{\alpha/(\alpha-1)}\Omega_g \subseteq \Omega_h \subseteq (1+\epsilon)^{\alpha/(\alpha-1)}\Omega_g.$$

Proof. Assumption (19) can be rewritten as

$$(1-\epsilon)x \in H_1 \text{ for every } x \in G_1$$

and

$$\frac{y}{1+\epsilon} \in G_1 \text{ for every } y \in H_1,$$

or equivalently

$$h((1-\epsilon)x) \leq 1 = g(x) \text{ for every } x \in \partial G_1$$

and

$$g((1+\epsilon)^{-1}y) \leq 1 = h(y) \text{ for every } y \in \partial H_1.$$

Thanks to the homogeneity of g and h, these yield

$$(1+\epsilon)^{-\alpha}g(x) \leq h(x) \leq (1-\epsilon)^{-\alpha}g(x) \quad x \in \mathbb{R}^N$$

and the conclusion follows from Lemma 4.2. □

Let us denote by ρ_1 and ρ_2 the radial functions of the starshaped sets G_1 and G_2 respectively, that is

$$\rho_i(\theta) = \sup\{\rho \geq 0 \,:\, \rho\theta \in F_i\}, \qquad \theta \in S^{N-1}\; i = 1,2$$

where $F_1 = G_1$ and $F_2 = H_1$ and set

$$r_i = \min_{S^{N-1}} \rho_i\,, \quad R_i = \max_{S^{N_1}} \rho_i\,, \quad i = 1,2,$$

$$r = \min\{r_1, r_2\} \quad R = \max\{R_1,\, R_2\}\,.$$

Now we are ready to state the following.

Theorem 4.4. *Let g and h satisfy* (10) *with the same $\alpha > 1$ and let G_1, H_1, r and R as above. Denote by Ω_g and Ω_h the solutions of problem (2) related to g and h respectively. Then there exists a constant $C > 0$ depending only on r and R such that*

$$d_H(\Omega_g, \Omega_h) \leq C\, d_H(G_1, H_1)\,, \tag{20}$$

for $d_H(G_1, H_1)$ small enough.

Proof. Set $d_H(G_1, H_1) = d$. Then

$$G_1 \subseteq H_1 + dB_1 \quad \text{and} \quad H_1 \subseteq G_1 + dB_1\,,$$

whence

$$G_1 \subseteq \left(1 + \frac{d}{r}\right) H_1 \quad \text{and} \quad H_1 \subseteq \left(1 + \frac{d}{r}\right) G_1\,.$$

Then Corollary 4.3 entails

$$\Omega_g \subseteq \left(1 + \frac{d}{r}\right)^{\alpha/(\alpha-1)} \Omega_h$$

and

$$\Omega_h \subseteq \left(1 + \frac{d}{r}\right)^{\alpha/(\alpha-1)} \Omega_g\,.$$

Since

$$\left(1 + \frac{d}{r}\right)^{\alpha/(\alpha-1)} \leq 1 + \frac{2\alpha d}{(\alpha-1)r}$$

for d small enough, we can write

$$\Omega_g \subseteq \Omega_h + \frac{2\alpha\, d}{(\alpha-1)r}\,\Omega_h \quad \text{and} \quad \Omega_h \subseteq \Omega_g + \frac{2\alpha\, d}{(\alpha-1)r}\,\Omega_g\,. \tag{21}$$

Notice now that, by the very definition of R, we have $G_1, H_1 \subseteq R\,B_1$, whence

$$g(x) \geq \left(\frac{|x|}{R}\right)^{\alpha} \quad \text{and} \quad h(x) \geq \left(\frac{|x|}{R}\right)^{\alpha}.$$

Since the solution of problem (2) for $g(x) = |x|^\alpha/R^\alpha$ is the ball centered at O with radius

$$\rho = \frac{R^{\alpha/(\alpha-1)}}{n^{1/(\alpha-1)}}\,,$$

Theorem 3.2 implies $\Omega_g, \Omega_h \subseteq \rho B_1$. Then (21) entails

$$\Omega_g \subseteq \Omega_h + \frac{2\alpha\rho}{(\alpha-1)r}\, d B_1 \quad \text{and} \quad \Omega_h \subseteq \Omega_g + \frac{2\alpha\rho}{(\alpha-1)r}\, d B_1 .$$

that are equivalent to (20) with

$$C = \frac{2\alpha\rho}{(\alpha-1)r} .$$

□

References

[1] A. Acker, *Interior free boundary problems for the Laplace equation*, Arch. Rational Mech. Anal. 75 (1980/81), n. 2, 157-168.

[2] A. Acker, *Uniqueness and monotonicity of solutions for the interior Bernoulli free boundary problem in the convex n-dimensional case*, Nonlinear Anal. 13 (1989), n. 12, 1409-1425.

[3] A. Aftalion, J. Busca, W. Reichel, *Approximate radial symmetry for overdetermined boundary value problems*, Adv. Diff. Eq. 4 n. 6 (1999), 907-932

[4] V. Agostiniani, R. Magnanini, *Symmetries in an overdetermined problem for the Green's function*, Discrete Contin. Dyn. Syst. Ser. S 4 (2011), n. **4**, 791-800.

[5] H.W. Alt, L.A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math., **325** (1981), 105-144.

[6] C. Bianchini, *A Bernoulli problem with non constant gradient boundary constraint*, Appl. Anal. **91**, n.3 (2012), 517-527.

[7] C. Bianchini, A. Henrot, P. Salani, *An overdetermined problem with non-constant boundary conditions*, preprint 2013.

[8] B. Brandolini, C, Nitsch, P. Salani, C. Trombetti, *On the stability of the Serrin problem*, J. Diff. Equations 245, Issue 6 (2008), 1566-1583.

[9] T. Briançon, *Regularity of optimal shapes for the Dirichlet's energy with volume constraint*, ESAIM: COCV, **10** (2004), 99-122.

[10] T. Briançon, M. Hayouni, M. Pierre, *Lipschitz Continuity of State Functions in Some Optimal Shaping*, Calculus of Variations and PDE's, **23** no 1 (2005), 13-32.

[11] D. Bucur, *Concentration-compacité et γ-convergence*, C. R. Acad. Sci. Paris, Série I, **327** (1998), p. 255-258.

[12] D. Bucur, *Minimization of the k-th eigenvalue of the Dirichlet Laplacian*, Arch. Rational Mech. Anal., 206-3 (2012), 1073-1083.

[13] L. A. Caffarelli, J. Spruck *Convexity properties of solutions to some classical variational problems*, Comm. Partial Differential Equations 7 (1982), n. 11, 1337-1379.

[14] A. Cianchi, P. Salani, *Overdetermined anisotropic elliptic problems*, Math. Ann. **345** n. 4 (2009), 859-881.

[15] G. Ciraolo, R. Magnanini, S. Sakaguchi, *Symmetry of minimizers with a level surface parallel to the boundary*, Preprint 2012.

[16] G. Ciraolo, R. Magnanini, S. Sakaguchi, *Solutions of elliptic equations with a level surface parallel to the boundary: stability of the radial configuration*, Preprint 2013.

[17] C. Enache, S. Sakaguchi, *Some fully nonlinear elliptic boundary value problems with ellipsoidal free boundaries*, Math. Nachr. **284** (2011), n. 14-15, 18721879.

[18] M. Crouzeix, *Variational approach of a magnetic shaping problem*, Eur. J. Mech. B Fluids 10, (1991), 527-536.

[19] B. Gustafsson, H. Shahgholian, *Existence and geometric properties of solutions of a free boundary problem in potential theory*, J. Reine Angew. Math., **473** (1996), 137-179.

[20] A. Henrot, G. A. Philippin, *Some overdetermined boundary value problems with elliptical free boundaries*, SIAM J. Math. Anal. **29** n. 2 (1998), 309-320.

[21] A. Henrot, M. Pierre, *Variation et Optimisation de forme, une analyse géométrique*, Mathématiques et Applications, vol. **48**, Springer 2005.

[22] A. Henrot, H. Shahgholian, *The one phase free boundary problem for the p-Laplacian with non-constant Bernoulli boundary condition*, Trans. Amer. Math. Soc. **354** no 6 (2002), 2399-2416.

[23] B. Kawohl, *Rearrangements and convexity of level sets in P.D.E.*, Lecture Notes in Mathematics, 1150, Springer, Berlin 1985.

[24] C.B. Morrey, *Multiple integrals in the calculus of variations*, Springer 1966.

[25] G. Pólya, *Torsional rigidity, principal frequency, electrostatic capacity and symmetrization*, Quart. Appl. Math., **6** (1948), 267-277.

[26] G. Pólya, G. Szegö, *Isoperimetric inequalities in mathematical physics*, Ann. Math. Studies, **27**, Princeton Univ. Press, 1951.

[27] P. Salani, *A Characterization of balls through optimal concavity for potential functions*, to appear in Proc. Amer. Math. Soc.

[28] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal., **43** (1971), 304-318.

[29] P. W. Schaefer, *On nonstandard overdetermined boundary value problems*, Proceedings of the Third World Congress of Nonlinear Analysis, Part 4 (Catania, 2000). Nonlinear Anal. **47** n. 4 (2001), 2203-2212.

[30] H. Shahgholian, *Diversifications of Serrin's and related problems*, in press in Complex Var. and Elliptic Eq.